FUBINI PROPERTIES OF IDEALS

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Abstract. Let I and J be σ-ideals on Polish spaces X and Y, respectively.

We say that the pair $\langle I, J\rangle$ has the Fubini Property (FP) if for every Borel subset B of $X \times Y$, if all its sections $B_x = \{y : \langle x, y\rangle \in B\}$ are in J, then its sections $B^y = \{x : \langle x, y\rangle \in B\}$ are in I, for every y outside a set from J. We study the question, which pairs of σ-ideals have the Fubini Property. We show, in particular, that:

– $\langle \mathrm{MGR}(X), J\rangle$ satisfies FP, for every J generated by any family of closed subsets of Y ($\mathrm{MGR}(X)$ is the σ-ideal of all meager subsets of X),

– $\langle \mathrm{NULL}_\mu, J\rangle$ satisfies FP, whenever J is generated by any of the following families of closed subsets of Y (NULL_μ is the σ-ideal of all subsets of X, having outer measure zero with respect to a Borel σ-finite continuous measure μ on X):

(i) all closed sets of cardinality ≤ 1,

(ii) all compact sets,

(iii) all closed sets in NULL_ν for a Borel σ-finite continuous measure ν on Y,

(iv) all closed subsets of a $\mathbf{\Pi}^1_1$ set $A \subseteq Y$.

We also prove that $\langle \mathrm{MGR}(X), \mathrm{MGR}(Y)\rangle$ and $\langle \mathrm{NULL}_\mu, \mathrm{NULL}_\nu\rangle$ are essentially the only cases of FP in the class of σ-ideals obtained from $\mathrm{MGR}(X)$ and NULL_μ by the operations of Borel isomorphism, product, extension and countable intersection.

1. Preliminaries

This paper is a contribution to the study of σ-ideals on Polish spaces. Our notation and terminology follow [9].

A *σ-ideal* on a metric space X is a family $I \subseteq \mathcal{P}(X)$ which is closed under taking subsets and countable unions. We always assume that the underlying space is uncountable, I is *proper*, i.e., $X \notin I$, contains all singletons and has a basis consisting of Borel sets. The latter means that every set from I is covered by a Borel set from I, i.e., for any $A \in I$ there is $B \in \mathbf{B}(X) \cap I$ with $A \subseteq B$. If, moreover, every set from I is covered by a $\mathbf{\Sigma}^0_2$ set from I, then we say that I is $\mathbf{\Sigma}^0_2$ supported. The most important example of a $\mathbf{\Sigma}^0_2$ supported σ-ideal on X is

$\mathrm{MGR}(X)$ = the collection of all meager subsets of X (provided $X \notin \mathrm{MGR}(X)$ and X has no isolated points, so that $\mathrm{MGR}(X)$ is proper and contains all singletons).

The most important example of a σ-ideal on X which is not $\mathbf{\Sigma}^0_2$ supported is

NULL_μ = the collection of all subsets of X, having outer measure zero with respect to a Borel σ-finite continuous measure μ on X.

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Given σ-ideals I and J on Polish spaces X and Y, respectively, we say that the pair $\langle I, J\rangle$ satisfies the *Fubini Property* (FP) if for every Borel subset B of the space $X\times Y$, if all its vertical sections $B_x=\{y:\ \langle x,y\rangle\in B\}$ are in J, then its horizontal sections $B^y=\{x:\ \langle x,y\rangle\in B\}$ are in I, for every y outside a set from J. Thus the Fubini theorem and the Kuratowski–Ulam theorem (see [9], 8.41) assert, in particular, that the pairs $\langle \mathrm{NULL}_\mu, \mathrm{NULL}_\nu\rangle$ and $\langle \mathrm{MGR}(X), \mathrm{MGR}(Y)\rangle$ satisfy FP. On the other hand, neither $\langle \mathrm{MGR}(X), \mathrm{NULL}_\nu\rangle$ nor $\langle \mathrm{NULL}_\mu, \mathrm{MGR}(Y)\rangle$ satisfies FP (see 3.6).

The aim of this paper is to decide for a number of other pairs of σ-ideals whether they satisfy FP. A special attention is given to the $\mathbf{\Sigma^0_2}$ supported σ-ideals. This class includes such important members as:

$[X]^{\leq\aleph_0}$ = the collection of countable subsets of X,

$\mathcal{K}^*_\sigma$= the σ-ideal generated by all compact subsets of a non-σ-compact X (if X is the Baire space $\mathbb{N}^\mathbb{N}$, then this is just the collection of σ-bounded subsets of $\mathbb{N}^\mathbb{N}$) (see [9], 21.24),

$\mathcal{E}$ = the σ-ideal generated by all closed measure zero subsets of the Cantor space $2^\mathbb{N}$ (see [3], 2.6).

Note that the three σ-ideals above are generated by $\mathbf{\Pi^1_1}$ (in the Effros Borel structure) hereditary families of closed sets (see [9], 35.G). This makes them much easier to handle, as we have the following

Lemma 1.1 *Let X and Y be Polish spaces and $\mathcal{F}$ a hereditary $\mathbf{\Pi^1_1}$ family of closed subsets of Y. Let I be an arbitrary σ-ideal and J be the σ-ideal generated by $\mathcal{F}$. Then the following conditions are equivalent:*

(i) the pair $\langle I,J\rangle$ satisfies FP,

(ii) for every Borel set $B\subseteq X\times Y$ with all sections B_x in $\mathcal{F}$, we have $\{y:\ B^y\notin I\}\in J$.

Proof. To prove the non-trivial direction (ii) $\Rightarrow$ (i), take an arbitrary Borel set $B\subseteq X\times Y$ such that $\forall x(B_x\in J)$. Then, the Burgess–Hillard theorem (see [9], 35.43) tells us that $B\subseteq\bigcup_n B_n$, with B_n Borel and $\forall n\forall x[(B_n)_x\in\mathcal{F}]$. By condition (ii), it follows that for each n we have $\{y:\ (B_n)^y\notin I\}\in J$. Finally,

$$\{y:\ B^y\notin I\}\subseteq\bigcup_n\{y:\ (B_n)^y\notin I\}\in J.\quad\square$$

What makes it still easier to work with σ-ideals which are $\mathbf{\Sigma^0_2}$ supported is the following basic structural result.

Theorem 1.2 (Kechris, Solecki [10]). *Let I be a $\mathbf{\Sigma^0_2}$ supported σ-ideal on a Polish space X. Then precisely one of the following possibilities holds:*

(i) There is a countable family $\{F_n : n\in\mathbb{N}\}$ of closed uncountable subsets of X such that, if for each n we put $I_n=\{B\subseteq X : B\cap F_n\in \mathrm{MGR}(F_n)\}$, then $I=\bigcap_{n\in\mathbb{N}} I_n$.

(ii) there is a homeomorphic embedding $\Phi: 2^\mathbb{N}\times\mathbb{N}^\mathbb{N}\to X$ such that $\Phi[\{\alpha\}\times\mathbb{N}^\mathbb{N}]\notin I$ for any $\alpha\in 2^\mathbb{N}$.

In fact, a revised form of the above will be especially useful for us.

We say that σ-ideals I and J on spaces X and Y, respectively, are *Borel isomorphic* ($I\equiv_B J$) if there exists a Borel isomorphism $f: X\to Y$ between X and Y such that for $A \in \mathbf{B}(X)$, $A \in I \iff f[A] \in J$; in this case we also write $J = f_*(I)$. According to a theorem of Sikorski (see [9], 15.10), if X and Y are uncountable Borel subsets of Polish spaces and both I and J contain uncountable sets, then $I \equiv_B J$ is equivalent to the fact that the corresponding Boolean algebras $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ and $\mathbf{B}(Y)/(J \cap \mathbf{B}(Y))$ are isomorphic. Recall that if X is an uncountable Polish space with no isolated points, then the Boolean algebra $\mathbf{B}(X)/(\mathrm{MGR}(X) \cap \mathbf{B}(X))$ is the unique, up to an isomorphism, complete atomless Boolean algebra that contains a countable dense subset (see e.g. [8], 25.4). It follows that all σ-ideals of the form MGR(B) for an uncountable Borel subset of a Polish space are pairwise Borel isomorphic (tacitly assuming, of course, that B has no isolated points and $B \notin \mathrm{MGR}(B)$). It is also well known that all σ-ideals of the form NULL_μ for a σ-finite continuous Borel measure μ on an uncountable Polish space are pairwise Borel isomorphic (see [9], 17.41).

We say that a σ-ideal I on a Polish space X fulfills the *countable chain condition* (shortly: is ccc), if there is no uncountable family of disjoint Borel sets outside I. We say that I has *property (M)*, if there exists (equivalently: for every) uncountable Polish space Y there is a Borel function $f : X \to Y$ such that $f^{-1}[\{y\}] \notin I$ for each $y \in Y$ (see [1]).

The revised form of Kechris–Solecki theorem, announced above, is formulated as follows

Proposition 1.3 *Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal on a Polish space X. Then precisely one of the following possibilities holds:*

(i) I is ccc and then it is Borel isomorphic to the σ-ideal $\mathrm{MGR}(2^{\mathbb{N}})$,

(ii) I has property (M).

Proof. It is not difficult to see that the two possibilities above correspond to the two clauses from 1.2. In the case of (ii) this is clear (see [1]). So it suffices to show that if $I = \bigcap_{n\in\mathbb{N}} I_n$ for a countable family $\{F_n : n \in \mathbb{N}\}$ of closed uncountable subsets of X, where $I_n = \{B \subseteq X : B \cap F_n \in \mathrm{MGR}(F_n)\}$, then I is Borel isomorphic to the σ-ideal $\mathrm{MGR}(2^{\mathbb{N}})$.

Clearly, for each n, the Boolean algebra $\mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$ is isomorphic to $\mathbf{B}(F_n)/\mathrm{MGR}(F_n)$ and so also to the algebra $\mathbf{B}(2^{\mathbb{N}})/\mathrm{MGR}(2^{\mathbb{N}})$ (note that since I contains all singletons, F_n has no isolated points). Next note that the function

$$\Phi([A]_I) = \langle [A]_{I_n} : \ n \in \mathbb{N}\rangle \text{ for } A \in \mathbf{B}(X)$$

is a complete embedding of the Boolean algebra $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ into the product algebra $\prod\limits_{n\in\mathbb{N}} \mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$ of the algebras $\mathbf{B}(X)/(I_n \cap \mathbf{B}(X))$. It then easily follows that $\mathbf{B}(X)/(I \cap \mathbf{B}(X))$ is a complete atomless Boolean algebra with a countable dense subset and thus it is also isomorphic to $\mathbf{B}(2^{\mathbb{N}})/\mathrm{MGR}(2^{\mathbb{N}})$. Hence, by Sikorski's theorem mentioned above, $I \equiv_B \mathrm{MGR}(2^{\mathbb{N}})$. (For a different proof see [2], 3.1). □

To realize the relevance of the dichotomy above to the study of FP for $\mathbf{\Sigma}^0_2$-supported σ-ideals let us note the following

Lemma 1.4 *Suppose that I_1, I_2, J_1 and J_2 are σ-ideals on Polish spaces X_1, X_2, Y_1 and Y_2, respectively.*

If $I_1 \equiv_B I_2$ and $J_1 \equiv_B J_2$, then $\langle I_1, J_1\rangle$ satisfies FP if and only if $\langle I_2, J_2\rangle$ does so.

Proof. Straightforward. □

Thus, from the point of view of FP, Borel isomorphic ideals are identical. In particular, the dichotomy above says that there is essentially only one $\mathbf{\Sigma}^0_2$ supported ccc σ-ideal, which will be referred to as *the category σ-ideal* and denoted simply by MGR.

In the first part of the paper we examine FP for pairs $\langle I, J\rangle$ of σ-ideals, at least one of which is $\mathbf{\Sigma}^0_2$ supported. The main results are:
– if J is $\mathbf{\Sigma}^0_2$ supported, then $\langle \mathrm{MGR}, J\rangle$ satisfies FP (theorem 2.4).
– if J is one of the σ-ideals $[Y]^{\leq\aleph_0}$, $\mathcal{K}^*_\sigma$, $\mathcal{E}$, then $\langle \mathrm{NULL}, J\rangle$ satisfies FP (theorems 2.1.(i) and 2.5). Here NULL stands for *the measure σ-ideal*, i.e. any of the Borel isomorphic σ-ideals of the form NULL_μ.

In the second part we concentrate on ccc σ-ideals. We try to find a pair of ccc σ-ideals which satisfies FP but is different from the obvious examples: $\langle \mathrm{MGR}, \mathrm{MGR}\rangle$ and $\langle \mathrm{NULL}, \mathrm{NULL}\rangle$. We show that there is no such pair in the class of σ-ideals obtained from MGR and NULL by the operations of Borel isomorphism, product, extension and countable intersection (theorem 3.9). The only examples we give require some additional set-theoretic assumptions (theorem 3.10).

2. Fubini Property for $\mathbf{\Sigma}^0_2$ supported σ-ideals.

Let us first deal with the simplest $\mathbf{\Sigma}^0_2$ supported σ-ideal, namely the one formed by countable sets. The idea of the proof of (i) below has been borrowed from [7] (see [7], 1.2).

Theorem 2.1 *Let X and Y be Polish spaces.*
(i) If I is an arbitrary ccc σ-ideal on X, then the pair $\langle I, [Y]^{\leq\aleph_0}\rangle$ satisfies FP.
(ii) If J is an arbitrary σ-ideal on Y, then the pair $\langle [X]^{\leq\aleph_0}, J\rangle$ does not satisfy FP.

Proof. (i) Let $B \subseteq X\times Y$ be a Borel set with all sections B_x countable. Then, by the Lusin-Novikov theorem (see [9], 18.10), B can be written as $\bigcup_n B_n$, where each B_n is the graph of a partial Borel function f_n. Note that $\{y : B^y \notin I\} = \bigcup_n \{y : (B_n)^y \notin I\}$. But for each n, $(B_n)^y = f_n^{-1}[\{y\}]$, so $|\{y : (B_n)^y \notin I\}| \leq \aleph_0$, by the ccc property of I. It follows that $|\{y : (B)^y \notin I\}| \leq \aleph_0$.
(ii) It is easy to find a Borel function $f : X \to Y$ such that $\forall y \in Y\ |f^{-1}[\{y\}]| > \aleph_0$. Let B be the graph of f and note that B violates FP for $\langle [X]^{\leq\aleph_0}, J\rangle$. □

Note that in proving 2.1 (ii) we use only the fact that $[X]^{\leq\aleph_0}$ has property (M). Thus we have

Proposition 2.2 *Let X and Y be Polish spaces. If I is a σ-ideal on X with property (M) and J is an arbitrary σ-ideal on Y, then the pair $\langle I, J\rangle$ does not satisfy FP.*

Combining this with 1.3 immediately gives

Theorem 2.3 *Let X and Y be Polish spaces. Let I be a $\mathbf{\Sigma}^0_2$ supported σ-ideal on X. If I is not ccc, then the pair $\langle I, J\rangle$ does not satisfy FP for any σ-ideal J on Y.*

Now let us turn to the case when I is ccc.

Theorem 2.4 *Let X and Y be Polish spaces. Let I and J be $\mathbf{\Sigma}^0_2$ supported σ-ideals on X and Y, respectively. If I is ccc, then the pair $\langle I, J\rangle$ satisfies FP.*

Proof. By 1.3 and 1.4 we can assume that $I = \mathrm{MGR}(X)$. Now suppose, towards a contradiction, that a Borel set $B \subseteq X \times Y$ is a counterexample to FP for

$\langle \mathrm{MGR}(X), J\rangle$, i.e. $\forall x\ B_x \in J$ but $Y' = \{y : B^y \notin \mathrm{MGR}(X)\} \notin J$. Since Y' is Borel (see [9], 16.1), we can extend the topology τ on Y to a new Polish topology τ' with the same Borel sets in which Y' is clopen, so Polish (in the relative topology – see [9], 13.1).

From now on let us work with the topology τ'. Note that J remains $\mathbf{\Sigma^0_2}$ supported.

Let $\{V_n :\ n \in \mathbb{N}\}$ be an open basis for Y. Consider the space $Z = Y' \setminus \bigcup\{V_n : V_n \in J\}$. Note that Z is closed and $Z \notin J$ (in fact $Y' \setminus Z \in J$), so Z is a nonempty Polish space. Moreover, for any open subset U of Y if $U \cap Z \neq \emptyset$, then $U \cap Z \notin J$.

Let $C = B \cap (X \times Z)$. Then $\forall z \in Z\ C^z \notin \mathrm{MGR}(X)$ so, by the Kuratowski-Ulam theorem applied to the product $X \times Z$, there is an $x_0 \in X$ such that $C_{x_0} \notin \mathrm{MGR}(Z)$. But $C_{x_0} \in J$ so there is $F \in J \cap \mathbf{\Sigma^0_2}$ with $C_{x_0} \subseteq F$. Then $F \cap Z$ is a non-meager $\mathbf{\Sigma^0_2}$ subset of Z (in the relative topology). It follows that there is an open subset U of Y such $U \cap Z \neq \emptyset$ and $U \cap Z \subseteq F$. But this implies that $F \notin J$ which contradicts the choice of F. □

This completes our discussion of FP for pairs in which both σ-ideals are $\mathbf{\Sigma^0_2}$ supported. We shall now consider the case, when I is the measure ideal NULL. Of course, we cannot hope to obtain the full analogue of 2.4, since we have the counterexample $\langle \mathrm{NULL}, \mathrm{MGR}\rangle$ (see 3.6). Nevertheless, some important cases of 2.4 are still valid.

Theorem 2.5 *Let X and Y be Polish spaces and let μ be a σ-finite Borel continuous measure on X. Then the pair $\langle \mathrm{NULL}_\mu, J\rangle$ satisfies FP whenever J is the σ-ideal generated by any of the following families of closed subsets of Y:*

(i) all compact sets (in this case Y is assumed to be non-σ-compact). Moreover, if $B \subset X \times Y$ is a Borel set with $B_x \in \mathcal{K}^*_\sigma$ for each $x \in X$, then there exists a set $Z \in \mathcal{K}^*_\sigma$ such that $B_x \subseteq Z$ for μ-almost all $x \in X$.

(ii) all closed sets in NULL_ν for a σ-finite Borel continuous measure ν on Y,

(iii) all closed subsets of a $\mathbf{\Pi^1_1}$ set $A \subseteq Y,\ A \neq Y$.

Proof. First note that in all three cases J is generated by a hereditary $\mathbf{\Pi^1_1}$ family $\mathcal{F}$ of closed subsets of Y so, by 1.1, it is enough to consider Borel sets B with all sections B_x in $\mathcal{F}$. We can also assume that the measure μ is finite.

(i) It clearly suffices to prove the "moreover" part. So let $B \subset X \times Y$ be a Borel set with $B_x \in K(Y)$ for each $x \in X$ and consider the function $\varphi : X \to K(Y)$ defined by

$$\varphi(x) = B_x \text{ for } x \in X,$$

where by $K(Y)$ we denote the space of all compact subsets of Y equipped with the Vietoris topology.

Since φ is Borel, Lusin's theorem (see [9], 17.12) tells us that there are compact sets $F_n \subseteq X,\ n \in \mathbb{N}$, such that $\forall n\ \varphi|F_n$ is continuous and $\mu(X \setminus \bigcup_n F_n) = 0$. Then $\forall n\ \varphi[F_n] \in K(K(Y))$, so by the continuity of the mapping $\bigcup :\ K(K(Y)) \to K(Y)$ (see [9], 4.29), $\bigcup \varphi[F_n] \in K(Y)$ and it suffices to define $Z = \bigcup_n(\bigcup \varphi[F_n])$.

(ii) and (iii) are easy consequences of the following result.

Lemma 2.6 (folklore?) *If $B \subseteq X \times Y$ is a Borel set such that every section B_x is closed, then $\{y :\ \mu(B^y) > 0\}$ is $\mathbf{\Sigma^0_2}$.*

Proof. (of 2.6) This seems to be essentially known but for the sake of completeness we shall sketch a simple argument, suggested by the referee.

We may assume that $\mu(X)$ is finite. It is enough to prove that in this case the sets $Y_a = \{y : \mu(B^y) \geq a\}$ are closed for every $a > 0$. Let $y_n \in Y_a$, $\lim_n y_n = y$. Then B^y contains every point belonging to infinitely many of the sets B^{y_n}, since the vertical sections of B are closed. That is, B^y contains $\bigcap_k \bigcup_{n\geq k} B^{y_n}$, which implies $\mu(B^y) \geq a$ and $y \in Y_a$. □

It may finally be pointed out that the part of 2.5.(i) stating FP for $\langle \mathrm{NULL}_\mu, \mathcal{K}^*_\sigma \rangle$ is a special case of 2.5.(iii), since every Polish space is a $\mathbf{\Pi^0_2}$, so $\mathbf{\Pi^1_1}$, subset of the Hilbert cube.

3. Fubini Property for ccc σ-ideals.

The two most frequently encountered ccc σ-ideals are the category σ-ideal MGR and the measure σ-ideal NULL. They also provide two classical cases, $\langle \mathrm{MGR}, \mathrm{MGR} \rangle$ and $\langle \mathrm{NULL}, \mathrm{NULL} \rangle$, of the Fubini Property phenomenon. Are there any other natural examples of pairs of ccc σ-ideals satisfying FP? In this section we negatively answer this question restricted to the realm of σ-ideals cooked up from MGR and NULL with the help of the operations of extension, countable intersection, Borel isomorphism and product.

Recall that given σ-ideals I on X and J on Y their *product* $I \otimes J$ is defined as the σ-ideal with the Borel basis consisting of Borel sets $B \subseteq X \times Y$ such that $\{x : B_x \notin J\} \in I$. Notice that the pair $\langle I, J \rangle$ has the Fubini property if and only if, for every Borel set $B \subset X \times Y$, $B \in I \otimes J$ implies $B^* \in J \otimes I$, where $B^* = \{(y, x) : (x, y) \in B\}$.

Since all measure (category, resp.) σ-ideals are Borel isomorphic, $\mathrm{MGR}\otimes\mathrm{MGR} \equiv_B \mathrm{MGR}$ and $\mathrm{NULL} \otimes \mathrm{NULL} \equiv_B \mathrm{NULL}$. Let $\mathbb{K}$ and $\mathbb{L}$ denote the σ-ideals of meager subsets in $\mathbb{R}$ and of Lebesgue measure zero subsets of $\mathbb{R}$, respectively. Gavalec [6] proved that if $I_1, \ldots, I_n \in \{\mathbb{K}, \mathbb{L}\}$, then it makes sense to write $I = I_1 \otimes \ldots \otimes I_n$, without brackets, and call I the product of n factors $I_1, \ldots, I_n$.

The next proposition summarizes some general properties of the product operation which will be needed in the sequel. The proofs are routine.

Proposition 3.1 *Let X, X', Y, Y' be Polish spaces.*

If I, $\bar{I}$ and I_n, $n \in \mathbb{N}$, are σ-ideals on X, J, $\bar{J}$ and J_n, $n \in \mathbb{N}$, are σ-ideals on Y, I' is a σ-ideal on X' and J' is a σ-ideal on Y', then

(i) $I \equiv_B I' \wedge J \equiv_B J' \Rightarrow I \otimes J \equiv_B I' \otimes J'$,
(ii) $(\bigcap_n I_n) \otimes J = \bigcap_n (I_n \otimes J)$,
(iii) $I \otimes (\bigcap_n J_n) = \bigcap_n (I \otimes J_n)$,
(iv) $I \subseteq I' \Rightarrow I \otimes J \subseteq I' \otimes J$,
(v) $J \subseteq J' \Rightarrow I \otimes J \subseteq I \otimes J'$.

Let $\overline{\mathcal{S}}$ be the smallest class of σ-ideals on Polish spaces such that:
(i) $\mathbb{K}$, $\mathbb{L} \in \overline{\mathcal{S}}$,
(ii) for any σ-ideals I_1, I_2, $(I_1 \in \overline{\mathcal{S}} \wedge I_1 \equiv_B I_2) \Rightarrow I_2 \in \overline{\mathcal{S}}$,
(iii) for any σ-ideals $I_n \in S$, $n \in \mathbb{N}$, on a space X, $\bigcap_n I_n \in \overline{\mathcal{S}}$,
(iv) for any σ-ideals I, $J \in S$, $I \otimes J \in \overline{\mathcal{S}}$.
(v) for any σ-ideals I, $\bar{I}$, $(I \in \overline{\mathcal{S}} \wedge I \subseteq \bar{I}) \Rightarrow \bar{I} \in \overline{\mathcal{S}}$.

Let $\mathcal{S}$ be the smallest class of σ-ideals satisfying conditions (i)–(iv) above. Since the operations involved in (ii)–(iv) are monotone, we have

$$\bar{I} \in \overline{\mathcal{S}} \iff \exists I \in \mathcal{S}\ I \subseteq \bar{I}$$

The elements of $\mathcal{S}$ can be represented in a useful normal form. Let $\mathcal{P}_n$ denote the family of all σ-ideals I on $\mathbb{R}^n$ which are the products of n factors taken from $\{\mathbb{K}, \mathbb{L}\}$.

Proposition 3.2 *If $I \in \mathcal{S}$ is a σ-ideal on a Polish space X, then there are: positive integers n_k, Borel isomorphisms φ_k between $\mathbb{R}^{n_k}$ and X, and σ-ideals $I_k \in \mathcal{P}_{n_k}$ on $\mathbb{R}^{n_k}$, $k \in \mathbb{N}$, such that:*

$$I = \bigcap_k (\varphi_k)_*(I_k).$$

Proof. It is enough to show that the class of σ-ideals of the above form is closed under the operations involved in the definition of $\mathcal{S}$.

In the case of the countable intersection this is obvious.

If $\psi : X \to X'$ is a Borel isomorphism between X and another Polish space X', then $\psi_*(I) = \psi_*\big(\bigcap_k (\varphi_k)_*(I_k)\big) = \bigcap_k (\psi \circ \varphi_k)_*(I_k)$.

Finally, let $J = \bigcap_l (\psi_l)_*(J_l)$, where for each l, J_l is the product of m_l factors from $\{\mathbb{K}, \mathbb{L}\}$ and ψ_l are Borel isomorphisms between $\mathbb{R}^{m_l}$ and Y . Then by 3.1 (i)–(iii), $I \otimes J = \big(\bigcap_k (\varphi_k)_*(I_k)\big) \otimes \big(\bigcap_l (\psi_l)_*(J_l)\big) = \bigcap_{k,l}\big((\varphi_k)_*(I_k) \otimes (\psi_l)_*(J_l)\big) = \bigcap_{k,l} (\vartheta_{k,l})_*(I_k \otimes J_l)$, where for each k, l, $\vartheta_{k,l}$ is a certain Borel isomorphism between $\mathbb{R}^{n_k+m_l}$ and $X \times Y$. $\square$

Gavalec [6] proved that $\forall n > 0$, every σ-ideal $I \in \mathcal{P}_n$ is ccc. Combining this with 3.2 and taking into account that the ccc property is preserved by Borel isomorphisms, countable intersections and extensions, we immediately get the following

Proposition 3.3 *Every σ-ideal $I \in \overline{\mathcal{S}}$ is ccc.*

Our next auxiliary result concerns extensions of ccc σ-ideals.

We say that σ-ideals I_1, I_2 on X are *orthogonal* and write $I_1 \perp I_2$, if there is no common extension of I_1, I_2 to a (proper!) σ-ideal on X. This is equivalent to the existence of a Borel set in I_1 whose complement is in I_2.

If I is a σ-ideal on X and $A \in \mathbf{B}(X) \setminus I$, then *the restriction of I to A*, denoted by $I|A$, is the σ-ideal on X given by

$$I|A = \{C \subseteq X : \ C \cap A \in I\}.$$

Clearly, $I \subseteq I|A$ for any $A \in \mathbf{B}(X) \setminus I$ and if I is, moreover, ccc, then every σ-ideal $\bar{I}$ on X extending I is of this form (it suffices to let $A = X \setminus \bigcup \mathcal{R}$, where $\mathcal{R}$ is any maximal family of pairwise disjoint Borel sets in $\bar{I} \setminus I$).

Proposition 3.4 *Let I_n, $n \in \mathbb{N}$, and $\bar{I}$ be ccc σ-ideals on a Polish space X. If $\bigcap_n I_n \subseteq \bar{I}$, then there exists a nonempty set $T \subseteq \mathbb{N}$ such that $\bigcap_{n \in T} I_n \subseteq \bar{I}$ and $I_n \not\subseteq \bar{I}$ for each $n \in T$.*

Proof. Put $I = \bigcap_n I_n$ and find $A \in \mathbf{B}(X) \setminus I$ such that $\bar{I} = I|A$. Let $T = \{n : \ A \notin I_n\}$.

Since $A \notin I$, $T \neq \emptyset$. We have

$$\bigcap_{n \in T} I_n \subseteq \big(\bigcap_{n \in T} I_n\big)|A = I|A = \bar{I}.$$

Finally, take $n \in T$. Since $A \notin I_n$, $I_n|A$ is a proper σ-ideal on X. But $I_n \subseteq I_n|A$ and $\bar{I} = I|A = (\bigcap_{n \in T} I_n)|A = \bigcap_{n \in T}(I_n|A) \subseteq I_n|A$, so $I_n|A$ witnesses that $I_n \not\perp \bar{I}$. $\square$

The special role the σ-ideals MGR and NULL play in the class $\overline{\mathcal{S}}$, as far as the Fubini Property is concerned, strongly depends on closure conditions which are summarized in the following

Proposition 3.5 (folklore?) *(i) The countable intersection of category (measure, resp.) σ-ideals is a category (measure, resp.) σ-ideal. More precisely, if X is an uncountable Polish space and σ-ideals I_n, $n \in \mathbb{N}$, on X are Borel isomorphic to* $\mathrm{MGR}(2^{\mathbb{N}})$ *(are of the form* NULL_{μ_n} *for certain probability Borel continuous measures μ_n on X, resp.), then the σ-ideal $I = \bigcap_n I_n$ is of the same form.*

(ii) The product of finitely many category (measure, resp.) σ-ideals is a category (measure, resp.) σ-ideal.

(iii) The extension of a category (measure, resp.) σ-ideal is a category (measure, resp.) σ-ideal.

Proof. (i). The category case has essentially been dealt with in the course of proving 1.3. In the measure case it is enough to define μ by

$$\mu(A) = \sum_n \frac{1}{2^{n+1}} \cdot \mu_n(A) \quad \text{for } A \in \mathbf{B}(X),$$

to get $\bigcap_n \mathrm{NULL}_{\mu_n} = \mathrm{NULL}_\mu$.

(ii). This follows from the fact that the product of n copies of $\mathbb{K}$ ($\mathbb{L}$, resp) is the σ-ideal of meager (Lebesgue measure zero, resp.) subsets of $\mathbb{R}^n$.

(iii). Let J be a σ-ideal on $\mathbb{R}$ and $I \subseteq J$, where $I \in \{\mathbb{K}, \mathbb{L}\}$. Find $A \in \mathbf{B}(\mathbb{R}) \setminus I$ with $J = I|A$. Then the Boolean algebras $\mathbf{B}(\mathbb{R})/\big(I|A \cap \mathbf{B}(\mathbb{R})\big)$ and $\mathbf{B}(\mathbb{R})/\big(I \cap \mathbf{B}(\mathbb{R})\big)$ are isomorphic. □

We return to the question, which pairs of $\langle I, J\rangle$ of members of $\overline{\mathcal{S}}$ have the Fubini Property. Let us first deal with the special case when $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$ for some fixed positive integers n, m. It will turn out that if $\langle I, J\rangle$ does not satisfy FP, then in fact FP is violated in a very strong sense. Here is a general formulation of the relevant definition.

Given σ-ideals I and J on Polish spaces X and Y, respectively, we say that a Borel set $B \subseteq X \times Y$ is a *0-1 counterexample to FP for* $\langle I, J\rangle$, if $\forall x \in X\ B_x \in J$ and $\forall y \in Y\ (X \setminus B^y) \in I$. It is straightforward to prove, that with regard to the existence of 0-1 counterexamples to FP, Borel isomorphic ideals are identical. Also note, that if B is a 0-1 counterexample to FP for $\langle I, J\rangle$, then $\{\langle y, x\rangle : \langle x, y\rangle \in (X \times Y) \setminus B\}$ is a 0-1 counterexample to FP for $\langle J, I\rangle$. Thus the existence of such a B contradicts FP for both $\langle I, J\rangle$ and $\langle J, I\rangle$.

The following construction is well-known.

Example 3.6 *Let $C \in \mathbf{B}(\mathbb{R})$ be such that $C \in \mathbb{L}$ and $\mathbb{R} \setminus C \in \mathbb{K}$. Then the set $B \subseteq \mathbb{R} \times \mathbb{R}$ defined by*

$$B = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : x + y \in C\},$$

is a 0-1 counterexample to FP for $\langle \mathbb{K}, \mathbb{L}\rangle$.

It has also been known that there exists a 0-1 counterexample to FP for $\langle \mathbb{K} \otimes \mathbb{L}, \mathbb{K} \otimes \mathbb{L}\rangle$ (see [7], 2.1). These facts generalize to

Lemma 3.7 *If $I \in \mathcal{P}_n$ and $J \in \mathcal{P}_m$, then the following conditions are equivalent:*

(i) $\langle I, J\rangle$ does not satisfy FP,

(ii) neither $I \equiv_B J \equiv_B$ MGR *nor $I \equiv_B J \equiv_B$* NULL,

(iii) there is a 0-1 counterexample to FP for $\langle I, J\rangle$.

Proof. Only (ii) $\Rightarrow$ (iii) requires proof. Let

$$I = I_1 \otimes \ldots \otimes I_n \text{ and } J = J_1 \otimes \ldots \otimes J_m,$$

with each $I_k,\ J_l \in \{\mathbb{K}, \mathbb{L}\}$. It follows, by 3.5, that we can find integers k and l such that $I_k \neq J_l$. Assume, w.l.o.g., that $k < l$, $I_k = \mathbb{K}$ and $J_l = \mathbb{L}$. Put $r = \max\{n+l-k, m\}$. Let $I', J' \in \mathcal{P}_r$ be the σ-ideals obtained by substituting for I_k (J_l, resp.) the product of $r-n$ copies of $\mathbb{K}$ ($r-m$ copies of $\mathbb{L}$, resp.). Then

$$I' = I'_1 \otimes \ldots \otimes I'_r \text{ and } J' = J'_1 \otimes \ldots \otimes J'_r,$$

with $I'_l = \mathbb{K}$ and $J'_l = \mathbb{L}$. Moreover, $I \equiv_B I'$ and $J \equiv_B J'$, so it suffices to find a 0-1 counterexample to FP for $\langle I', J' \rangle$.

First note that $I' \perp J'$. For that purpose take a Borel set $A \subseteq \mathbb{R}$ such that $A \in \mathbb{L}$ and $\mathbb{R} \setminus A \in \mathbb{K}$. Define a Borel set $C \subseteq \mathbb{R}^r$ by

$$C = \{(x_1, \ldots, x_r) : \ x_l \in A\}.$$

It is not difficult to prove that $C \in J'$ and $(\mathbb{R}^r \setminus C) \in I'$.

We complete the proof exactly as in 3.6. Namely, the set $B \subseteq \mathbb{R}^r \times \mathbb{R}^r$ defined by

$$B = \{\langle \bar{x}, \bar{y} \rangle \in \mathbb{R}^r \times \mathbb{R}^r : \bar{x} + \bar{y} \in C\},$$

($+$ is the ordinary addition in $\mathbb{R}^r$) is a 0-1 counterexample to FP for $\langle I', J' \rangle$. $\square$

Our last auxiliary fact concerns the question of what happens to FP when we pass to larger σ-ideals.

Proposition 3.8 *Let $I,\ \bar{I}$ and $J,\ \bar{J}$ be σ-ideals on Polish spaces X and Y, respectively.*

(i) If $I \subseteq \bar{I}$, then FP for $\langle I, J \rangle$ implies FP for $\langle \bar{I}, J \rangle$,

(ii) If $I \subseteq \bar{I},\ J \subseteq \bar{J}$ and J is, moreover, ccc, then FP for $\langle I, J \rangle$ implies FP for $\langle \bar{I}, \bar{J} \rangle$.

Proof. (i) follows immediately from the fact, that for any $B \in \mathbf{B}(X \times Y)$, $\{y : B^y \notin \bar{I}\} \subseteq \{y : \ B^y \notin I\}$.

(ii). By (i), we can assume that $\bar{I} = I$. Fix $A \in \mathbf{B}(Y) \setminus J$ such that $\bar{J} = J|A$.

Take an arbitrary Borel set $B \subseteq X \times Y$ with $\forall x\ B_x \in \bar{J}$. Let $D = B \cap (X \times A)$ and note that $\forall x\ D_x = B_x \cap A \in J$. By FP for $\langle I, J \rangle$, $\{y : \ D^y \notin I\} \in J$. But $\{y : \ D^y \notin I\} = \{y : \ B^y \notin I\} \cap A$, so $\{y : \ B^y \notin I\} \in \bar{J}$. $\square$

We are now ready to state the main result of this section.

Theorem 3.9 *For every σ-ideals $I,\ J \in \overline{\mathcal{S}}$ the following conditions are equivalent:*

(i) $\langle I, J \rangle$ satisfies FP,

(ii) either $I \equiv_B J \equiv_B$ MGR or $I \equiv_B J \equiv_B$ NULL.

Proof. Only the implication (i) $\Rightarrow$ (ii) requires proof. So assume that $I,\ J \in \overline{\mathcal{S}}$ and $\langle I, J \rangle$ satisfies FP. By 3.2,

$$\bigcap_k (\varphi_k)_*(I_k) \subseteq I \text{ and } \bigcap_l (\psi_l)_*(J_l) \subseteq J$$

for certain positive integers $n_k,\ m_l$, Borel isomorphisms $\varphi_k : \mathbb{R}^{n_k} \to X$, $\psi_l : \mathbb{R}^{m_l} \to X$ and σ-ideals $I_k \in \mathcal{P}_{n_k},\ J_l \in \mathcal{P}_{m_l}$, respectively.

By 3.3 and 3.4, there are nonempty sets $T,\ W \subseteq \mathbb{N}$ such that $\forall k \in T\ (\varphi_k)_*(I_k) \not\subseteq I$, and $\forall l \in W\ (\psi_l)_*(J_l) \not\subseteq J$, and

$$\bigcap_{k\in T}(\varphi_k)_*(I_k) \subseteq I \text{ and } \bigcap_{l\in W}(\psi_l)_*(J_l) \subseteq J.$$

Now suppose, towards a contradiction, that neither $I \equiv_B J \equiv_B \mathrm{MGR}$ nor $I \equiv_B J \equiv_B \mathrm{NULL}$. By 3.5, we may find integers $k \in T$ and $l \in W$ such that neither $I_k \equiv_B J_l \equiv_B \mathrm{MGR}$ nor $I_k \equiv_B J_l \equiv_B \mathrm{NULL}$. Since $(\varphi_k)_*(I_k) \not\subseteq I$ ($(\psi_l)_*(J_l) \not\subseteq J$, resp.), let $\bar{I}$ ($\bar{J}$, resp.) be a σ-ideal extending $(\varphi_k)_*(I_k)$ and I ($(\psi_l)_*(J_l)$ and J, resp.). Then, by 3.7, there is a 0-1 counterexample B to FP for $\langle(\varphi_k)_*(I_k), (\psi_l)_*(J_l)\rangle$. Clearly, B is also a 0-1 counterexample to FP for $\langle\bar{I}, \bar{J}\rangle$. But on the other hand, since $\langle I, J\rangle$ satisfies FP, so does $\langle\bar{I}, \bar{J}\rangle$, by 3.8.(ii) and 3.3. This contradiction completes the proof. □

We will finally present the promised examples of pairs of ccc σ-ideals which satisfy FP but are different from $\langle\mathrm{MGR}, \mathrm{MGR}\rangle$ and $\langle\mathrm{NULL}, \mathrm{NULL}\rangle$. Recall that a cardinal κ is called *quasi-measurable* if it is uncountable and there is a proper ω_1-saturated (i.e., there is no uncountable family of disjoint subsets of κ outside the ideal) κ-additive ideal of $\mathcal{P}(\kappa)$ containing singletons (see [4], 9C). It is well-known that "ZFC + Martin's Axiom + there is a quasi-measurable cardinal $\kappa < \boldsymbol{c}$" is equiconsistent with "ZFC + there is a two-valued-measurable cardinal" (see [4], 9G).

The σ-ideal J below has previously been considered by several authors (see e.g. [5] and [7]).

Theorem 3.10 *Assume Martin's Axiom + there is a quasi-measurable cardinal $\kappa < \boldsymbol{c}$. Let $A \subseteq \mathbb{R}$ be a set of cardinality κ and let $\mathcal{J}$ be a proper, ω_1-saturated κ-additive ideal on A containing singletons. If J is the σ-ideal on $\mathbb{R}$ with the basis consisting of Borel sets $B \subseteq \mathbb{R}$ such that $B \cap A \in \mathcal{J}$, then:*

(i) J is ccc,

(ii) the pairs $\langle\mathbb{K}, J\rangle$ and $\langle\mathbb{L}, J\rangle$ satisfy FP,

(iii) the pairs $\langle J, \mathbb{K}\rangle$ and $\langle J, \mathbb{L}\rangle$ satisfy FP,

(iv) $J \not\equiv_B \mathrm{MGR}$ and $J \not\equiv_B \mathrm{NULL}$.

Proof. (i) immediately follows from the ω_1-saturation of $\mathcal{J}$.

(ii). Let $I \in \{\mathbb{K}, \mathbb{L}\}$. Suppose, towards a contradiction, that a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$ is a counterexample to FP for $\langle I, J\rangle$, i.e. $\forall x\ B_x \in J$ but $Y' = \{y : B^y \notin I\} \notin J$. Since Y' is Borel (see [9], 16.1 and 17.25), $Y' \cap A \notin \mathcal{J}$.

Let $D = \bigcup_{q\in\mathbb{Q}}(B + \langle q, 0\rangle)$. Then $\forall x \in \mathbb{R}\ D_x \in J$ and $\forall y \in Y'\ \mathbb{R} \setminus D^y \in I$, since $D^y = \bigcup_{q\in\mathbb{Q}}(B^y + q)$ and $B^y \in \mathbf{B}(\mathbb{R}) \setminus I$. Let $C = Y' \cap A$. The rest of the argument is well-known. Since $|C| \leq \kappa < \boldsymbol{c}$ and $\forall y \in C\ (\mathbb{R} \setminus D^y) \in I$, MA implies that $\bigcap_{y\in C} D^y \neq \emptyset$. But if $x \in \bigcap_{y\in C} D^y$, then $C \subseteq D_x \in J$, contradicting the fact that $C \notin J$.

(iii) Let $I \in \{\mathbb{K}, \mathbb{L}\}$. Let $B \subseteq \mathbb{R} \times \mathbb{R}$ be a Borel set with $\forall x\ B_x \in I$. Set $E = \bigcup_{x\in A} B_x$. We have $\{y :\ B^y \notin J\} \subseteq E$, since for each y, $B^y \notin J \Rightarrow B^y \cap A \neq \emptyset$. But $|A| = \kappa < \boldsymbol{c}$, so MA implies that $E \in I$.

(iv) follows from (ii) and 3.6. □

Of course the result above is not a satisfactory solution to the problem of finding a new pair of *natural* ccc σ-ideals with FP or, even better, a single ccc σ-ideal I

having FP with itself and such that MGR $\not\equiv_B I \not\equiv_B$ NULL. The latter is apparently open even if the word "natural" is omitted.

References

[1] M. Balcerzak, *Can ideals without ccc be interesting?*, Topology and Appl. 55 (1994), 251–260.
[2] M. Balcerzak, D. Rogowska, *Making some ideals meager on sets of size of the continuum*, Topology Proc. 21 (1996), 1–13.
[3] T. Bartoszyński, H. Judah, Set Theory. *On the structure of the real line*, A K Peters 1995.
[4] D.H. Fremlin, *Real-valued-measurable cardinals*, in: Set theory of the reals, Haim Judah Ed., Israel Math. Conf. Proc. 6 (1993), 151–304.
[5] D.H. Fremlin and J. Jasinski, *G_δ-covers and large thin sets of reals*, Proc. London. Math. Soc. (3) 53 (1986), 518-538.
[6] M. Gavalec, *Iterated products of ideals of Borel sets*, Coll. Math. 50 (1985), 39–52.
[7] H. Judah, A. Lior, I. Reclaw, *Very small sets* , Coll. Math. 72 no. 2 (1997), 207-213.
[8] W. Just, M. Weese, *Discovering modern set theory. II*, Graduate studies in math. 18, AMS 1997.
[9] A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Math. 156, Springer-Verlag 1995.
[10] A.S. Kechris, S. Solecki, *Approximating analytic by Borel sets and definable chain conditions*, Israel J. Math. 89(1995), 343–356.

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