Interpreted nets

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X - universe of *variables*, called *net-places,*

A - universe of *values* of the variables

Transition over ***X***: $t = ({}^{\rightarrow}t, t^{\rightarrow})$

${}^{\rightarrow}t,\ t^{\rightarrow} \in \boldsymbol{X}^{*}$: *ordered pre/post-sets*

T - universe of all transitions

Net-structure: any subset of ***T***

${}^{\bullet}t,\ t^{\bullet}$ - sets of elements occurring in ${}^{\rightarrow}t,\ t^{\rightarrow}$

${}^{\bullet}t^{\bullet} = {}^{\bullet}t \cup t^{\bullet}$ - *neighbourhood* of t

For $T \subseteq \boldsymbol{T}$: ${}^{\bullet}T^{\bullet} = \bigcup_{t \in T} {}^{\bullet}t^{\bullet}$

Valuation of ${}^{\bullet}T^{\bullet}$: any total function $M: {}^{\bullet}T^{\bullet} \rightarrow \boldsymbol{A}$ (marking)

$\boldsymbol{M}_T$ - set of all markings

I_T - interpretation of T with each transition t associates a partial function $t^{I_T} : \boldsymbol{M}_T \to \boldsymbol{M}_T$ for $M, M' \in dom(t^{I_T})$ with:

(i) $t^{I_T}(M)|{}^\bullet T^\bullet \setminus {}^\bullet t^\bullet = M|{}^\bullet T^\bullet \setminus {}^\bullet t^\bullet$ (no effect outside ${}^\bullet t^\bullet$)

(ii) If $M|{}^\bullet t^\bullet = M'|{}^\bullet t^\bullet$ then $t^{I_T}(M)|{}^\bullet t^\bullet = t^{I_T}(M')|{}^\bullet t^\bullet$ (effect depends on ${}^\bullet t^\bullet$ only)

For $t = (\varepsilon, \varepsilon)$ (isolated): $t^{I_T}(M) = M$

$M \in dom(t^{I_T})$: *t is firable at M in T*

$P = \langle T, I_T \rangle$: *interpreted* net

Semantics of P: binary relation $P^{I_T} = \bigcup_{t \in T} t^{I_T}$

$(P^{I_T})^*$: *reachability* relation in P.

Example

Place/Transition nets with arrow-weights 1 and unbounded capacity of places: $\boldsymbol{A} = \{0, 1, 2, \dots\}$

Interpretation t^I of transition $t \in T$:

$M' = t^I(M)$ iff

$$(\forall x \in {}^\bullet t\colon\ M(x) > 0) \wedge M'(x) = \begin{cases} M(x) - 1 & \textit{if } x \in {}^\bullet t \\ M(x) + 1 & \textit{if } x \in t^\bullet \\ M(x) & \textit{else} \end{cases}$$

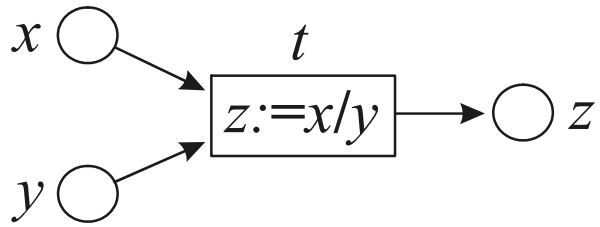
Example

Transition $t = ({}^{\rightarrow}t, t^{\rightarrow})$ may represent an assign. statement $\langle y_1, y_2, \ldots, y_m \rangle := f(x_1, x_2, \ldots, x_n)$ with $x_1 x_2 \ldots x_n = {}^{\rightarrow}t$ and $y_1 y_2, \ldots y_m = t^{\rightarrow}$, computing value of the atomic term $f(x_1, x_2, \ldots, x_n)$ and storing it in all $y_1, y_2, \ldots, y_m$.

Interpretation t^I: $M' = t^I(M)$ iff

$$M'(x) = \begin{cases} f(x_1, x_2, \ldots, x_n) & \textit{if } x \text{ occurs in } t^{\rightarrow} \\ M(x) & \textit{else} \end{cases}$$

$z := x/y$ quotient of numbers stored in $x, y = {}^{\rightarrow}t$ and returning result in $z = t^{\rightarrow}$

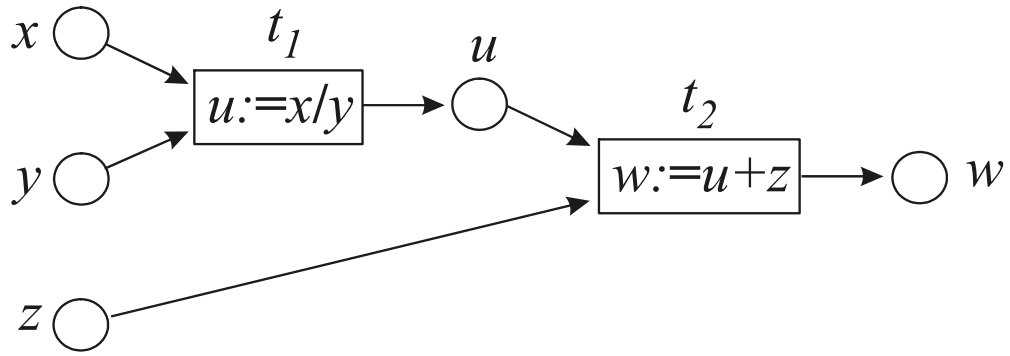


$$t^I(M)(x) = M(x), \quad t^I(M)(y) = M(y),$$

$$t^I(M)(z) = \begin{cases} \frac{M(x)}{M(y)} & \text{if } M(y) \neq 0 \\ \perp & \text{else} \end{cases}$$

7 General assign. statement (nested term):

If $t_1 = (xy, u)$ is $u := x/y$, $t_2 = (uz, w)$ is $w := u + z$, then $\{t_1, t_2\}$ is $w := x/y + z$:



$$t_1^I(M)(s) = \begin{cases} \frac{M(x)}{M(y)} & \text{if } s = u \wedge M(y) \neq 0 \\ \perp & \text{if } s = u \wedge M(y) = 0 \\ M(s) & \text{else} \end{cases}$$

$$t_2^I(M)(s) = \begin{cases} M(u) + M(z) & \text{if } s = w \\ M(s) & \text{else} \end{cases}$$

Interpretation of nets for assignment statements conforms to their computer's realisation - if an order of executing transitions is imposed by another net modelling instruction counter: fetching a value from a memory cell does not change its content!

Concatenation of nets

- Concatenation of $t_1 = ({}^{\rightarrow}t_1, t_1^{\rightarrow})$ and $t_2 = ({}^{\rightarrow}t_2, t_2^{\rightarrow})$:

$t_1 \bullet t_2 = ({}^{\rightarrow}t_1 {}^{\rightarrow}t_2,\ t_1^{\rightarrow} t_2^{\rightarrow})$

- Transitions in all net structures are labelled:

$l_T : \mathrm{L}_T \xrightarrow{1-1} T$ - labelling function in T,

L_T - set of labels in T. Notation: $a{:}t \in T$ means $l_T(a) = t$

- $t \in T_1 \bullet T_2$ iff

either (1) $t = t_1 \bullet t_2$ and $a{:}t_1 \in T_1,\ a{:}t_2 \in T_2$ for a certain label $a \in \mathrm{L}_{T_1} \cap \mathrm{L}_{T_2}$ and transitions $t_1,\ t_2$

or (2) $a{:}t \in T_i$ and $a \notin L_{T_j}$ for $j \neq i$

For $P_1 = \langle T_1, I_{T_1} \rangle, \quad P_2 = \langle T_2, I_{T_2} \rangle,$ let ${}^\bullet T_1^\bullet \cap {}^\bullet T_2^\bullet = \varnothing$.
Interpretation I_T of transition t in $T = T_1 \bullet T_2$:

$$t^{I_T}(M) = t_1^{I_{T_1}}(M|{}^\bullet T_1^\bullet) \cup t_2^{I_{T_2}}(M|{}^\bullet T_2^\bullet)$$

where $t_i \in T_i$ $(i = 1,2)$ is such that $t = t_1 \cdot t_2$ in case (1) and $t_i = t$ in case (2). In case (2): $t_j = (\varepsilon, \varepsilon)$ for $j \neq i$.

Concatenation of P_1 and P_2 :

$$P = \langle T, I_T \rangle = P_1 \bullet P_2 = \langle T_1 \bullet T_2, I_{T_1 \bullet T_2} \rangle.$$

This conforms to the intuitively expected behaviour of P because:

$$t^{I_T}(M) = \begin{cases} t_1^{I_{T_1}}(M|^\bullet T_1^\bullet) \cup t_2^{I_{T_2}}(M|^\bullet T_2^\bullet) & \text{if } t = t_1 \bullet t_2 \\ t_1^{I_{T_1}}(M|^\bullet T_1^\bullet) \cup M|^\bullet T_2^\bullet & \text{if } t_1 = t \\ t_2^{I_{T_2}}(M|^\bullet T_2^\bullet) \cup M|^\bullet T_1^\bullet & \text{if } t_2 = t \end{cases}$$

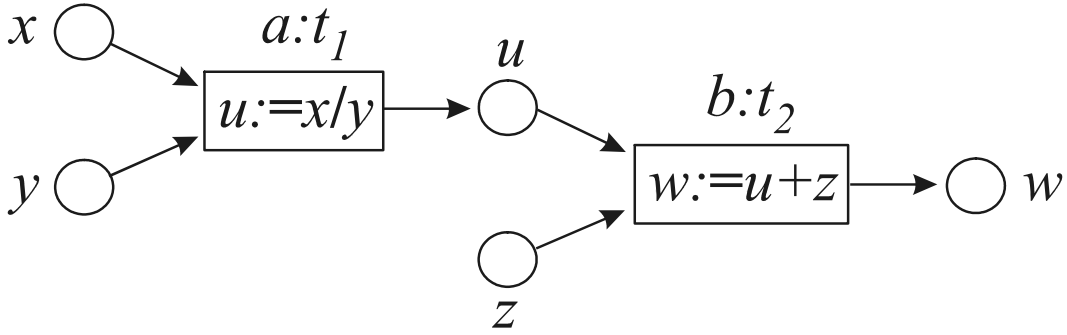
Associativity:

For nets $P_i = \langle T_i, I_{T_i} \rangle$ $(i = 1,2,3)$ let ${}^\bullet T_i^\bullet \cap {}^\bullet T_j^\bullet = \emptyset$ when $i \neq j$.

Then: $P_1 \bullet (P_2 \bullet P_3) = (P_1 \bullet P_2) \bullet P_3$.

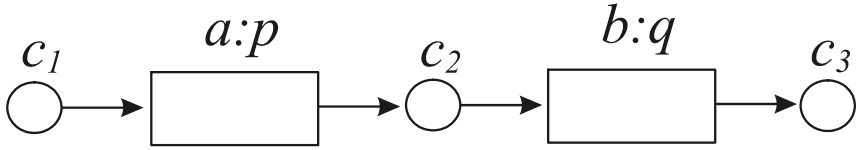
Example.

Net $P_1 = \{a{:}(xy, u),\ b{:}(uz, w)\}$:



computes $u := x/y$ and $w = u + z$ in unspecified order.

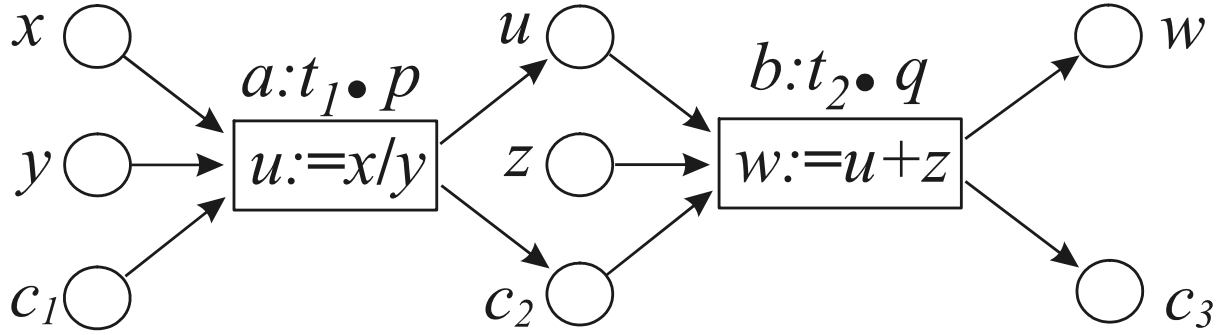
Net $P_2 = \{a{:}(c_1, c_2),\ b{:}(c_2, c_3)\}$:



with Petri net interpretation represents a control (instruction counter) for P_1.

Net

$P_3 = P_1 \bullet P_2 = \{a{:}(xyc_1, uc_2),\ b{:}(uzc_2, wc_3)\}{:}$



correctly computes statement $w := x/y + z$

in the desired order of firing transitions, if the initial marking of P_2 is $M(c_1) = 1,\ M(c_2) = M(c_3) = 0$.

Decomposition of nets wrt partition of places

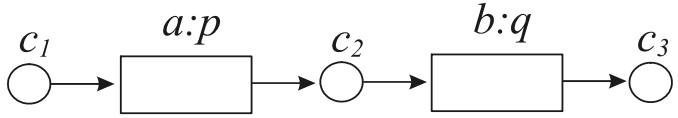
Conditions ensuring that behaviour of a net can be expressed by behaviour of its components.

Problem:

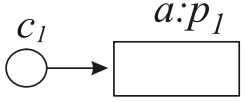
For $P = \langle T, I_T \rangle$, what conditions a partition $\{S_1, \ldots, S_n\}$ of ${}^\bullet T^\bullet$ must satisfy, so that:

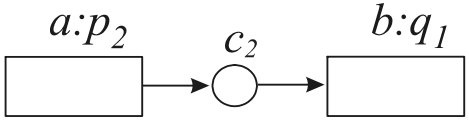
${}^\bullet T_i^\bullet = S_i$ for $i = 1, .., n$ and

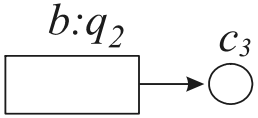
$P = P_1 \bullet \ldots \bullet P_n$ where $P_i = \langle T_i, I_{T_i} \rangle$?

$P_2 =$ 

can be decomposed as $P_2 = Q_1 \bullet Q_2 \bullet Q_3$

$Q_1 =$ 

$Q_2 =$ 

$Q_3 =$ 

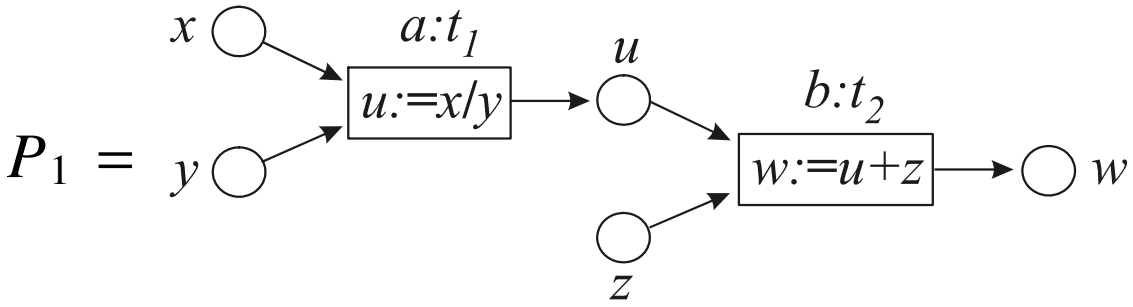
Q_i with Petri net interpretation.

Decomposition into Q_1, Q_2, Q_3 corresponds to partition $\{\{c_1\},\{c_2\},\{c_3\}\}$ of the set $\{c_1, c_2, c_3\}$. For any partition of this set there is a corresponding decomposition of P_2.

16 **Proposition**

Each Petri net can be decomposed into as many ways as there are partitions of its places: for each partition, there is a corresponding decomposition of the net.

On the other hand, the net



representing $w := x/y + z$ cannot be decomposed: (1) neither a:t_1 nor b:t_2 can be split so that concatenation of its components would yield interpretation of t_1 as division or t_2 as addition. (2) split into $\{a{:}t_1\}$ and $\{b{:}t_2\}$ is wrong since both nets share place u. The only partition of $\{x, y, z, u, w\}$ such that there is a decomposition of P_1 is trivial: $\{\{x, y, z, u, w\}\}$.

Problem: for which partitions there exist decompositions of a given net?

Definition (functional partitions)

Given net $P = \langle T, I_T \rangle$, a partition $\{S_1, \ldots, S_n\}$ of ${}^\bullet T^\bullet$ is *functional* wrt P iff for every $t \in T$:

(a) If $M|S_i = M'|S_i$ then $t^{I_T}(M)|S_i = t^{I_T}(M')|S_i$ provided that $M, M' \in dom(t^{I_T})$ for $i = 1, .., n$

(b) If $M_1, \ldots, M_n \in dom(t^{I_T})$ then $M \in dom(t^{I_T})$ where $M = \bigcup_{i=1}^{n} M_i|S_i$

18 **Theorem**

Net $P = \langle T, I_T \rangle$ is decomposable wrt a partition $\{S_1, \ldots, S_n\}$ of ${}^\bullet T^\bullet$ iff the partition is functional wrt P.

Corollary

If $P = P_1 \bullet \ldots \bullet P_n$ then for any permutation $k_1, \ldots, k_n$ of $1, \ldots, n$: $P = P_{k_1} \bullet \ldots \bullet P_{k_n}$. Hence: concatenation of nets works as operator "||" on Petri nets - the parallel composition with merged transitions identically labelled.

A net may be decomposed in several ways. However:

Proposition

For a fixed (functional) partition $\{S_1, \ldots, S_n\}$ of places in a net $P = \langle T, I_T \rangle$, decomposition $P = \langle T_1, I_{T_1} \rangle \bullet \ldots \bullet \langle T_n, I_{T_n} \rangle$ with ${}^\bullet T_i^\bullet = S_i$ is unique (up to the order of components).

Theorem

The atomic decomposition of a net

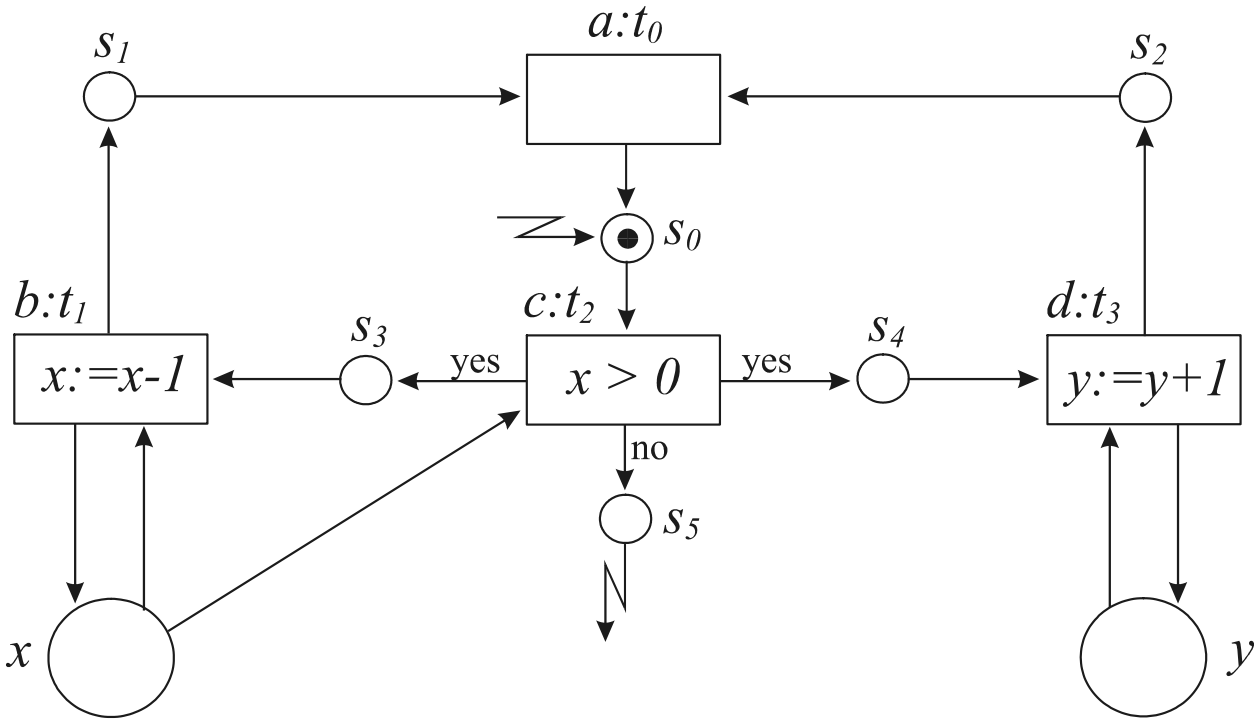
$P = \langle T, I_T \rangle$ is unique, i.e. if

$P = P_1 \bullet \ldots \bullet P_n$

$P = P'_1 \bullet \ldots \bullet P'_m$

are two of its atomic decompositions then sequences $P_1, \ldots, P_n$ and $P'_1, \ldots, P'_m$ are permutations of each other.

Example. The net $P_{paradd} = \langle T, I_T \rangle$:



is a parallel computation of $x + y$ of non-negative integer numbers by adding and substracting 1 and comparing with 0. The initial values are in x and y, the result - in y.

The net-structure is $T = \{a{:}t_0, b{:}t_1, c{:}t_2, d{:}t_3\}$. Transition $a{:}t_0$ represents empty statement. Interpretation:

$$(M, M') \in t_0^{I_T} \Leftrightarrow M(s_1) = M(s_2) = 1 \wedge$$

$$M'(s) = \begin{cases} 0 & \text{if } s = s_1 \vee s = s_2 \\ 1 & \text{if } s = s_0 \\ M(s) & \text{else} \end{cases}$$

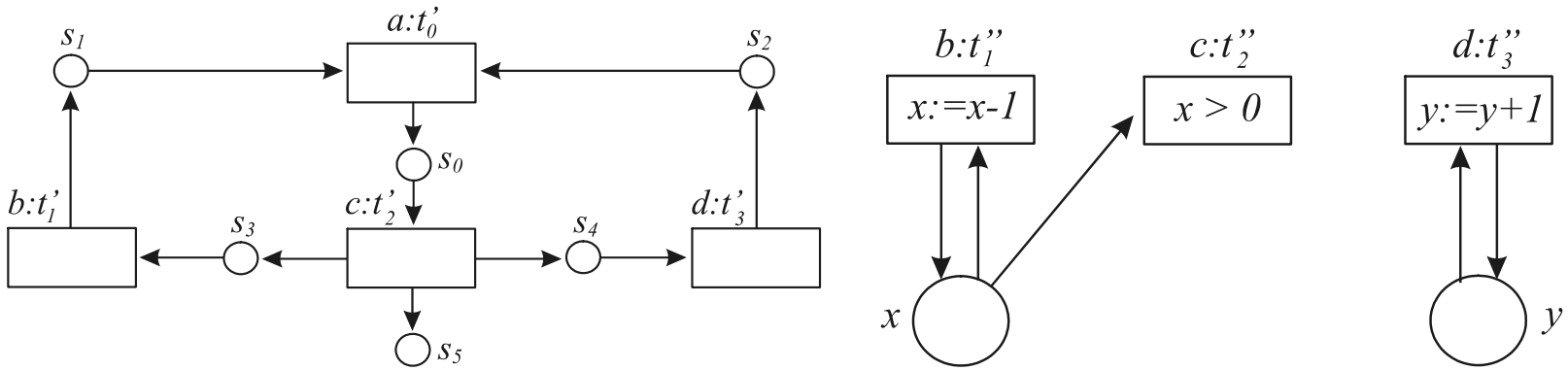
$$(M, M') \in t_1^{I_T} \Leftrightarrow M(s_3) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } s = s_3 \\ 1 & \text{if } s = s_1 \\ M(s) - 1 & \text{if } s = x \\ M(s) & \text{else} \end{cases}$$

$$(M, M') \in t_2^{I_T} \Leftrightarrow M(s_0) = 1 \wedge$$

$$M'(s) = \begin{cases} 0 & \text{if } s = s_0 \\ 1 & \text{if } (s = s_3 \vee s = s_4) \wedge M(x) > 0 \\ 1 & \text{if } s = x_5 \wedge M(x) = 0 \\ M(s) & \text{else} \end{cases}$$

$$(M, M') \in t_3^{I_T} \Leftrightarrow M(s_4) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } s = s_4 \\ 1 & \text{if } s = s_2 \\ M(s) + 1 & \text{if } s = y \\ M(s) & \text{else} \end{cases}$$

23 Decomposition into control P_{cont} and computation P_{comp}:

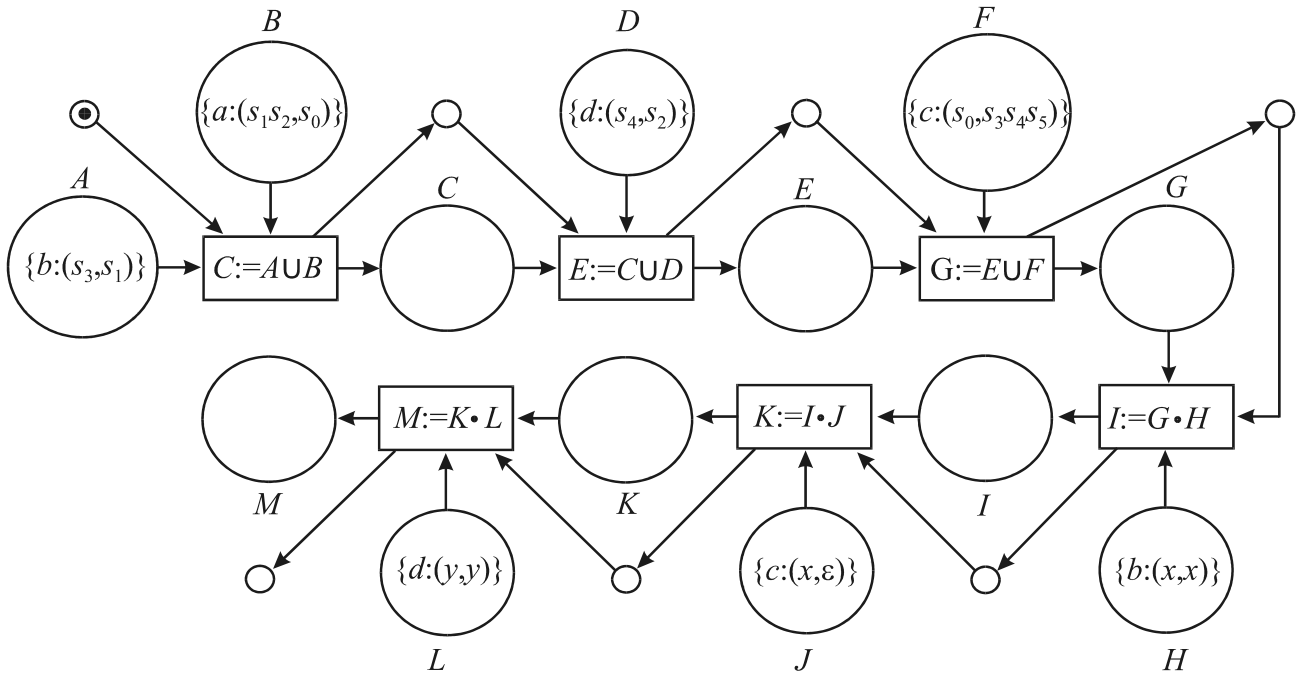


$P_{paradd} = P_{cont} \bullet P_{comp}$

Partition of places: $\{\{s_0, s_1, s_2, s_3, s_4, s_5\}, \{x, y\}\}$

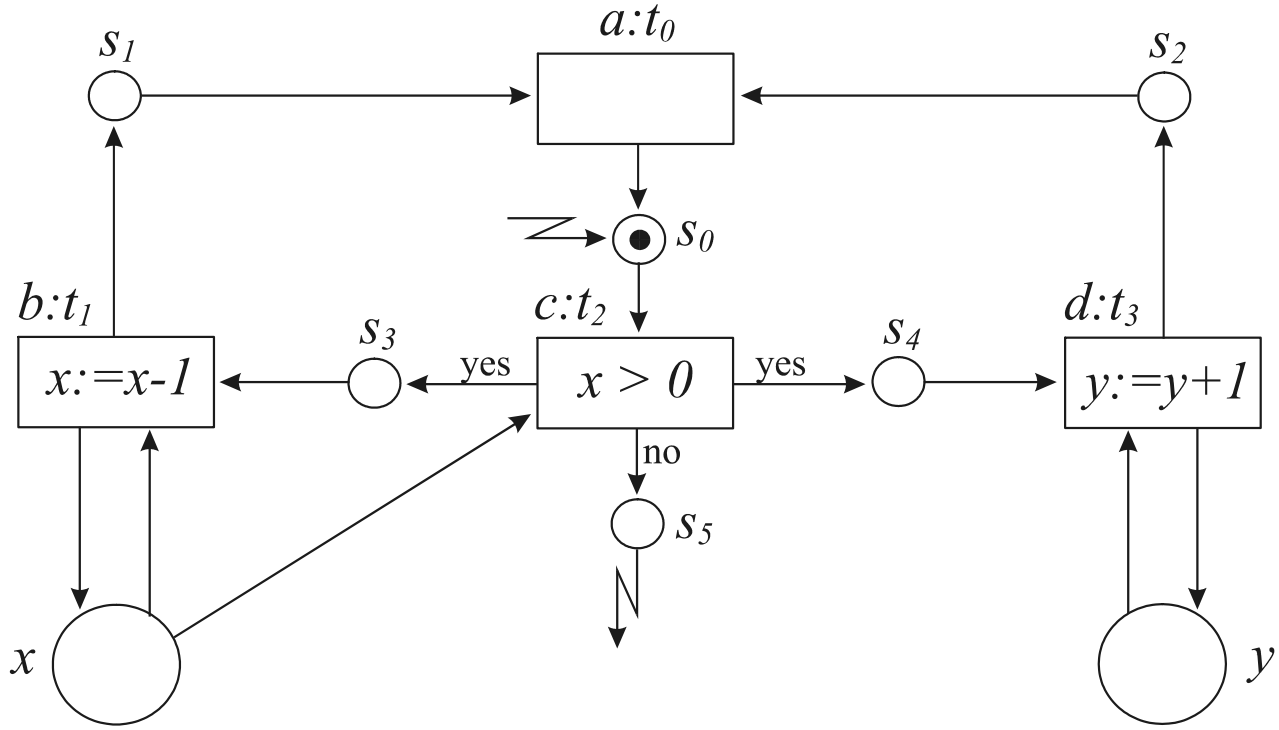
Decomposition of transitions: $t_0 = t_0' \bullet (\varepsilon, \varepsilon)$, $t_1 = t_1' \bullet t_1''$, $t_2 = t_2' \bullet t_2''$, $t_3 = t_3' \bullet t_3''$.

Example



Universe of values of places: $\boldsymbol{A} = 2^{\boldsymbol{T}} \cup \{0, 1\}$. Large places hold nets. Small places hold 0 or 1 (absence or presence of control token). The data structure is $\langle \boldsymbol{A}, \bullet, \cup, +, - \rangle$, with concatenation and union of nets and addition and subtraction of 1.

On termination, in the place *M* the net structure



appears.