Interpreted nets

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Abstract. The nets considered here are an extension of Petri nets in two aspects. In the semantical aspect, there is no one firing rule common to all transitions, but every transition is treated as an operator on data stored in its entry places and return results in its exit places. A state (marking) is a mapping of places (variables) into a given data structure, while interpretation is a mapping of transitions into a set of state transformers. Locality of transition's activity is like in Petri nets. In the structural aspect, entry and exit places to a transition are ordered. A concatenation of such nets is defined, hence their calculus (a monoid). This allows for combining small nets into large ones, in particular designing a computation and control parts separately, then putting them together into one. Such extended nets may produce, in particular, other nets. A number of properties of the operation on nets and their decomposition are demonstrated.

1. Introduction

Consider a Petri net structure, that is a bipartite directed graph. Imagine that places may hold arbitrary data, while transitions represent arbitrary operators on the data stored in their entry places and return results in exit places. In such a model the state (marking) of the net is a mapping of places into a given data structure, while interpretation is a mapping of transitions into a set of state transformers. To ensure locality of transition's activity, we require that it has no effect outside entry and exit places. The main feature of such "abstract" nets is that each transition is assigned an individual firing rule, in contrast to one firing rule common to all transitions as in Petri nets. Although transitions may assume various interpretations, one may regard a union of these interpretations as one interpretation of the net. The passage from the Petri net firing rule to interpretation as a mapping of a set (of transitions) into a set (of state transformers) reminds historical development of a notion of function: from one expression with a free variable(s), via a set of expressions conditionally selected, to a subset of Cartesian product of two sets. A data structure may be a collection of arbitrary objects with appropriate operations the state transformers may perform, e.g. a set of real numbers with arithmetic operations as in Example 3.1, or even nets with some composition operations as in Example 5.1. Thus, arbitrarily complex actions on whatever data are incorporated into net's semantics, not only rules imposing partial order of transition firing. The ability to impose such order is encoded in semantics of Petri nets - their principal feature - while account of computing is (sometimes) put down in comments attached to transitions and does not influence net behaviour at all. The nets developed here resemble programs (parallel in general), i.e. collections of statements, data structures and control mechanisms. Only the latter are present in Petri nets, that become a special case of the "abstract" nets. Although there are a number of the so-called higher level types of nets (e.g. [Gen-Lau 81], [Jen 81]), or object nets (e.g. [Val 98]), the level of abstraction considered here seems to go further.

Another issue explored is a composition of nets, called concatenation, since it retains the main properties of an ordinary concatenation. This allows for combining small nets into large ones, in particular designing a computation and control parts separately, then putting them together into one by concatenation. A number of properties of the operation on nets are demonstrated. Composition (and decomposition) of nets has been treated extensively in literature (e.g. [Bra 80], [Maz 84]), however it concerned nets with "Petri-like" interpretations, in which case the question of decomposability has a rather trivial answer (Proposition 4.1). It turns out that such nets are decomposable in any possible way, but usually it is not so with other interpretations.

The idea of interpreted nets originates in [Cza 85], but here is presented in a different setting and further developed.

2. Interpreted nets with ordered pre/post-sets of transitions

Let $\mathbb{X}$ and $\mathbb{A}$ be sets - a universe of *variables*, called *net-places,* and a universe of *values* the variables may assume, respectively. A *transition* over $\mathbb{X}$ is a pair $t = ({}^{\rightarrow}t, t^{\rightarrow})$ where ${}^{\rightarrow}t, t^{\rightarrow}$ are finite, possibly empty (denoted ε) sequences of distinct elements from $\mathbb{X}$, i.e. ${}^{\rightarrow}t, t^{\rightarrow} \in \mathbb{X}^*$. ${}^{\rightarrow}t$ is an *ordered pre-set* and $t^{\rightarrow}$ an *ordered post-set* of t. By $\mathbb{T}$ the universe of all transitions is denoted. *A net-structure* is any subset of $\mathbb{T}$. Denote by ${}^{\bullet}t, t^{\bullet}$ sets of elements occurring in the sequences ${}^{\rightarrow}t, t^{\rightarrow}$ respectively and let us call ${}^{\bullet}t^{\bullet} = {}^{\bullet}t \cup t^{\bullet}$ a *neighbourhood* of t. For a net-structure $T \subseteq \mathbb{T}$ let ${}^{\bullet}T^{\bullet} = \bigcup_{t \in T} {}^{\bullet}t^{\bullet}$. A *valuation* of the set ${}^{\bullet}T^{\bullet}$, called a marking of the net-structure T, is any total function $M : {}^{\bullet}T^{\bullet} \rightarrow \mathbb{A}$ and let $\mathbb{M}_T$ be the set of all markings. An *interpretation* I_T of T with each transition $t \in T$ associates a partial function $I_T(t) : \mathbb{M}_T \rightarrow \mathbb{M}_T$ denoted for short by t^{I_T} and, for $M, M' \in dom(t^{I_T})$ (a domain of t^{I_T}) satisfying:

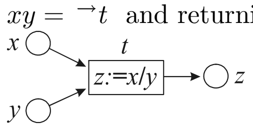
(i) $t^{I_T}(M)|{}^{\bullet}T^{\bullet} \setminus {}^{\bullet}t^{\bullet} = M|{}^{\bullet}T^{\bullet} \setminus {}^{\bullet}t^{\bullet}$ (activity of t has no effect outside ${}^{\bullet}t^{\bullet}$)
(ii) If $M|{}^{\bullet}t^{\bullet} = M'|{}^{\bullet}t^{\bullet}$ then $t^{I_T}(M)|{}^{\bullet}t^{\bullet} = t^{I_T}(M')|{}^{\bullet}t^{\bullet}$ (activity of t depends on ${}^{\bullet}t^{\bullet}$ only)

It follows from *(i)* that empty, i.e. isolated transition $t = (\varepsilon, \varepsilon)$ does not change marking, i.e. $t^{I_T}(M) = M$. If $M \in dom(t^{I_T})$ then say "*t is firable at M in the net-structure T*". An *interpreted* net is $P = \langle T, I_T \rangle$, its *semantics* is a binary relation $P^{I_T} = \bigcup_{t \in T} t^{I_T}$. ($t^{I_T}$ is treated as a binary relation defined by $(M, M') \in t^{I_T}$ iff $M' = t^{I_T}(M)$). The reflexive and transitive closure $(P^{I_T})^*$ is a *reachability* relation in the net P.

Example 2.1. For Place/Transition nets with arrow-weights 1 and unbounded capacity of places: $\mathbb{A} = \{0, 1, 2, ...\}$ and the interpretation t^I of transition $t \in T$ is defined by $M' = t^I(M)$ iff

$$(\forall x \in {}^\bullet t : M(x) > 0) \wedge M'(x) = \begin{cases} M(x) - 1 \; \textit{if} \;\; x \in {}^\bullet t \\ M(x) + 1 \; \textit{if} \;\; x \in t^\bullet \\ M(x) \qquad \textit{else} \end{cases}$$

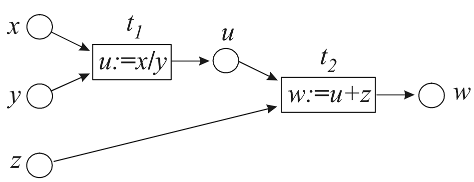
Example 2.2. A transition $t = ({}^\rightarrow t, t^\rightarrow)$ may represent an assignment statement $\langle y_1, y_2, ..., y_m \rangle := f(x_1, x_2, ..., x_n)$ with $x_1 x_2 ... x_n = {}^\rightarrow t$ and $y_1 y_2, ... y_m = t^\rightarrow$, computing a value of the atomic term $f(x_1, x_2, ..., x_n)$ and storing it in all $y_1, y_2, ..., y_m$. Thus, the interpretation t^I is defined by $M' = t^I(M)$ iff
$$M'(x) = \begin{cases} f(x_1, x_2, ..., x_n) \; \textit{if} \;\; x \text{ occurs in } t^\rightarrow \\ M(x) \qquad\qquad \textit{else} \end{cases}$$
For instance, the statement $z := x/y$ computing a quotient of real numbers (with a symbol of "undefined", i.e. $\mathbb{A} = \mathbb{R} \cup \{\perp\}$) stored in variables (places) $xy = {}^\rightarrow t$ and returning result in $z = t^\rightarrow$ is a transition



Here it is seen that the order of the pre-set places matters.

Thus $t^I(M)(x) = M(x)$, $t^I(M)(y) = M(y)$, $t^I(M)(z) = \begin{cases} \frac{M(x)}{M(y)} \text{ if } \; M(y) \neq 0 \\ \perp \quad \text{else} \end{cases}$

A set of such transitions is a net which may (but not always does!) represent a general assignment statement, that is such that the term on the right side of ":=" may be nested, i.e. not atomic. For instance, if $t_1 = (xy, u)$ represents $u := x/y$, $t_2 = (uz, w)$ represents $w := u + z$, then the net-structure $\{t_1, t_2\}$ representing $w := x/y + z$ and pictured as:



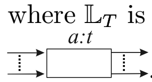
is interpreted as:

$$t_1^I(M)(s) = \begin{cases} \frac{M(x)}{M(y)} \text{ if } \;\; s = u \wedge M(y) \neq 0 \\ \perp \qquad \text{if } \;\; s = u \wedge M(y) = 0 \\ M(s) \text{ else} \end{cases} \quad t_2^I(M)(s) = \begin{cases} M(u) + M(z) \text{ if } \;\; s = w \\ M(s) \qquad\qquad \text{else} \end{cases}$$

Interpretation of nets for assignment statements conforms to their computer's realisation (if an order of executing transitions is imposed by another net modelling instruction counter, see Example 3.1): fetching a value from a memory cell (place) does not change its contents.

Since a net-structure T is defined as a set of transitions, any partition of this set is a trivial decomposition of T. If any component net-structure is interpreted by interpretation I_T restricted to the component, then we may speak of a decomposition of the net $P = \langle T, I_T \rangle$ wrt a partition of the set T (of its transitions). In the following, we examine decomposition wrt a partition of the set ${}^\bullet T^\bullet$ (of places).

3. Concatenation of nets

First, let us define concatenation of transitions $t = ({}^{\rightarrow}t,\, t^{\rightarrow})$ $t' = ({}^{\rightarrow}t',\, t'^{\rightarrow})$: $t \cdot t' = ({}^{\rightarrow}t\ {}^{\rightarrow}t',\ t^{\rightarrow}\ t'^{\rightarrow})$ if each sequence ${}^{\rightarrow}t\ {}^{\rightarrow}t'$ and $t^{\rightarrow}\ t'^{\rightarrow}$ contains distinct elements (i.e. ${}^\bullet t \cap {}^\bullet t' = \emptyset = t^\bullet \cap t'^\bullet$) and undefined otherwise.
Second, let transitions in a net-structure T be labelled: $l_T : \mathbb{L}_T \rightarrow T$ is a labelling function in T, where $\mathbb{L}_T$ is a set of labels in T. Say then "a labels t" if $l_T(a) = t$, pictorially: 

. In what follows, we will consider labelled net-structures only and denote them also by T, possibly with indices.
Third, for the labelled net-structures T_1, T_2 with ${}^\bullet T_1^\bullet \cap {}^\bullet T_2^\bullet = \emptyset$ and labelling l_{T_1}, l_{T_2} respectively, define their concatenation: $t \in T_1 \bullet T_2$ iff **either** ($t = t_1 \cdot t_2$ where $t_1 \in T_1$, $t_2 \in T_2$ and for a certain label $a \in \mathbb{L}_{T_1} \cap \mathbb{L}_{T_2}$: $l_{T_1}(a) = t_1$, $l_{T_2}(a) = t_2$) **or** ($t \in T_1 \cup T_2$ otherwise). In words: $T_1 \bullet T_2$ is a set of concatenations of transitions identically labelled in both constituents; a transition in T_i for which there is no transition in T_j (when $j \neq i$) labelled identically, remains in $T_1 \bullet T_2$ as it is in T_i. Labelling $l_{T_1 \bullet T_2}$ in $T_1 \bullet T_2$ is: $l_{T_1 \bullet T_2}(a) = t_1 \cdot t_2$ in the first case and $l_{T_1 \bullet T_2}(a) = l_{T_i}(a)$ if $l_{T_i}(a) \in T_i$ $(i = 1, 2)$ in the second. Note that due to ${}^\bullet T_1^\bullet \cap {}^\bullet T_2^\bullet = \emptyset$ the concatenation of labelled net-structures is well defined. Finally, given nets $P_1 = \langle T_1, I_{T_1} \rangle$, $P_2 = \langle T_2, I_{T_2} \rangle$, interpretation I_T of a transition $t \in T = T_1 \bullet T_2$ is defined by $t^{I_T}(M) = t_1^{I_{T_1}}(M|{}^\bullet T_1^\bullet) \cup t_2^{I_{T_2}}(M|{}^\bullet T_2^\bullet)$ where $t_i \in T_i$ $(i = 1, 2)$ is such that $t = t_1 \cdot t_2$ in the first case and $t_i = t$ in the second. In the latter case $t_j = (\varepsilon, \varepsilon)$ for $j \neq i$. Concatenation of P_1 and P_2 is defined as $P = \langle T, I_T \rangle = P_1 \bullet P_2 = \langle T_1 \bullet T_2, I_{T_1 \bullet T_2} \rangle$. This conforms to the intuitively expected behaviour of P because of:

Proposition 3.1 For $i = 1, 2$:

$$t^{I_T}(M) = \begin{cases} t_1^{I_{T_1}}(M|{}^\bullet T_1^\bullet) \cup t_2^{I_{T_2}}(M|{}^\bullet T_2^\bullet) & \text{if } t = t_1 \cdot t_2 \text{ and } \exists a\colon l_{T_i}(a) = t_i \in T_i \\ t_1^{I_{T_1}}(M|{}^\bullet T_1^\bullet) \cup M|{}^\bullet T_2^\bullet & \text{if } t = t_1 \in T_1 \text{ and not the first case} \\ t_2^{I_{T_2}}(M|{}^\bullet T_2^\bullet) \cup M|{}^\bullet T_1^\bullet & \text{if } t = t_2 \in T_2 \text{ and not the first case} \end{cases}$$

Proof: obvious due to ${}^\bullet T_1^\bullet \cap {}^\bullet T_2^\bullet = \emptyset$ and *(i)* in Section 2 □

A desired property of net concatenation is associativity:

Proposition 3.2 For nets $P_i = \langle T_i, I_{T_i} \rangle$ $(i = 1, 2, 3)$ let ${}^\bullet T_i^\bullet \cap {}^\bullet T_j^\bullet = \emptyset$ when $i \neq j$. Then: $P_1 \bullet (P_2 \bullet P_3) = (P_1 \bullet P_2) \bullet P_3$.

Proof: For $t_i \in T_i$, $l_{T_i} : \mathbb{L}_{T_i} \rightarrow T_i$ let e.g. $l_{T_1}(a) = t_1$ and no transition in T_2 and T_3 is labelled by a, $l_{T_2}(b) = t_2$, $l_{T_3}(b) = t_3$ and no transition in T_1 is labelled by b. Thus, $l_{T_2 \bullet T_3}(b) = t_2 \cdot t_3$, hence $l_{T_1 \bullet (T_2 \bullet T_3)}(a) = t_1$, $l_{T_1 \bullet (T_2 \bullet T_3)}(b) = t_2 \cdot t_3$. On the other hand, $l_{T_1 \bullet T_2}(a) = t_1$, $l_{T_1 \bullet T_2}(b) = t_2$, hence $l_{(T_1 \bullet T_2) \bullet T_3}(a) = t_1$, $l_{(T_1 \bullet T_2) \bullet T_3}(b) = t_2 \bullet t_3$. Considering other cases of labelling, we get analogous outcome. Therefore, $T_1 \bullet (T_2 \bullet T_3) = (T_1 \bullet T_2) \bullet T_3$. By definition of $I_{T_1 \bullet T_2}$ we immediately get $I_{T_1 \bullet (T_2 \bullet T_3)} = I_{(T_1 \bullet T_2) \bullet T_3}$ $\square$

Due to Proposition 3.2, one may write $P_1 \bullet P_2 \bullet ... \bullet P_n$ denoted $\bigodot_{i=1}^{n} P_i$. Note that for the Petri net interpretation (Example 2.1) operator $\bigodot_{i=1}^{n}$ reduces to the commonly known (commutative) parallel composition of nets with "gluing together" transitions labelled identically in different constituents cf. [Maz 95].

Example 3.1. Net $P_1 = \{a{:}(xy, u), b{:}(uz, w)\}$ in Fig.3.1 interpreted as in Example 2.2, represents computation of statements $u := x/y$ and $w = u + z$ in unspecified order. Net $P_2 = \{a{:}(c_1, c_2), b{:}(c_2, c_3)\}$ in Fig.3.2 with Petri net interpretation as in Example 2.1, represents a control (instruction counter) for P_1.

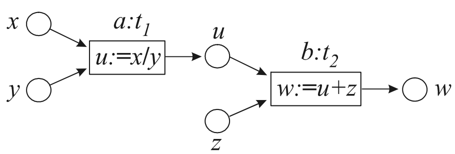


Fig.3.1 Net P_1

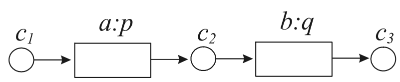


Fig.3.2 Net P_2

Net $P_3 = P_1 \bullet P_2 = \{a{:}(xyc_1, uc_2), b{:}(uzc_2, wc_3)\}$ in Fig.3.3, represents correct computation of statement $w := x/y + z$ in the desired order of firing transitions, if the initial marking of P_2 is $M(c_1) = 1$, $M(c_2) = M(c_3) = 0$.

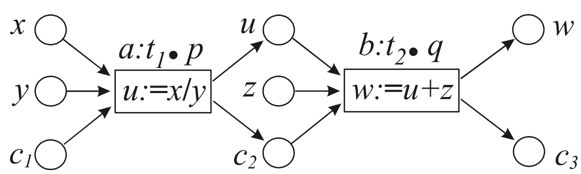
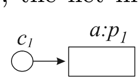
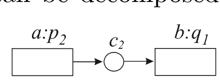
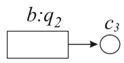


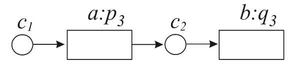
Fig.3.3 Net $P_3 = P_1 \bullet P_2$

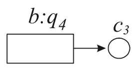
4. Decomposition of nets wrt partition of (sets of) places

We shall formulate conditions ensuring that behaviour of a net can be expressed by behaviour of its components. Given net $P = \langle T, I_T \rangle$, let us look for nets $P_i = \langle T_i, I_{T_i} \rangle$, $(i = 1, ..., n)$ with ${}^{\bullet}T_i^{\bullet} \cap {}^{\bullet}T_j^{\bullet} = \emptyset$ when $i \neq j$, such that $P = \bigodot_{i=1}^{n} P_i$. For instance, the net in Fig.3.2 can be decomposed as $P_2 = Q_1 \bullet Q_2 \bullet Q_3$ where $Q_1 =$ 

, $Q_2 =$ 

$Q_3 =$ 

, each Q_i with Petri net interpretation. Decomposition into Q_1, Q_2, Q_3 corresponds to partition $\{\{c_1\}, \{c_2\}, \{c_3\}\}$ of the set $\{c_1, c_2, c_3\}$. Obviously, for any partition of this set there is a corresponding decomposition of P_2, e.g. to $\{\{c_1, c_2\}, \{c_3\}\}$ corresponds decomposition into 

and 

. In general, we have:

Proposition 4.1 Each Petri net can be decomposed in as many ways as there are partitions of (the set of) its places. In other words, for each partition, there exists a corresponding decomposition of the net.

Proof: For a Petri net $P = \langle T, I_T \rangle$ let $\{S_1, ..., S_n\}$ be a partition of ${}^\bullet T^\bullet$. For S_i take transitions adjacent to places in S_i, change their names but retain labels, remove places not belonging to S_i and define a net-structure T_i to be the set of such modified transitions. Define interpretation I_{T_i} by $I_{T_i}(M|S_i) = I_T(M)|S_i$ where M is a marking of T. Obviously, $P = P_1 \bullet ... \bullet P_n$ □

On the other hand, the net P_1 in Fig.3.1 (not a Petri net!) cannot be decomposed at all, since neither transition $a{:}t_1$ nor $b{:}t_2$ can be split so that concatenation of its components would yield interpretation of t_1 as division or t_2 as addition. Thus, the only partition of $\{x, y, z, u, w\}$ for which there exists a decomposition of P_1 is trivial: $\{\{x, y, z, u, w\}\}$.

Each decomposition $\bigodot_{i=1}^{n} P_i$ of $P = \langle T, I_T \rangle$ defines a unique partition $\{{}^\bullet T_1^\bullet, ..., {}^\bullet T_n^\bullet\}$ of ${}^\bullet T^\bullet$ where $P_i = \langle T_i, I_{T_i} \rangle$. But for some partitions, no corresponding non-trivial decomposition of P may exist, as shows the latter example. We aim at characterising partitions for which there exist decompositions of a given net.

Definition 4.1 (functional partitions)
Given net $P = \langle T, I_T \rangle$, a partition $\{S_1, ..., S_n\}$ of ${}^\bullet T^\bullet$ is *functional* wrt P iff for every $t \in T$:
(a) If $M|S_i = M'|S_i$ then $t^{I_T}(M)|S_i = t^{I_T}(M')|S_i$ provided that $M, M' \in dom(t^{I_T})$ for $i = 1, ..., n$
(b) If $M_1, ..., M_n \in dom(t^{I_T})$ then $M \in dom(t^{I_T})$ where $M = \bigcup_{i=1}^{n} M_i|S_i$

For example, the only functional partition of the set of places in the net in Fig.3.1 is $\{\{x, y, z, u, w\}\}$, while every partition of the set of places in the net in Fig.3.2 is functional.

Theorem 4.1 A net $P = \langle T, I_T \rangle$ is decomposable wrt a partition $\{S_1, ..., S_n\}$ of ${}^\bullet T^\bullet$ iff the partition is functional wrt P.

Proof. ($\Rightarrow$). Let P be decomposable wrt $\{S_1, ..., S_n\}$: there are nets P_i with $P = \bigodot_{i=1}^{n} P_i$, $P_i = \langle T_i, I_{T_i} \rangle$, ${}^\bullet T_i^\bullet = S_i$. By definition of net concatenation:

$t^{I_T}(M) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M|S_i)$ for any marking $M \in dom(t^{I_T})$. Since $t_i^{I_{T_i}}$ are functions then $M|S_k = M'|S_k \Rightarrow t_k^{I_{T_k}}(M|S_k) = t_k^{I_{T_k}}(M'|S_k)$ provided that $M, M' \in dom(t^{I_T})$. By property of restriction "|" and by $S_i \cap S_k = \emptyset$ for $i \neq k$:
$t^{I_T}(M)|S_k = (\bigcup_{i=1}^{n} t_i^{I_{T_i}}(M|S_i))|S_k = t_k^{I_{T_k}}(M|S_k)$
$t^{I_T}(M')|S_k = (\bigcup_{i=1}^{n} t_i^{I_{T_i}}(M'|S_i))|S_k = t_k^{I_{T_k}}(M'|S_k)$
Therefore, we get $M|S_k = M'|S_k \Rightarrow t^{I_T}(M)|S_k = t^{I_T}(M')|S_k$ that is point *(a)* in Definition 4.1. To prove *(b)*, suppose $M_1, ..., M_n \in dom(t^{I_T})$ and $M = \bigcup_{k=1}^{n} M_k|S_k$. Thus, $M|S_i = M_i|S_i$, $i = 1, ..., n$. Since $t^{I_T}(M) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M|S_i) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M_i|S_i)$ and $t^{I_T}(M_k) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M_k|S_i)$, then $M \in dom(t^{I_T})$.
($\Leftarrow$). Let $\{S_1, ..., S_n\}$ be a functional partition of ${}^\bullet T^\bullet$ wrt P. We look for nets $P_i = \langle T_i, I_{T_i}\rangle$, $i = 1, ..., n$ such that ${}^\bullet T_i^\bullet = S_i$, $t^{I_T}(M) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M|S_i)$, that is $P = \bigodot_{i=1}^{n} P_i$. For a transition $t = ({}^\rightarrow t, t^\rightarrow)$ denote $t{\downarrow}S_i = ({}^\rightarrow t{\downarrow}S_i, t^\rightarrow{\downarrow}S_i)$, where ${}^\rightarrow t{\downarrow}S_i$, etc. is projection of the sequence ${}^\rightarrow t$ onto the set S_i (i.e. ${}^\rightarrow t{\downarrow}S_i$ is ${}^\rightarrow t$ with all members $\notin S_i$ deleted), and let $T{\downarrow}S_i = \{t{\downarrow}S_i\colon\ t \in T\}$. Define $T_i = T{\downarrow}S_i$ and $t_i^{I_{T_i}}(M_i) = t^{I_T}(M)|S_i$, where M is arbitrary marking in P in which t is firable and satisfying $M|S_i = M_i$. Obviously, $T = T_1 \bullet ... \bullet T_n$, so it remains to show $t^{I_T}(M) = t_1^{I_{T_1}}(M_1) \cup ... \cup t_n^{I_{T_n}}(M_n)$. By definition of $t_i^{I_{T_i}}$: $t_i^{I_{T_i}}(M_i) = t^{I_T}(M_i')|S_i$, where $M_i' : {}^\bullet T^\bullet \rightarrow \mathbb{A}$ is a certain marking in P in which t is firable and satisfying $M_i'|S_i = M_i$. Since $M|S_i = M_i'|S_i$ then applying points *(a)* and *(b)* in Definition 4.1 and $t_i^{I_{T_i}}(M_i) = t^{I_T}(M_i')|S_i$, we get $t^{I_T}(M)|S_i = t^{I_T}(M_i')|S_i = t_i^{I_{T_i}}(M_i)$ which implies $t^{I_T}(M) = t_1^{I_{T_1}}(M_1) \cup ... \cup t_n^{I_{T_n}}(M_n)$ $\square$

In the "$\Leftarrow$" part of the proof we have taken arbitrary ordering of $S_1, ..., S_n$ and defined sets of places in the components P_i as ${}^\bullet T_i^\bullet = S_i$. Thus, the composition $\bigodot_{i=1}^{n} P_i$ does not depend on the ordering. This property accounts for behaviour of the whole net P as localised in its components P_i and implies immediately:

Corollary 4.1 If $P = P_1 \bullet ... \bullet P_n$ then for any permutation $k_1, ..., k_n$ of indices $1, ..., n$: $P = P_{k_1} \bullet ... \bullet P_{k_n}$. In such cases concatenation of nets works as operator "||" on Petri nets - the parallel composition with merged transitions identically labelled.

In general, a net may be decomposed in several ways, e.g. look back at Proposition 4.1. However we have:

Proposition 4.2 For a fixed (functional) partition $\{S_1, ..., S_n\}$ of the (set of) places in a net $P = \langle T, I_T\rangle$, decomposition $P = \bigodot_{i=1}^{n}\langle T_i, I_{T_i}\rangle$ with ${}^\bullet T_i^\bullet = S_i$ is unique (up to the order of components).

Proof: Suppose $P = \bigodot_{i=1}^{n}\langle T_i', I_{T_i'}\rangle$ with ${}^\bullet T_i'^\bullet = S_i$ (we assume the same order of components corresponding to the same sets S_i in both decompositions of P). Then $T'_i = T_i$, since $S_j \cap S_k = \emptyset$ when $j \neq k$. If M is a marking of T such that a transition $t \in T$ is firable at M then $t^{I_T}(M) = \bigcup_{i=1}^{n} t_i^{I_{T_i}}(M|S_i)$ for some $t_1, ..., t_n \in T_i$ and $t^{I_T}(M) = \bigcup_{i=1}^{n} {t'_i}^{I_{T'_i}}(M|S_i)$ for some $t'_1, ..., t'_n \in T'_i$. But $t'_i = t_i$, because t'_i and t_i are components (both corresponding to the set S_i, i.e. ${}^\bullet t'^\bullet_i = {}^\bullet t^\bullet_i \subseteq S_i$) of the same transition t split in the two supposed decompositions of P. Thus ${t'_i}^{I_{T'_i}}(M|S_i) = t_i^{I_{T_i}}(M|S_i)$ since $T'_i = T_i$. □

If a decomposition is *atomic*, that is such that no component is decomposable, then it is unique (up to the order of components):

Theorem 4.2 The atomic decomposition of a net $P = \langle T, I_T \rangle$ is unique, i.e. if $P = \bigodot_{i=1}^{n} P_i = \bigodot_{i=1}^{m} P'_i$ are two of its atomic decompositions then $n = m$ and sequences $P_1, ..., P_n$ and $P'_1, ..., P'_m$ are permutations of each other.

Proof. Due to Theorem 4.1, these decompositions yield some functional partitions $\mathbb{S} = \{S_1, ..., S_n\}$, $\mathbb{S}' = \{S'_1, ..., S'_m\}$, of the set ${}^\bullet T^\bullet$ with classes S_i and S'_j atomic, i.e. not further fractionisable (in other words, atomic nets P_i and P'_j yield only trivial decomposition $\{S_i\}$ and $\{S'_j\}$ of the set of their places). Suppose $\mathbb{S} \neq \mathbb{S}'$ that is, either certain $S_i \notin \mathbb{S}'$ or certain $S'_j \notin \mathbb{S}$. Let, e.g. $S_i \notin \mathbb{S}'$ and let $x \in S_i$. Obviously $x \in S'_j \in \mathbb{S}'$ for a certain S'_j (because $\mathbb{S}$ and $\mathbb{S}'$ are partitions of the same set) and $x \notin S_k$ and $x \notin S'_l$ for $k \neq i$, $l \neq j$, (because $S_i \cap S_k = \emptyset$, $S'_j \cap S'_l = \emptyset$). Suppose $S_i \neq S'_j$. Then, e.g. $y \in S_i$ and $y \notin S'_j$. Now, apply assumption on the atomicity of (sub)nets P_i and P'_j: x and y as places belonging to one atomic class, cannot be separated by another functional partition of ${}^\bullet T^\bullet$ (otherwise interpretation I_T could not be composed of interpretations of the component subnets determined by this partition), which happens in partition $\mathbb{S}'$. Therefore $S_i = S'_j$ implying $\mathbb{S} = \mathbb{S}'$. By Proposition 4.2 we get the uniqueness of the atomic decomposition of P. □

Example 4.1 The net $P_{paradd} = \langle T, I_T \rangle$ in Fig.4.1 represents parallel computation of the sum $x + y$ of non-negative integer numbers x and y. The initial values are stored in places x and y, the result on termination - in y. A small place holding a token points to an operation (transition) to be executed. Initially s_0 only holds a token, on termination - s_5. Permitted operations are adding and subtracting 1 (if possible) and comparing with 0. The net-structure is $T = \{a{:}t_0, b{:}t_1, c{:}t_2, d{:}t_3\}$. Transition $a{:}t_0$ represents empty statement. Interpretation I_T of the (labelled) transitions is:

$$(M, M') \in t_0^{I_T} \Leftrightarrow M(s_1) = M(s_2) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } s = s_1 \vee s = s_2 \\ 1 & \text{if } s = s_0 \\ M(s) & \text{else} \end{cases}$$

$$(M, M') \in t_1^{I_T} \Leftrightarrow M(s_3) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } s = s_3 \\ 1 & \text{if } s = s_1 \\ M(s) - 1 & \text{if } s = x \\ M(s) & \text{else} \end{cases}$$

$$(M, M') \in t_2^{I_T} \Leftrightarrow M(s_0) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } \ s = s_0 \\ 1 & \text{if } \ (s = s_3 \vee s = s_4) \wedge M(x) > 0 \\ 1 & \text{if } \ s = x_5 \wedge M(x) = 0 \\ M(s) & \text{else} \end{cases}$$

$$(M, M') \in t_3^{I_T} \Leftrightarrow M(s_4) = 1 \wedge M'(s) = \begin{cases} 0 & \text{if } \ s = s_4 \\ 1 & \text{if } \ s = s_2 \\ M(s) + 1 & \text{if } \ s = y \\ M(s) & \text{else} \end{cases}$$

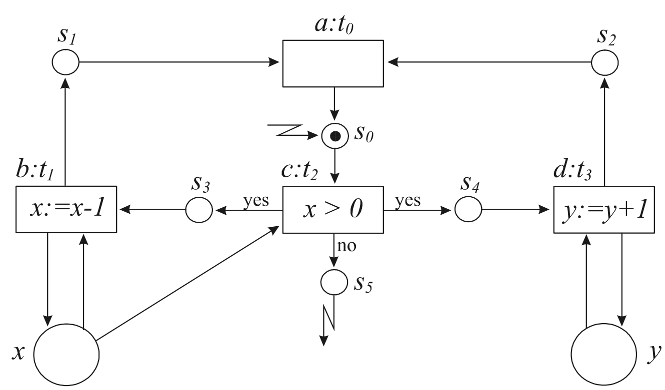


Fig.4.1 Parallel computation of $y := x + y$

Decomposition of the net in Fig.4.1 into control (P_{cont}) and computation (P_{comp}) part is in Fig.4.2 and Fig.4.3: $P_{paradd} = P_{cont} \bullet P_{comp}$. The respective partition of the set of places is $\{\{s_0, s_1, s_2, s_3, s_4, s_5\}, \{x, y\}\}$, decomposition of transitions is $t_0 = t_0^{'} \cdot (\varepsilon, \varepsilon)$, $t_1 = t_1^{'} \cdot t_1^{''}$, $t_2 = t_2^{'} \cdot t_2^{''}$, $t_3 = t_3^{'} \cdot t_3^{''}$. Obviously, each part is further decomposable.

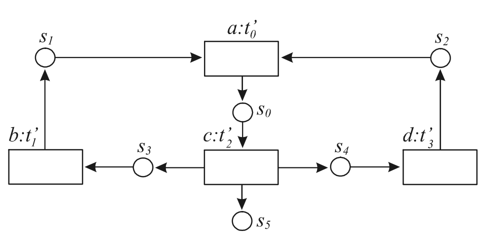


Fig.4.2 Control part

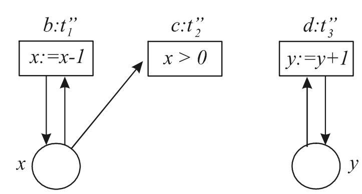


Fig.4.3 Computation part

5. Final remarks

Concatenation of nets defined in Section 3 is a partial operation: it is defined only when sets of places in both nets are disjoint. Otherwise multiple occurrences of the same place in a transition might happen. For example, if

$\{a{:}(x,y)\} \bullet \{a{:}(x,z)\} = \{a{:}(xx,yz)\}$, one may face a confusion when interpretation is imposed (e.g. if x in the net $\{a{:}(x,y)\}$ holds a real number, while in $\{a{:}(x,z)\}$ - a truth value). A dual operation on nets is their union: since a net-structure is a set of transitions and its interpretation is a mapping of the transitions into relations between markings, the set-theoretic union may be applied to nets. But to avoid another confusion, disjointness of sets of transition labels in both operands must be required. Otherwise different transitions might be assigned to the same label. For example, if $\{a{:}(x,y)\} \cup \{a{:}(x,z)\} = \{a{:}(x,y), a{:}(x,z)\}$, transitions (x,y) and (x,z) are labelled by a in the result. This violates requirement that labelleing is a function. One may ask whether the distribution law holds if nets P_1 and P_2 have different transition labels and P_3 has different than P_1 and P_2 places, i.e. whether $P_3 \bullet (P_1 \cup P_2) = P_3 \bullet\ P_1 \cup P_3 \bullet\ P_2$? The answer is no, as shows the counterexample with $P_1 = \{a{:}(x,y)\}$, $P_2 = \{b{:}(x,z)\}$, $P_3 = \{a{:}(u,v)\}$. Therefore, the restrictions imposed on concatenation and union of nets are too severe if one expects to get a semiring (even partial) of nets with such operations. Nonetheless, concatenation and union are quite powerfull means to combine small nets into large ones and to infer behaviour of the latter from behaviour of the former. Consider a ("meta") net in Example 5.1 specifying production of the net in Example 4.1. It visualises, by the way, how "abstract" nets may be used to construct nets.

Example 5.1 Let the universe of values (contents) of places be $\mathbb{A} = 2^{\mathbb{T}} \cup \{0,1\}$. In Fig.5.1 a "meta" net is depicted whose large places hold nets while small - 0 or 1 (absence or presence of control token). The data structure is $\langle \mathbb{A}, \bullet, \cup, +, - \rangle$, with operations: concatenation and union of nets and addition and subtraction of 1. If the net initially is marked as in Fig.5.1 then on termination, in the place M the net structure in Fig.4.1 will appear. Transition labels and names of control places in the "meta" net are immaterial thus omitted. Interpretation of the control part is as in Petri nets, interpretation of the computing part is mutatis mutandis analogous to that in Example 4.1.

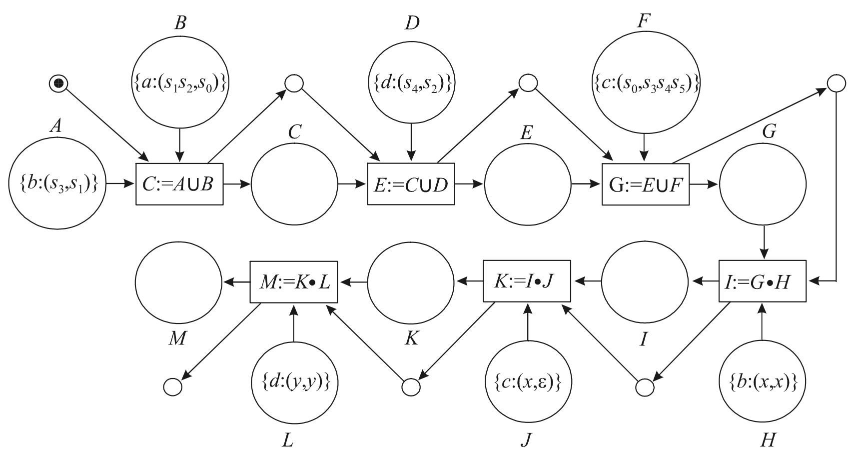


Fig.5.1 Net specifying step-by-step construction of net in Fig.4.1

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