Analiza - klasa 3a Zestaw 7. 5.12.2023

Zadania różne

1. Rozwiąż układy równań:

(a) $\begin{cases} x-y=6 \\ x^3-y^3=126 \end{cases}$

(b) $\begin{cases} xy+24=\dfrac{x^3}{y} \\ xy-6=\dfrac{y^3}{x} \end{cases}$

(c) $\begin{cases} (x-y)(y-1)=6 \\ (x+2)(y+2)=24 \end{cases}$

(d) $\begin{cases} 7x^3+11xy^2=6 \\ x^3+x^2y+y^3=1 \end{cases}$

(e) $\begin{cases} (x^2+x+1)(y^2+y+1)=3 \\ (1-x)(1-y)=6 \end{cases}$

(f) $\begin{cases} \dfrac{xy}{x+2y}+\dfrac{x+2y}{xy}=2 \\ \dfrac{xy}{x-2y}-6\dfrac{x-2y}{xy}=1 \end{cases}$

(g) $\begin{cases} x^3+x^3y^3+y^3=17 \\ xy+x+y=5 \end{cases}$

(h) $\begin{cases} x^3+y^3=1 \\ x^4+y^4=1 \end{cases}$

(i) $\begin{cases} (1+x)(1+x^2)(1+x^4)=1+y^7 \\ (1+y)(1+y^2)(1+y^4)=1+x^7 \end{cases}$

(j) $\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{9}{2} \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{2} \end{cases}$

(k) $\begin{cases} x^2+y^2=x+y \\ x^4+y^4=\dfrac{(x+y)^2}{2} \end{cases}$

(l) $\begin{cases} \dfrac{x+y}{xyz}=3 \\ \dfrac{y+z}{xyz}=4 \\ \dfrac{z+x}{xyz}=5 \end{cases}$

2. Uprość wyrażenia

(a) $\left(\dfrac{a\sqrt{a}+b\sqrt{b}}{\sqrt{a}+\sqrt{b}}-\sqrt{ab}\right)\cdot\left(\dfrac{\sqrt{a}+\sqrt{b}}{a-b}\right)^2$, dla $a,b>0$, $a\neq b$.

(b) $\dfrac{(a^2-b^2)(\sqrt[3]{a}-\sqrt[3]{b})}{\sqrt[3]{a^4}+\sqrt[3]{ab^3}-\sqrt[3]{a^3b}-\sqrt[3]{b^4}}$, dla $|a|\neq|b|$.

(c) $\sqrt{\left(\dfrac{1}{\sqrt{ab}}+\dfrac{1}{\sqrt[6]{a}}\left(\dfrac{1}{\sqrt[6]{b^5}}-\dfrac{1}{\sqrt[3]{a}\sqrt[2]{b}}\right)\right)^3}$, dla $a,b>0$.

(d) $\sqrt[3]{(a^2+1)\sqrt{1+\dfrac{1}{a^2}}}+\sqrt[3]{(a^2-1)\sqrt{1-\dfrac{1}{a^2}}}$, dla $a>1$.

(e) $\left(\dfrac{a-\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}-\dfrac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right)\cdot\dfrac{b^2}{\sqrt{a^4-a^2b^2}}$.

3. Rozwiąż równania

(a) $\sqrt{1-|5-x^2|}=2$

(b) $\sqrt{4x+8}-\sqrt{3x-2}=2$

(c) $\sqrt{x^2+x+1}+\sqrt{x^2-x+1}=4$

(d) $\sqrt{1+x^2}-\dfrac{1}{\sqrt{1+x^2}}=\sqrt{|x^2-1|}$

(e) $\sqrt[3]{x+45}-\sqrt[3]{x-16}=1$,

(f) $\sqrt{x^3+x^2-1}+\sqrt{x^3+x^2+2}=3$

(g) $\sqrt{x+2}+\sqrt{3x+8}=\sqrt{2x+6}$

(h) $\sqrt{x-3}+\sqrt{7-x}=\sqrt[4]{8(x-4)(6-x)}$

(i) $\dfrac{\sqrt{x^2+x+6}+\sqrt{x^2-x-4}}{\sqrt{x^2+x+6}-\sqrt{x^2-x-4}}=5$

(j) $\dfrac{x\sqrt[3]{x}-1}{\sqrt[3]{x^2}-1}-\dfrac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}-1}=12$

(k) $\sqrt{4+x\sqrt{x^2+40}}=x+2$

(l) $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\dfrac{3}{2}\sqrt{\dfrac{x}{x+\sqrt{x}}}$

4. Rozwiąż nierówności

(a) $\dfrac{1}{x+2}\leqslant\dfrac{3}{x-3}$,

(b) $\dfrac{1+x^3}{x^2-4}<x$,

(c) $\dfrac{(x-1)(x-2)(x-3)}{(x+1)(x+2)(x+3)}>1$,

(d) $\left|\dfrac{x^2-5x+3}{x^2-1}\right|<1$,

(e) $\dfrac{3}{|x+3|-1}\geqslant|x+2|$,

(f) $x^2+\dfrac{1}{x^2}+1\geqslant4\left(x-\dfrac{1}{x}\right)$,

(g) $(x-2)\sqrt{x^2+6}\leqslant x^2-4$,

(h) $\sqrt{2x+1}<\sqrt{x^3-4x^2+x+5}$,

(i) $\dfrac{x^2-16}{\sqrt{35-2x-x^2}}\leqslant|x|+4$,

5. Udowodnij, że

(a) $\left(x^4+\dfrac{1}{x}\right)\left(x^3+\dfrac{1}{x^2}\right)<\left(x^4+\dfrac{1}{x^2}\right)\left(x^3+\dfrac{1}{x}\right)$, dla $x>1$;

(b) $\left(x+\dfrac{1}{x}\right)^6-\left(x^3+\dfrac{1}{x^3}\right)^2\geqslant6\left(\left(x+\dfrac{1}{x}\right)^3+\left(x^3+\dfrac{1}{x^3}\right)\right)$, dla $x>0$.

6. Niech $a\in\mathbb{R}$. Rozwiąż równanie

$$\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+x}}}}=x.$$

7. Udowodnij, że dla dowolnych $n,m\in\mathbb{N}$, $n,m>1$, prawdziwa jest nierówność

$$\frac{1}{\sqrt[n^2]{mn+1}}+\frac{1}{\sqrt[m^2]{mn+1}}>1.$$