Algorithmic Coalitional Game Theory

Lecture 6: Representations

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Representations

How to represent a coalitional game?

A naive solution is to enumerate the payoffs of each set of players which requires space exponential in the number of players.

A representation is evaluated by three criteria:

- *Expressivity* – how many games can it represent?
- *Conciseness* – how much space is required to represent a game?
- *Efficiency* – how fast algorithms we can develop for it?

Representations

- *Efficiency* – we will consider the following computational problems:
  1. Checking whether the core is empty
  2. Checking whether an imputation is NOT in the core
  3. Computing the Shapley value

Complexity

"We propose another criterion for judging whether a proposed solution concept is appropriate: The computational complexity of the problems associated with it should not be too great.

There is something unfair about a concept of "fairness" that requires a supercomputer in order to test whether it applies in a given situation, or in order to produce an example of an allocation that is fair according to the concept."

Deng and Papadimitriou, 1994
"On the complexity of cooperative solution concepts"

Complexity

(…)

“But more importantly, our proposed criterion can be seen as an instance of the thesis of bounded rationality. Bounded rationality is the hypothesis that decisions by realistic economic agents cannot involve unbounded resources for reasoning.”

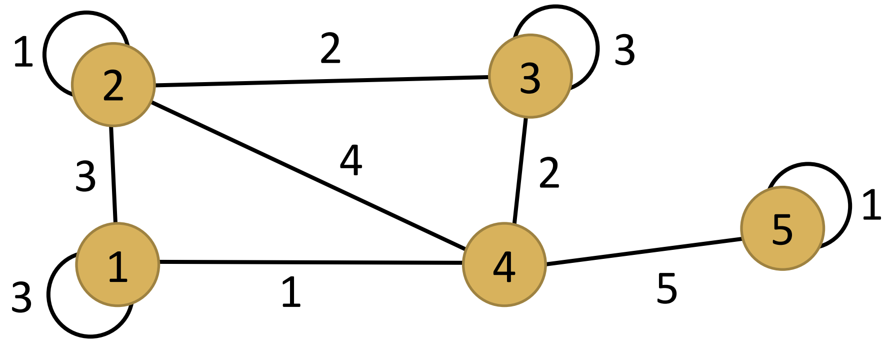
Deng and Papadimitriou, 1994
“On the complexity of cooperative solution concepts”

Induced Subgraph Games

Induced Subgraph Games [Deng & Papadimitriou 1994]

A game is represented by an undirected, weighted graph $G = (N, E, w)$, where w_{ij} is the weight of edge $\{i, j\}$ (with possible self-loops). The value of coalition $S \subseteq N$ is the sum of weights of edges in $G[S]$, i.e., the subgraph induced by S:

$$v_G(S) = \sum_{i,j \in S} w_{ij}\,.$$



Induced Subgraph Games

- Expressiveness: it is not fully expressive
- Conciseness: size of the representation is $O(|N|^2)$.
- Efficiency:

Induced Subgraph Games [Deng & Papadimitriou 1994]

1. The Shapley value equals $SV_i(v_G) = w_{ii} + \sum_{j \in N \setminus \{i\}} w_{ij}$; hence, it can be computed in polynomial time.
2. Checking whether an imputation is not in the core is NP-complete.
3. Checking whether the core is empty is NP-complete.

Proof: On the blackboard.

Induced Subgraph Games

<u>*Sketch of the proof:*</u>

From Additivity, we can consider each edge seperately. Assume $E = \{\{i, j\}\}$. Clearly $N \setminus \{i, j\}$ are null-players so from Null-Player and Efficiency we get $SV_i(v_G) = SV_j(v_G) = \frac{1}{2} w_{ij}$. If $i = j$, then analogously $SV_i(v_G) = w_{ii}$.

The core is non-empty if and only if there is no negative-cut in the graph: if $\{S, N \setminus S\}$ is a negative cut, then $v(S) + v(N \setminus S) > v(N)$; on the other hand, if there is no negative cut, then the Shapley value is in the core.

Induced Subgraph Games

<u>*Sketch of the proof (continued):*</u>

Now, checking if a negative-cut exists is NP-complete (reduction from MAX-CUT).

MAX-CUT
Input: Graph $G = (V, E)$, $w: E \to \mathbb{R}_{\geq 0}$ and a real value K *Question:* Is there exists a cut $\{S, N \setminus S\}$ with the total weight $> K$.

Synergy Coalition Groups

Synergy Coalition Groups [Conitzer & Sandholm 2006]

A game is represented as a list, L, of coalitions (that includes all singletons) and their values: $\big(S_1, v(S_1)\big), \dots, \big(S_k, v(S_k)\big)$. The value of coalition $S \subseteq N$ is the maximal value of all partitions of S into coalitions $S_1, \dots, S_k$:

$$v_L(S) = \max_{P=\{S_{i_1},\dots,S_{i_m}\}\in\mathcal{P}(S)} v(P).$$

$$L = \big((\{1\}, 1), (\{2\}, 0), (\{3\}, 0), (\{1,2\}, 2), (\{123\}, 3)\big)$$

Synergy Coalition Groups

- Expressiveness: it can express all superadditive games
- Conciseness: if there are only few groups that can collaborate productively
- Efficiency:

Synergy Coalition Groups [Conitzer & Sandholm 2006]

1. Checking whether an imputation is in the core can be done in polynomial time.
2. Checking whether the core is empty is NP-complete.
3. Computing the value of a coalition is NP-hard.

Proof: On the blackboard.

Synergy Coalition Groups

<u>*Sketch of the proof:*</u>

The imputation is not in the core if and only if one of the listed coalitions is getting less than its value.

EXACT-COVER-BY-3-SETS
Input: Set S of size $3m$, subsets $\{S_1, \dots, S_k\}$ of S of size 3. *Question:* Is there exists a subset $J \subseteq \{1, \dots, k\}$ s.t. $\lvert J \rvert = m$ and $\bigcup_{i \in J} S_i = S$.

Consider a game represented as the following list:

$$(S_1, 3), \dots, (S_k, 3), (S \cup \{x\}, 6m), (S \cup \{y\}, 6m), (\{x, y\}, 6m)$$

Now, the core is non-empty if and only if $v(S \cup \{x, y\}) = 9m$, i.e., if and only if $v(S) = 3m$.

Marginal Contribution Nets

Marginal Contribution Nets [Ieong & Shoham 2005]

A game is represented as a list of rules R of the form:

$$(p_1 \land \cdots \land p_m \land \neg n_1 \land \cdots \land \neg n_k) \to \omega,$$

where $p_i, n_i \in N$ and $\omega \in \mathbb{R}$ is the weight. The value of coalition $S \subseteq N$ is the sum of weights of rules that S satisfies, i.e., such that $p_i \in S$ for every $p_i \in \{p_1, \dots, p_m\}$ and $n_i \notin S$ for every $n_i \in \{n_1, \dots, n_k\}$:

$$v_R(S) = \sum_{(r \to \omega) \in R : S \text{ satisfies } r} \omega \,.$$

$$R = \{1 \land 2 \to 7, \qquad 1 \land \neg 3 \to 3, \qquad 3 \to 2\}$$

Marginal Contribution Nets

- Expressiveness: it is fully expressive (because we can specify value of each coalition with a separate rule)
- Conciseness: if value of a coalition is determined by the presence or absence of small groups of players
- Efficiency:

Marginal Contribution Nets [Ieong & Shoham 2005]

1. Shapley value can be computed in polynomial time.
2. Checking whether an imputation is not in the core is NP-complete.
3. Checking whether the core is empty is NP-hard.

Proof: On the blackboard.

Marginal Contribution Nets

Sketch of the proof:

From Additivity, we can consider each rule seperately. Clearly, all players that do not appear in the rule are null-players. From Null-Player Out we can consider a game without them. Now, the only coalition with non-zero value is $\{p_1, p_2, \ldots, p_m\}$.

Results for the core follows from the results for Induced Subgraph Games.

Conclusions

	Induced Subgraph Games	Synergy Coalition Groups	Marginal Contribution Nets
Fully expressive	✗ NO	✗ NO	✓ YES
Checking whether an inputation is not in the core	✗ NP-complete	✓ P	✗ NP-complete
Checking whether the core is empty	✗ NP-complete	✗ NP-complete	✗ NP-hard
Computing the Shapley value	✓ P	✗ NP-hard	✓ P

References

- [Conitzer & Sandholm 2006] V. Conitzer, T. Sandholm. *Complexity of constructing solutions in the core based on synergies among coalitions.* Artificial Intelligence 170, pp. 607-619, 2006.
- [Deng & Papadimitriou 1994] X. Deng, C.H. Papadimitriou. *On the complexity of cooperative solution concepts.* Mathematics of Operations Research 19, pp. 257-266, 1994.
- [Ieong & Shoham 2005] S. Ieong, Y. Shoham. *Marginal contribution nets: A compact representation scheme for coalitional games.* Proceedings of the 6th ACM Conference on Electronic Commerce (ACM-EC), 193-202, 2005.