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Shapley Value for Games with Externalities and Games on Graphs

PhD dissertation

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June 2014

Author's declaration:
I hereby declare that this dissertation is my own work.

June 9, 2014 ....

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The dissertation is ready to be reviewed

June 9, 2014 ....

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Shapley Value for Games with Externalities and Games on Graphs

The Shapley value [46] is one of the most important solution concepts in coalitional game theory. It was originally defined for classical model of a coalitional game, which is relevant to a wide range of economic and social situations. However, while in certain cases the simplicity is the strength of the classical coalitional game model, it often becomes a limitation. To address this problem, a number of extensions have been proposed in the literature. In this thesis, we study two important such extensions – to games with externalities and graph-restricted games.

Games with externalities [53] are a richer model of coalitional games in which the value of a coalition depends not only on its members, but also on the arrangement of other players. Unfortunately, four axioms that uniquely determine the Shapley value in classical coalitional games are not enough to imply a unique value in games with externalities. In this thesis, we study a method of strengthening the Null-Player Axiom by using α-parameterized definition of the marginal contribution in games with externalities. We prove that this approach yields a unique value for every α. Moreover, we show that this method is indeed general, in that all the values that satisfy the direct translation of Shapley's axioms to games with externalities can be obtained using this approach.

Graph-restricted games [36] model naturally-occurring scenarios where coordination between any two players within a coalition is only possible if there is a communication channel between them. Two fundamental solution concepts that were proposed for such a game are the Shapley value and its particular extension – the Myerson value. In this thesis we develop algorithms to compute both values. Since the computation of either value involves visiting all connected induced subgraphs of the graph underlying the game, we start by developing a dedicated algorithm for this purpose and show that it is the fastest known in the literature. Then, we use it as the cornerstone upon which we build algorithms for the Shapley and Myerson values.

Keywords: Shapley value, coalitional games, externalities, graph-restricted games, centrality metrics, Algorithmic Game Theory

ACM Classification: J.4, I.2.1

Wartość Shapleya w Grach z Efektami Zewnętrznymi i Grach na Grafach

Wartość Shapleya [46] jest jedną z najważniejszych metod podziału w teorii gier koalicyjnych. Oryginalnie została zdefiniowana w klasycznym modelu gier koalicyjnych, który jest dobrą ilustracją wielu ekonomicznych i społecznych sytuacji. Chociaż prostota jest w wielu przypadkach siłą klasycznego modelu gier koalicyjnych, często staje się jednak też jego ograniczeniem. Aby poradzić sobie z tym problemem, kilka rozszerzeń gier koalicyjnych zostało zaproponowanych w literaturze. W tej rozprawie zajmujemy się dwoma ważnymi rozszerzeniami – do gier z efektami zewnętrznymi oraz gier ograniczonych grafem (ang. *graph-restricted games*).

Gry z efektami zewnętrznymi [53] są bogatszym modelem gier koalicyjnych, w którym wartość koalicji zależy nie tylko od jej członków, ale także od rozmieszczenia innych graczy. Niestety, cztery aksjomaty, które implikują wartość Shapleya w klasycznych grach koalicyjnych, nie są wystarczające, aby implikować unikalną wartość w grach z efektami zewnętrznymi. W tej rozprawie badamy metodę polegającą na wzmocnieniu Aksjomatu Gracza Zerowego (ang. *Null-Player Axiom*), używając α-parametryzowanej definicji wkładu marginalnego. Udowadniamy, że takie podejście daje unikalną wartość dla każdego α. Ponadto, pokazujemy że jest ono ogólne: wszystkie wartości, które spełniają bezpośrednie tłumaczenie aksjomatów Shapleya, mogą być uzyskane z użyciem tego podejścia.

Gry ograniczone grafem [36] modelują naturalnie pojawiające się sytuacje, w których koordynacja dwóch graczy w ramach koalicji jest możliwa tylko wtedy, gdy istnieje kanał komunikacji między nimi. Dwie podstawowe koncepcje podziału, które zostały zaproponowane dla takich gier to wartość Shapleya oraz jej rozszerzenie – wartość Myersona. W tej rozprawie proponujemy algorytmy do obliczania obu wartości. Ponieważ obliczenie ich opiera się na enumerowaniu wszystkich spójnych indukowanych podgrafów w grafie gry, zaczynamy od opracowania algorytmu dedykowanego do tego celu i pokazujemy, że jest szybszy niż inne algorytmy w literaturze. Potem używamy go jako podstawę algorytmów do obliczania wartości Shapleya i Myersona.

Słowa kluczowe: wartość Shapleya, gry koalicyjne, efekty zewnętrzne, gry ograniczone grafem, miary centralności, algorytmiczna teoria gier

Klasyfikacja ACM: J.4, I.2.1

Acknowledgements

I would like to thank Dr. Tomasz Michalak, my mentor, co-author of all my papers, and a friend, for introducing me to research and his guidance through all these years. He has always motivated me to work, and when I was working - to work even harder. I thank for all the skype discussions and showing me "how a good paper should be written".

I would like to thank my supervisor, Prof. Andrzej Szałas. I could always count on his help and advice when I needed it.

I would like to thank my family for their support. They have always believed in me. In particular, I thank my mom who always argued with my bad reviews, my dad for the passion for mathematics which I inherited from him, and my sister, Patrycja, for adding the articles in every single English sentence written by me back in the day.

Last but not least, I would like to thank my wife, Milena, for pulling me away from work and pushing me back whenever it was needed. For everything I cannot name.

Contents

Overview

The problem of how to fairly divide a surplus obtained through cooperation is one of the most fundamental issues studied in coalitional game theory. It is relevant to a wide range of economic and social situations, from sharing the cost of a local wastewater treatment plant, through dividing the annual profit of a joint venture enterprise, to determining power in voting bodies. Assuming that the coalition of all the players (*i.e.* the *grand coalition*) forms, Shapley [46] defined a unique division scheme, called the Shapley value, which satisfies four intuitive axioms: it distributes all payoff among the players (*Efficiency*) in a linear way (*Additivity*) while treating symmetric players equally (*Symmetry*) and ignoring players with no influence on payoffs (*Null-Player Axiom*).

The Shapley value has been originally defined for classical coalitional games, which are built on the following two simple assumptions: every group of players is allowed to form a coalition and performance of every coalition is to be rewarded with a real-valued payoff being completely independent from the performance or payoffs of other coalitions. However, while in certain cases the simplicity is the strength of the classical coalitional game model, it often becomes a limitation. Indeed, the classical model is too simple to adequately represent various real-life situations.

To address the above problem, a number of extensions have been proposed in the literature. One especially vivid direction of research is how to extend the classical coalitional game model so to account for externalities from coalition formation. Such externalities, as formalized by Thrall and Lucas [53], occur in all the situations where the value of a group depends not only on its members, but also on the arrangement of other players. In other words, there is an *external* impact on the value of a group. As a matter of fact, externalities are common in many real-life situations, e.g., the merger of two companies affects the profit of its competitor. Unfortunately, in such settings four axioms proposed by Shapley are not enough to imply a unique value. For the last fifty years the problem how to extend Shapley value to games with externalities has not been resolved. This issue is the focus of Part I of our work.

In Part II we depart from games with externalities and study games on graphs. In the model proposed by Myerson [36] called *graph-restricted games* agents (or players) can communicate and cooperate only with agents that they know or are connected to. Such restrictions emerge in social networks analysis, but also in sensor networks, telecommunications or trade agreements, which makes it one of the most recent application of coalitional games. Now, if we extend the game defined only for connected groups to full coalitional game, then by calculating Shapley value we can obtain a value of a player in graph-restricted game. Unfortunately, calculating Shapley value in general requires enumerating of all 2^n coalitions. To this end, we show that traversing only connected coalitions is sufficient to calculate Shapley value in graph-restricted games and propose dedicated algorithms for this purpose.

Coalitional Games

We finish the overview by introducing basic notations of coalitional games. We will associate players with natural numbers, thus set of players is $N = \{1, 2, \dots, n\}$. By a *coalition* (denoted by S) we mean any non-empty subset of N. Now, a *game* is given by a function v that associates a real value with every coalition, i.e., $v : 2^N \to \mathbb{R}$. As is customary in the literature, we assume that the coalition of all players (i.e., *grand coalition*) will form. Then the outcome of the game (or the *value of the game*) is some distribution of jointly achieved payoff $v(N)$ between them – φ denotes a vector of payoffs and φ_i is the share of player i. Now, Shapley's axioms are formalized as follows:

- *Efficiency* (the entire available payoff is distributed among players): $\sum_{i \in N} \varphi_i(v) = v(N)$ for every game v;
- *Symmetry* (payoffs do not depend on the players' names): $\varphi(f(v)) = f(\varphi)(v)$ for every game v and every bijection $f : N \to N$ where $f(v)(S) \stackrel{\text{def}}{=} v(\{f(i) \mid i \in S\})$ and $f(\varphi) \stackrel{\text{def}}{=} \varphi_{f(i)}$;[1]

[1]Function f is a permutation, but we reserve this word to interpreting a sequence. Formally, $f(S)$ is an image of S: $f(S) \stackrel{\text{def}}{=} \{f(i) \mid i \in S\}$. Now, game v and value φ are functions, thus

- *Additivity* (the sum of payoffs in two separate games equals the payoff in a combined game): $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$ for all the games v_1, v_2, $(v_1 + v_2)(S) \stackrel{\text{def}}{=} v_1(S) + v_2(S)$;
- *Null-Player Axiom* (players that do not have an impact on the value of any coalition should get nothing): if $\forall_{S \subseteq N, i \in S} v(S \cup \{i\}) - v(S) = 0$ then $\varphi_i(v) = 0$ for every game v and player $i \in N$.

As a rationalization of his value, Shapley [46, p. 39] presented the following *bargaining process* (or bargaining procedure). Assume that players enter the grand coalition in a random order. As a player enters, he receives a payoff that equals his marginal contribution to the group of players that he joins (*i.e.*, $v(S \cup \{i\}) - v(S)$ when i joins coalition S). Now, the Shapley value is the expected outcome of player's contributions over all orders (permutations).

To formalize this definition, let us denote the set of all permutations of S by $\Omega(S)$. As is common in combinatorics, we identify permutation $\pi \in \Omega(S)$ with a corresponding ordering. Formally, $\pi : \{1, 2, \ldots, |S|\} \to S$. We will denote the set of agents that appear in permutation π after i by C_i^π, *i.e.*, $C_i^\pi \stackrel{\text{def}}{=} \{\pi(j) \mid j > \pi^{-1}(i)\}$. Now, the Shapley value can be calculated with the following formula:

$$SV_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Omega(N)} v(C_i^\pi \cup \{i\}) - v(C_i^\pi). \tag{1}$$

As we look closely on the bargaining process we can see that a given marginal contribution of player i to coalition S is counted multiple times: for every permutation in which players S comes before i and all others, i.e., $N \setminus (S \cup \{i\})$, after, player i will contribute $v(S \cup \{i\}) - v(S)$. This observation leads to the more concise form of the formula for Shapley value:

$$SV_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)). \tag{2}$$

We will use both formulas later.

$f(v)$ and $f(\varphi)$ are function compositions: $(f(v))(S) = v(f(S))$ and $f(\varphi_i) = \varphi_{f(i)}$. Intuitively, value of S in game $f(v)$ equals the value of a coalition obtained by replacing all players i from S with $f(i)$. For example, if f exchange 1 and 2, then $f(v)(\{1,3\}) = v(\{2,3\})$.

Publications

Almost all results included in this thesis have been published or being under submission and presented multiple times at the top international conferences.

- Chapters 1, 2, 4, 5, 6 are based on the paper *Reconsidering the Shapley Value in Games with Externalities* [51], which I presented at the International Conference on Game Theory at the The 24th Stony Brook Game Theory Festival in New York. The paper was invited for resubmission to Theoretical Economics.

- Chapter 3 is based on the paper *Steady Marginality: A Uniform Approach to Shapley Value for Games with Externalities* [48] which I presented at the Fourth Symposium on Algorithmic Game Theory (SAGT'11) and was published in the proceedings.

- Chapter 7 is based on the unpublished manuscript *The Shapley Axiomatization for Values in Partition Function Games* [50].

- Chapters 8, 9, 10 and 11 are based on the paper *Algorithms for the Myerson and Shapley Values in Graph-restricted Games* [49] which I presented at the 13th International Conference on Autonomous Agents and Multiagent Systems (AAMAS'13) and was published in the proceedings.

- In Chapter 12, the first part (Section 12.1) presents my contribution to the paper *Computational Analysis of Connectivity Games with Applications to Terrorist Networks* [34] published in the proceedings of Twenty-Third International Joint Conference on Artificial Intelligence (IJCAI'13).

- In Chapter 12, the second part (Section 12.2) presents my contribution to the paper *A Shapley value-based Approach to Determine Gatekeepers in Social Networks with Applications* [44] accepted for the Twenty-First European Conference on Artificial Intelligence (ECAI'14).

All theorems, lemmas and algorithms included in this thesis[2] have been obtained by myself.

[2] Only the algorithm from Section 12.1 was designed in a cooperation with Piotr Szczepański.

PART I

GAMES WITH EXTERNALITIES

Chapter 1

Introduction to Part I

Shapley studied environments when one cooperative arrangement does not impose any externalities on any other cooperative arrangements. However, such an assumption is clearly untenable in many practical economic situations of interest. For example, on the oligopolistic market, joint R&D projects increase the competitive edge of cooperating companies. Similarly, the extent of pollution reduction achieved by an international treaty depends not only on the signatories to the treaty, but also on similar agreements among non-participants. Extending the Shapley value to all such settings has been a subject of ongoing debate in the literature for more than fifty years. This issue is also the focus of Part I of our work.

A natural requirement for a fair division scheme is that it remunerates the players of a coalitional game based on their *contribution* to the surplus generated through cooperation. For example, in Shapley's axiomatization, the Null-Player Axiom requires that no payoff will be allocated to players that make zero contribution to any possible coalition in the game. The key issue, then, is how such a contribution should be measured.

In the context of cooperative games, the marginal contribution of a player to a coalition is the difference between the value of this coalition with and without the player. It can be also understood as a loss incurred by the remaining players should the player leave a given coalition. Considering this latter intuition, the Shapley value is defined as the average marginal contribution of a player, taken over all possible ways to dissolve the grand coalition by removing players one after the other in a queue (*i.e.* permutation) until the empty coalition is obtained. In any given permutation, the marginal contribution of a particular player is assigned *deterministically* as it does not play a role in what a player does after leaving a coalition. This is, however, not the case in games with externalities, where the definition of the marginal contribution becomes much more intricate.

When externalities are present, the value of the coalition that a player has left may be influenced by which coalition, if any, this player subsequently joins. In other words, the choice of a player's action *after it leaves a coalition* may result in different values of the player's marginal contribution to that coalition. One way to account for all such values is to assume that a player can choose to join different coalitions with different probabilities – we will denote the set of such probabilities (or weights) by α. Then, in games with externalities, the sequential dissolution of the grand coalition according to a given permutation of players can be viewed as a *stochastic* process, rather than a deterministic one. The marginal contribution of a player is then the difference between the value of the coalition with the player and the *expected* value of this coalition when the player has left.

In games with externalities, not only the definition of the marginal contribution but also the axiomatization of the value becomes more involved, and it can be easily shown that the standard translation of Shapley's axioms to games with externalities does not yield a unique value. A number of methods have been developed in the literature to address this issue. Some, such as [11] and [25], obtain uniqueness by modifying some of Shapley's original axioms. Other contributors add new axioms (and sometimes drop some of the original ones), moving increasingly further away from Shapley's original axiomatization. For instance, Grabisch and Funaki used Markovian and Ergodic Axioms and modified the Symmetry and the Null-Player Axioms [19]. Yet another method is to build extensions to games with externalities relying on alternative axiomatizations of the original Shapley value, such as Myerson's [38] balanced-contribution axiomatization or Young's [55] monotonicity axiomatization.

Now, in all of the above approaches, the payoff scheme can be defined in terms of marginal contributions parameterized by α. This naturally raises the question of the extent to which the α-weighting approach can be used to capture other values. Until now, the most general result in this spirit was obtained for the third method: Fujinaka [15] proved that, for any α, Young's monotonicity axiomatization, parametrized with α, guarantees a unique value. However, no such study for Shapley's original axiomatization exists in the literature.

Therefore, in our work, we focus on the first method, that is, we study how

Shapley's original axiomatization can be adapted to games with externalities using marginal contributions parametrized with α-weights. We will refer to this approach as the *marginality approach*.

We begin by proving that, *for every value of* α, Shapley's original axioms of Efficiency, Symmetry, Additivity and the α-parametrized Null-Player Axiom yield a unique extension of the Shapley value for games with externalities. We will refer to this value as the α-value. The results of [11, 25], focusing on two particular sets of weights α, can be considered as special cases of this general theorem. Furthermore, the theorem is a counterpart of Fujinaka's result for Young's axiomatization [15]. We then extend the analysis of the marginality approach as follows.

A fundamental question arising with respect to α-value is: which values – either among those already proposed in the literature or any new potential ones – can be defined as an α-value? A key result of our work is that we prove the marginality approach encompasses *all values* that satisfy Shapley's original axiomatization and exactly those.

Next, we analyze how properties of an α-value translate into properties of weights α. In particular, we focus on the axioms known as *Weak Monotonicity*, *Strong Monotonicity*, *Strong Symmetry*, and *Strong Null-Player*. Weak (Strong) Monotonicity states that, if we increase the value of a coalition containing a player, the payoff of this player will not decrease (will increase). We prove that α-value satisfies Weak (Strong) Monotonicity if and only if weights α are non-negative (positive). The Strong Symmetry axiom requires that the value of any coalition has a symmetric influence not only on the payoffs of its members but also on the payoffs of all non-members. We prove that the α-value satisfies Strong Symmetry if and only if weights α are such that the permutation in which players leave the grand coalition does not affect the probability that a given coalition structure is eventually created. We say that weights α satisfying this condition are *interlace resistant*. As a corollary to this result we have that the *average approach* to translating the Shapley value to games with externalities, proposed by Macho-Stadler *et al.* [29], is a subclass of the marginality approach and is equivalent to the marginality approach used with interlace resistant weights.

The Strong Null-Player Axiom states that a player who does not have an impact on the values of coalitions in the game does not affect the payoff division – that is, if we remove a null-player the payoffs from the game will stay the same. We prove that if α-value satisfies Strong Symmetry then it satisfies the Strong Null-Player Axiom if and only if weights α are such that the probability of joining a particular coalition depends only on the other coalitions in the coalition structure and not on the coalition that is being left. This condition on weights α we call *expansion resistance*.

Although the interlace and expansion resistance conditions may at first appear somewhat arbitrary, they are in fact key to understanding the relationship between the α-parameterized Shapley axiomatization and the Myerson axiomatization based on the concept of balanced contributions extended to games with externalities. In this respect, we prove that the α-value satisfies Myerson's axioms (Efficiency, α-parametrized Balanced Contributions) if and only if α is interlace and expansion resistant.

The remainder of Part I is organized as follows:

- In Chapter 2, we present the basic definitions and notation that we use throughout the paper and formally introduce the marginality approach.
- In Chapter 3, we present marginality approach on a case study: we propose a new definition of weights α and derive an existing value using marginality approach.

Next, our main results are presented in the following chapters:

- In Chapter 4, we prove that α-parametrized Shapley's axiomatization (Efficiency, Symmetry, Additivity and α-parametrized Null-Player Axiom) yields a unique value for every weighting α (called α-value). Furthermore, we show that every value which satisfies Shapley's original axiomatization can be derived using α-parametrized axiomatization with some weights α.
- In Chapter 5, we characterize what conditions must be met for value to satisfy Weak Monotonicity, Strong Monotonicity, Strong Symmetry or the Strong Null-Player Axiom. To this end, we introduce a notion of weights' interlace and expansion resistance. Then, we prove that Myerson's α-parametrized axiomatization is equivalent to Shapley's if and only if weights are interlace and expansion resistant.
- In Chapter 6, we analyze all the existing definitions of weights α in the context of properties analyzed in the previous chapter and discuss different axiomatization.
- In Chapter 7, we develop a general approximation algorithm to calculate α-value for any α, which is the first approximation algorithm to calculate any extension of Shapley value to games with externalities.

CHAPTER 2

MARGINALITY APPROACH

IN this chapter we introduce marginality approach to extending Shapley value to games with externalities. We start with basic definitions and our notation.

Note: This chapter is based on [51].

2.1 PRELIMINARIES

In the overview of our work we already introduced the set of players $N = \{1, 2, \ldots, n\}$ and the notion of a coalition (denoted S) as a subset of N. Now, a *partition* of players (denoted P) can be formalized as a partition of N, that is, a set of disjoint coalitions whose union is N. Here, for technical convenience, we will assume that every partition contains artificial, empty coalition, i.e., $\emptyset \in P$ for every partition P.[1] The number of non-empty coalitions in P is denoted $|P|$. Now, a pair (S, P), where P is a partition of N and $S \in P$, is called an *embedded coalition*. The set of all partitions and the set of all embedded coalitions over N are denoted by $\mathcal{P}(N)$ and $EC(N)$ (or simply, $\mathcal{P}$ and EC when the set of players is clear from the context).

In the basic definition of a coalitional game a value is assigned to every coalition. That way it can only model environments in which coalition have the same value no matter how other players are arranged, i.e., what coalitions they form. To model externalities we introduce *games in partition-function*

[1] This common assumption is convenient whenever we talk about a transfer of a player – creating a new coalition can be considered a transfer to the empty one.

form: here function v associates a real number with every embedded coalition, i.e., $v : EC \to \mathbb{R}$. In Part I, whenever we talk about game, we mean game in a partition function form.

In various parts of the paper we will make use of the class of simple games $\langle e^{(S,P)} \rangle_{(S,P) \in EC}$ where only one coalition in one partition has non-zero payoff:

$$e^{(S,P)}(S,P) = 1 \textit{ and } e^{(S,P)}(\tilde{S},\tilde{P}) = 0 \textit{ otherwise.}$$

We use a shorthand notation for set subtraction and set union operations: $N_{-S} \stackrel{\text{def}}{=} N \setminus S$ and $S_{+\{i\}} \stackrel{\text{def}}{=} S \cup \{i\}$. Often, we omit brackets for a singleton set and simply write S_{+i}. For partitions, if $P \in \mathcal{P}(N)$, then P_{-i} (or P_{-S}) is a partition of players over $N \setminus S$ (or $N \setminus \{i\}$). To denote the partition obtained by the transfer of player i to coalition T in partition P, we introduce the following notation:

$$\tau_i^T(P) \stackrel{\text{def}}{=} P \setminus \{P(i), T\} \cup \{P(i)_{-i}, T_{+i}\},$$

where $P(i)$ denotes i's coalition in P. In particular, $\tau_i^{\emptyset} \stackrel{\text{def}}{=} P_{-i} \cup \{i\}$. Finally, the partition obtained from P by the transfer of a group of players S to a new coalition will be denoted $P_{[S]}$: $P_{[S]} \stackrel{\text{def}}{=} \{T \setminus S \mid T \in P\} \cup \{S\}$.

We end this section by translating Shapley's axiom to the environment with the externalities. This set of axioms we call a *direct translation* of Shapley's axioms.

- *Efficiency* (the entire available payoff is distributed among players): $\sum_{i \in N} \varphi_i(v) = v(N, \{N, \emptyset\})$ for every game v;
- *Symmetry* (payoffs do not depend on the players' names): $\varphi(f(v)) = f(\varphi)(v)$ for every game v and every bijection $f : N \to N$;[2]
- *Additivity* (the sum of payoffs in two separate games equals the payoff in a combined game): $\varphi(\beta_1 v_1 + \beta_2 v_2) = \beta_1 \varphi(v_1) + \beta_2 \varphi(v_2)$ for all the games v_1, v_2 and scalars $\beta_1, \beta_2 \in \mathbb{R}$, $(v_1 + v_2)(S,P) \stackrel{\text{def}}{=} v_1(S,P) + v_2(S,P)$ and $(\beta v)(S,P) \stackrel{\text{def}}{=} \beta \cdot v(S,P)$;

[2]Please see the overview of the thesis for formal definition of $f(v)$ and $f(\varphi)$. Here, we specify the missing part: $f(P) \stackrel{\text{def}}{=} \{f(S) \mid S \in P\}$ and $f(S,P) \stackrel{\text{def}}{=} (f(S), f(P))$.

- *Null-Player Axiom* (players that do not have an impact on the value of any coalition should get nothing):
  if $\forall_{(S,P)\in EC, i\in S}\forall_{T\in P} v(S,P) - v(S_{-i}, \tau_i^T(P)) = 0$ then $\varphi_i(v) = 0$ for every game v and player $i \in N$.

Translation of Efficiency and Symmetry is straightforward. Translation of Additivity is consistent with [7, 29, 33]. In the original axiomatization, Shapley used a weaker version: $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$. In games without externalities, it was enough to imply that the value is linear; thus, the payoff division does not depend on the unit it is calculated with (*i.e.*, $\varphi(\beta v) = \beta\varphi(v)$). However, in games with externalities, the weaker version of Additivity combined with Shapley's other three axioms implies that the value can be scaled, but only by rational numbers (see [29] for details). While we are not aware of real-life applications in which irrational values of coalitions occur, for consistency with the literature, we allow irrational numbers in the function domain; thus, we strengthen Additivity by the linearity condition. However, we retain the name Additivity, as we feel this is a natural translation of this axiom to games with externalities. Finally, translation of the Null-Player Axiom is a strict definition that corresponds to the understanding that a null-player is not supposed to have any impact on the game. We will discuss stronger version of the Null-Player Axiom in the next section.

2.2 Parametrized axiomatization

In this section we introduce the marginality approach to extending the notion of the Shapley value to games with externalities. We begin by presenting the origins of this approach, traces of which can be already found in Bolger [7], and which have been then subsequently used by a few authors to develop their particular extensions [11, 25].

Let us start by considering the following example of a simple game with externalities, where it is easily visible that Shapley's original axiomatization does not imply a unique value.

Example 1. *Let $N = \{1,2,3\}$ be the set of players and the partition function v be defined as follows: $v(\{1,2\},\{\{1,2\},\{3\},\emptyset\}) = a$, $v(\{2\},\{\{1\},\{2\},\{3\},\emptyset\}) = b$, $v(\{2\},\{\{1,3\}, \{2\},\emptyset\}) = c$, and $v(S,P) = 0$ for all the remaining embedded coalitions. Thus, only coalitions $\{1,2\}$ and $\{2\}$ have non-zero value in this game.*

Now, let us consider a payoff $\varphi_1(v)$ of player 1 in this game: according to the extensions of the Shapley value to games with externalities by Pham do and Norde [11], McQuillin [33], Bolger [7], Macho-Stadler et al. [29], and Hu and Yang [25], respectively, equals:

$$\varphi_1(v) = \begin{cases} \frac{1}{6}(a-b) & \textit{in the case of Pham do and Norde} \\ \frac{1}{6}(a-c) & \textit{in the case of McQuillin} \\ \frac{1}{6}(a-\frac{b+c}{2}) & \textit{in the case of Bolger and Macho-Stadler et al.} \\ \frac{1}{6}(a-\frac{3b+2c}{5}) & \textit{in the case of Hu and Yang} \end{cases}$$

As can be seen, although each of these extensions satisfies the direct translation of all four original axioms to games with externalities,[3] they yield very different payoffs.

The main challenge in constructing a value for games with externalities comes from the fact that it is not straightforward to evaluate the role played by particular players in a setting where evaluating coalitions can be ambiguous, *i.e.*, where embedded coalitions may have different values depending on the partition they are embedded within. All extensions in the literature, including the ones in Example 1, are, in fact, methods to address this problem.

The marginality approach aims to extend the notion of the Shapley value with an axiomatization which is as close to the original one as possible. Whereas, as we have seen in Section 2.1, the translation of Efficiency, Symmetry and Additivity to games with externalities is straightforward, this is not entirely truth with the Null-Player Axiom. In the direct (or strict) translation of this axiom, a player is called a null-player if he never has any effect on the value of any coalition. It means that all his transfers outside a given coalition should not change the value of this coalition.

Example 2. *In Example 1, player 1 is a null-player in a strict sense if $a = b = c$. This is because the marginal contribution of player 1 to $\{1,2\}$ in $\{\{1,2\},\{3\},\emptyset\}$ considering its transfer to coalition $\{3\}$ is equal to $v(\{1,2\},\{\{1,2\},\{3\},\emptyset\}) - v(\{1\},\{\{1\},\{2,3\},\emptyset\}) = a - c = 0$, under the assumption that $a = c$. Analogously, creating a new coalition (i.e., transferring to an empty one) yields no contribution. Furthermore, it is not difficult to observe that this holds for any other marginal contribution associated with a transfer of player 1 within any partition as payoffs before and after such a transfer equal zero.*

[3] See below for discussion on the Null-Player Axiom.

We will call marginal contributions associated with a given transfer within a partition (such as those considered in Example 2) *elementary marginal contributions.*

Now, the marginality approach is based on the more general view of the contribution: given a partition, although particular transfers may change the value of the embedded coalition, the *overall* marginal contribution in this partition may still equal zero. For instance, let us set $b = a + r$ and $c = a - r$ in Example 1. Despite the fact that elementary marginal contributions are non-zero, the overall marginal contribution is zero, if we assume that both transfers are evaluated equally. It is, then, a less strict translation of the Null-Player Axiom in which we require that the overall marginal contribution is zero but not necessarily elementary marginal contributions.

From the above analysis it follows that to specify the marginal contribution in the marginality approach, one needs to indicate which transfers are considered and in which proportion. Although, in theory, any function of elementary marginal contributions is admissible, we restrict ourselves only to affine[4] combinations of elementary marginal contributions. This assumption is justified, as we show that these weights are enough to obtain every value that meets Shapley's axioms (see Theorem 4 for details). Formally,

$$[mc_i^{\alpha}(v)](S,P) \stackrel{\text{def}}{=} \sum_{T \in P_{-S}} \alpha_i(S_{-i}, \tau_i^T(P))[v(S,P) - v(S_{-i}, \tau_i^T(P))],$$

where $\alpha_i : \{(S,P) \in EC \mid i \notin S\} \to \mathbb{R}$ denotes weights of a given transfer under the assumptions that:

(a) $\alpha_i(S,P) = \alpha_{f(i)}(f(S), f(P))$ for every bijection $f : N \to N$ and $(S,P) \in EC$ such that $i \notin S$ (to satisfy Symmetry); and

(b) $\sum_{T \in P_{-S}} \alpha_i(S_{-i}, \tau_i^T(P)) = 1$ for every $(S,P) \in EC$ such that $i \in S$ (for normalization).

Note that $\alpha_i(S,P)$ is the weight associated with i's transfer from coalition S_{+i} that results in partition P. For example, the weight of transfer of player 1 from $(\{1,2\}, \{\{1,2\},\{3\},\emptyset\})$ to $\{3\}$ is denoted by $\alpha_1(\{2\}, \{\{2\},\{1,3\},\emptyset\})$. The definition of marginal contribution based on the weights α will be called

[4]Linear combination $\sum \alpha_i x_i$ is called *affine* when weights sum up to one: $\sum \alpha_i = 1$.

α-marginality.[5]

Now, we define the α-parametrized version of the Null-Player Axiom as follows:

> *α-Null-Player Axiom* (players that do not contribute to the value of any coalition should get nothing): if $\forall_{(S,P)\in EC, i\in S}[mc_i^\alpha(v)](S,P) = 0$ then $\varphi_i(v) = 0$ for every game v and player $i \in N$.

As we will see in Chapter 4, Shapley's α-parametrized axiomatization, that is, Efficiency, Symmetry, Additivity, and α-Null-Player Axiom, is enough to obtain uniqueness for every α. Defining an extension of the Shapley value to games with externalities with such strengthening constitutes the *marginality approach*.

We end this chapter with an example of weights. In the simplest and chronologically the first definition by Bolger [7], all transfers are considered equally important:

$$[mc_i^{\alpha^B}(v)](S,P) \stackrel{\text{def}}{=} \sum_{T\in P_{-S}} \frac{1}{|P_{-i}|}(v(S,P) - v(S_{-i}, \tau_i^T(P))).$$

However, this equality has been questioned, as a change caused by forming a new coalition can be argued a better or worse assessment of the contribution of a player than transfer to other existing coalition (see the next chapter for details). Moreover, transfer to a bigger coalition can be considered *more likely* than a the small one. Because of that, including steady-marginality from Chapter 3, four other weightings α have been proposed in the literature [11, 29, 25]. We will discuss all of them in Chapter 6.

[5] For the generality of the approach, we allow negative weights. However, we show in Section 5.2 that negative weights produce non-monotonic values; thus, we believe that they should not be used. Non-negative weights have a natural interpretation as the probability of transfer and we will refer to this intuition freely.

Chapter 3

Steady Marginality: A Case Study

Two important extensions of the Shapley value to games with externalities are *externality-free* value proposed by Pham Do and Norde [11] and McQuillin's value *full-of-externalities* [33]. Both can be considered as reference points for other extensions, as, under certain conditions, they limit the space of possible extensions at two opposite extremes.

While the externality-free value has been proposed using the marginality approach, McQuillin to derive his value added three non-marginality based axioms. The key of them was recursion that required the value to be a fixed-point solution (i.e. if we consider a value to be a game by itself, then the value computed for such a game should not change). Nevertheless, marginality-based axiomatization of the McQuillin value that connects to the original Shapley value has remained unknown.

In this chapter we close this gap by proposing a new definition of a marginal contribution; we present new weighting α that allows us to derive the extension of the Shapley value proposed by McQuillin. Our weights are dual to the one proposed by Pham Do and Norde and recently, in an important publication, considered by De Clippel and Serrano. In other words, we close the picture, so that the two opposite values for games with externalities that limit the space of many other extensions, are now based on the marginality principle.

Our new approach to marginality, which we call a *steady marginality*, differs from those proposed earlier in the literature ([7, 11, 25]). To compute player's marginal contribution to a coalition we compare the value of the coalition with the specific player with the value of the coalition obtained by the transfer of the player to another coalition, existing in the partition (so the number of coalitions is *steady*). Thus, we do not include the value of a coalition in a partition when a specific player forms a new singleton coalition.

The rest of this chapter is organized as follows. In Section 3.1, we present *externality-free* value and *full-of-externalities* value. In Section 3.2, we introduce our new definition of marginality. In Section 3.3, we define a new class of games and prove they form a basis of space of games with externalities. In Section 3.4, we present the main result of this chapter – we prove that there exists only one value which satisfies all our axioms and that this value is equal to one proposed earlier by McQuillin. A comparison of all existing definitions of marginality, including the steady marginality, will be the subject of Chapter 6.

Note: This chapter is based on [48].

3.1 Two values

In this section we introduce *externality-free* and *full-of-externalities* values. Both values can be obtained using a minor modification of Shapley's formula (2). In this formula, we iterate over all coalitions and for each one we add/remove its value multiplied by some weight. This technique cannot be directly applied to games with externalities, as a value of a coalition depends on the partition of outside players. Both values overcome this problem by taking the value of a coalition from only one partition. Thus, they do not take into consideration all information about the players' performance provided by the game.

In the *externality-free* value, proposed by Pham Do and Norde, the value of a coalition is taken from the partition in which all other players are alone, hence no externalities from merging coalitions affect it:

$$\hat{v}^{free}(S) \stackrel{\text{def}}{=} v(S, \{S\} \cup \{\{j\} \mid j \in N \setminus S\}).$$

The value takes the form:

$$\varphi_i^{free}(v) \stackrel{\text{def}}{=} SV_i(\hat{v}^{free}) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(\hat{v}^{free}(S \cup \{i\}) - \hat{v}^{free}(S)).$$

The second value, proposed by McQuillin, called by us *full-of-externalities*, is dual to the externality-free one. Here, the value of S is taken from the partition, in which all other players are in one coalition:

$$\hat{v}^{McQ}(S) \stackrel{\text{def}}{=} v(S, \{S, N \setminus S, \emptyset\}).$$

Thus, the value of S is affected by all externalities from merging coalitions. Now, McQuillin value takes the form:

$$\varphi_i^{McQ}(v) \stackrel{\text{def}}{=} SV_i(\hat{v}^{McQ}) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (\hat{v}^{McQ}(S \cup \{i\}) - \hat{v}^{McQ}(S)).$$

In a more general approach the value of a coalition S can be a weighted average over all possible values that coalition obtains in different partitions. This approach was formalized by Macho-Stadler *et al.* as an *average approach* [29]. We will describe it in Section 5.3. Both values described in this section are the borderline cases in this approach.

3.2 Steady marginality

In this section, we introduce our new definition of marginality.

We start by presenting the approach that leads to the externality-free value. Pham Do and Norde [11] used only one non-zero weight for the transfer of i to the empty coalition: $\alpha_i^{free}(S, P) = 1$ if $\{i\} \in P$ and $\alpha_i^{free}(S, P) = 0$ otherwise. Their definition of the marginal contribution takes the form:

$$[mc_i^{\alpha^{free}}(v)](S, P) \stackrel{\text{def}}{=} v(S, P) - v(S_{-i}, P \setminus S \cup \{S_{-i}, \{i\}\}).$$

De Clippel and Serrano [10] justified this approach by treating the transfer as a two-step process. In the first step, player i leaves the coalition S and for a moment remains alone (i.e., creates a singleton coalition). An optional second step consists of player i joining some coalition from $P \setminus S$ (in terms of coalitions, $\{i\}$ merges with another one). Although both steps may change the value of S_{-i}, the authors argue that only the first one corresponds to the *intrinsic* marginal contribution – the influence from the second step comes rather from the external effect of merging coalitions, not from i leaving S. Discarding the impact of merging coalitions in marginal contribution leads to the *externality-free* value.

We will consider the transfer of i in a different way. Our first step will consist of leaving coalition S and joining one of the other coalitions in partition. In the second step, player i can exit his new coalition and create his own. Thus, we look at the *creating of a new coalition* as an extra action, which should not be included in the effect of i leaving coalition S. According to this, the natural way to define the *steady marginal contribution* of player i to (S, P), is to take into account only the transfer to the other existing coalition.

Definition 1. *The* steady marginal contribution *of player* $i \in S$ *to embedded coalition* $(S, P) \in EC$ *such that* $S \neq N$ *is defined as:*

$$[mc_i^{\alpha^{full}}(v)](S, P) \stackrel{\text{def}}{=} \sum_{\substack{T \in P_{-S} \\ T \neq \emptyset}} \frac{1}{|P| - 1}(v(S, P) - v(S_{-i}, \tau_i^T(P)))$$

and $[mc_i^{\alpha^{full}}(v)](N, \{N, \emptyset\}) \stackrel{\text{def}}{=} v(N, \{N, \emptyset\}) - v(N_{-i}, \{N_{-i}, \{i\}, \emptyset\})$.

In our new definition presented above the special case with $S = N$ takes place when creating a new coalition is the only option, i.e., no other coalition exists. To diminish the importance of this special case let us introduce the modified version of the size operator of the partition: $|P|_\emptyset \stackrel{\text{def}}{=} |P|$ if $|P| \geq 2$ and $|\{N, \emptyset\}|_\emptyset \stackrel{\text{def}}{=} 2$. Thus, the only difference is that we treat partition $\{N, \emptyset\}$ as a two-coalition partition (usually we count only non-empty coalitions and $|\{N, \emptyset\}| = 1$). That way we can include the special case of $\{N, \emptyset\}$ in a concise definition: in the steady marginality we take into account only transfers that do not modify $|P|_\emptyset$ of the partition.

Our approach can be justified by these real life examples in which creation of a new coalition is rare and not likely. These include political parties or million-dollar industries (such as oil oligopoly). In all such situations, our approach is likely to lead to more proper results.

3.3 Constant-Coalition Games

In this section we will introduce a new class of simple games – *constant-coalition games*. These games will play a key role in the next section where we prove the uniqueness of the α^{full}-parametrized axiomatization. The name comes from the fact that in a given game every partition in which a coalition with non-zero value is embedded has exactly the same number of coalitions. We show that the collection of such games is a basis of partition function games.

First, we need some additional notation.

Definition 2. *($R_1 \preceq R_2$) Let R_1, R_2 be two proper, non-empty subsets of two partitions. We say that R_2* can be reduced *to R_1 (denoted $R_1 \preceq R_2$) if three conditions are met:*

(a) all players which appear in R_1, appear in R_2;

(b) two players which are in the same coalition in R_1, are in the same coalition in R_2;

(c) two players which are not in the same coalition in R_1 are not in the same coalition in R_2.

Assume $R_1 \preceq R_2$. Based on the presented conditions, as we delete players from R_2 which are not in R_1 we get exactly the R_1 configuration. This observation can be expressed in an alternative definition of the $\preceq$-operator:

$$R_1 \preceq R_2 \Leftrightarrow R_1 \cup \{T\} = (R_2)_{[T]} \quad \textit{for} \quad T = \bigcup_{T_1 \in R_1} T_1 \setminus \bigcup_{T_2 \in R_2} T_2.$$

For example $\{\{1,2\},\{3\}\} \preceq \{\{1,2,4\},\{3\},\{5\}\}$ but $\{\{1,2\},\{3\}\} \not\preceq \{\{1,2,3\}\}$ and $\{\{1,2\},\{3\}\} \not\preceq \{\{1\},\{3,4\}\}$.

Now we can introduce a new basis for games with externalities.

Definition 3. *For every embedded coalition (S,P), the* constant-coalition game *$c^{(S,P)}$ is defined by*

$$c^{(S,P)}(\tilde{S},\tilde{P}) \stackrel{\text{def}}{=} \begin{cases} (|P|_\emptyset - 1)^{-|\tilde{S}\setminus S|} & \textit{if } (\tilde{P}_{-\tilde{S}} \preceq P_{-S}) \textit{ and } (|P|_\emptyset = |\tilde{P}|_\emptyset), \\ 0 & \textit{otherwise,} \end{cases}$$

for every $(\tilde{S},\tilde{P}) \in EC$.

Note that $(\tilde{P}_{-\tilde{S}} \preceq P_{-S})$ implies $S \subseteq \tilde{S}$ as we get $\tilde{N} \setminus \tilde{S} \subseteq N \setminus S$ from the (a) condition in Definition 2. Thus, in game $c^{(S,P)}$ non-zero values have only embedded coalitions formed from (S,P) by some transition of players from $P \setminus \{S\}$ to S which does not change the number of the coalitions (in a $|\cdot|_\emptyset$ operator sense).

Lemma 1. *The collection of constant-coalition games is a basis of the partition function games.*

Proof. Let $c = (c^{(S,P)})_{(S,P)\in EC}$ be the vector of all games.

First, we show that the constant-coalition games are linearly independent. Suppose the contrary. Then, there exists a vector of weights $\lambda = (\lambda_{(S,P)})_{(S,P)\in EC}$ with at least one non-zero value such that $\lambda \times c = \sum_{(S,P)\in EC} \lambda_{(S,P)} c^{(S,P)}$ is a zero vector. Let (S^*, P^*) be the embedded coalition with a non-zero weight $\lambda_{(S^*,P^*)} \neq 0$ and minimal S^* (i.e. (S^*, P^*) is the minimal element of the embedded-coalition relation r: $(S_1, P_1)r(S_2, P_2) \Leftrightarrow S_1 \subseteq S_2$). Thus, for any other embedded coalition (S, P) either $\lambda_{(S,P)} = 0$ or $S \nsubseteq S^* \Rightarrow c^{(S,P)}(S^*, P^*) = 0$ (this implication follows from the remarks after Definition 3). Then,

$$\sum_{(S,P)\in EC} \lambda_{(S,P)} c^{(S,P)}(S^*, P^*) = \lambda_{(S^*,P^*)} c^{(S^*,P^*)}(S^*, P^*) = \lambda_{(S^*,P^*)} \neq 0,$$

contradicts our previous assumption.

The size of a collection of all constant-coalition games is equal to the dimension of the space of partition function games. Thus, a class of constant-coalition games is a basis. □

3.4 Uniqueness of the α^{full}-value

In this section we show that there exists only one value that satisfies α^{full}-parametrized Shapley's axiomatization and that it is equivalent to the value proposed by McQuillin.

Theorem 1. *There is a unique value $\varphi^{\alpha^{full}}$ satisfying Efficiency, Symmetry, Additivity and the α^{full}-Null-Player Axiom.*

Proof. We will prove that in every constant-coalition game there exists only one value which satisfies these axioms. Based on Additivity and Lemma 1 this implies our thesis.

Let $d \cdot c^{(S,P)}$ be a constant-coalition game multiplied by scalar d. We will show that any player i from the coalition different than S is an α^{full}-null-player. Based on the definition we have to prove that $mc_i^{\alpha^{full}}(d \cdot c^{(S,P)})$ is a zero vector. To this end, let us transform formula for steady marginal contribution using $|\cdot|_\emptyset$ notation:

$$[mc_i^{\alpha^{full}}(v)](S,P) \stackrel{\text{def}}{=} \sum_{\substack{T\in P_{-S} \\ |\tau_i^T(P)|_\emptyset = |P|_\emptyset}} \frac{1}{|P|_\emptyset - 1}(v(S,P) - v(S_{-i}, \tau_i^T(P))) \tag{3.1}$$

Thus, for every $(\tilde{S},\tilde{P}) \in EC$ such that $i \in \tilde{S}$ we want to show that $mc_i^{\alpha^{full}}(d \cdot c^{(\tilde{S},\tilde{P})}) = 0$ which, based on formula (3.1), is equivalent to

$$\sum_{\substack{T\in\tilde{P}_{-\tilde{S}},\\ |\tau_i^T(\tilde{P})|_\emptyset=|\tilde{P}|_\emptyset}} c^{(S,P)}(\tilde{S},\tilde{P}) - c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P})) = 0.$$

We divide the proof into two cases with zero and non-zero value of $c^{(S,P)}(\tilde{S},\tilde{P})$.

Lemma 2. *If* $c^{(S,P)}(\tilde{S},\tilde{P}) = 0$ *then, for every* $T \in \tilde{P} \setminus \{\tilde{S}\}$*,*

$$c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P})) = 0.$$

Proof. Based on the definition of the constant-coalition games we can deduce that at least one of the following conditions occurs:

- $\tilde{P}_{-\tilde{S}} \not\preceq P_{-S}$: from the definition of $\preceq$-operator we know that there is a player in $\tilde{P}_{-\tilde{S}}$ which is not in P_{-S}, or there is a pair of players which are together in one coalition structure and separated in the other one; it is easy to see, that adding player i to some coalition in $\tilde{P}_{-\tilde{S}}$ will not fix any of these anomalies;

- $|P|_\emptyset \neq |\tilde{P}|_\emptyset$: if i is alone ($\tilde{S} = \{i\}$) then, for every $T \in \tilde{P} \setminus \{\tilde{S}\}$,

  $$c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P})) = c^{(S,P)}(\emptyset,\tau_i^T(\tilde{P})) = 0;$$

  otherwise, as we only consider the transfer of player i to the other existing coalition, the number of the coalitions remains intact: $|P|_\emptyset \neq |\tau_i^T(\tilde{P})|_\emptyset$. □

Lemma 3. *If* $c^{(S,P)}(\tilde{S},\tilde{P}) = x$ *for* $x > 0$ *then there exists only one* $T \in \tilde{P} \setminus \{\tilde{S}\}$ *such that* $c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P}))$ *has non-zero value. Moreover, this value is equal to* $x(|P|-1)$.

Proof. If $c^{(S,P)}(\tilde{S},\tilde{P}) > 0$ and $c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P})) > 0$ then from the definition:

$$\begin{aligned} c^{(S,P)}(\tilde{S}_{-i},\tau_i^T(\tilde{P})) &= (|P|-1)^{-|\tilde{S}_{-i}\setminus S|} = (|P|-1)\cdot(|P|-1)^{-|\tilde{S}\setminus S|} \\ &= (|P|-1)\cdot c^{(S,P)}(\tilde{S},\tilde{P}) \end{aligned}$$

which proves the values equality part.

First, we will consider a special case when transfer to the empty coalition is allowed: if $\tilde{P} = \{N,\emptyset\}$, then $(\tilde{S},\tau_i^T(\tilde{P}))$ equals $(N_{-i},\{N_{-i},\{i\}\})$. As we assumed $i \notin S$, we know that $\{i\} \preceq P_{-S}$ and we are not changing the partition size ($|\{N, \emptyset\}|_\emptyset = |\{N_{-i}, \{i\}\}|_\emptyset$) it follows that $c^{(S,P)}(N_{-i}, \{N_{-i}, \{i\}\}) > 0$ which finishes this case.

Now, assume that $\tilde{P} \neq \{N, \emptyset\}$. Let $T_i \in P \setminus S$ be the coalition of player i. From the definition of constant-coalition games (Definition 3) we know that P_{-S} can be reduced to $\tilde{P}_{-\tilde{S}}$ and both partitions have the same number of the coalitions: $|P_{-S}| = |\tilde{P}_{-\tilde{S}}|$. As players from one coalition cannot be separated, there must be some non-empty coalition $\tilde{T}_i$ in $\tilde{P}_{-\tilde{S}}$ which can be reduced from T_i by deleting players from $\tilde{S}$. Moreover, it must contain at least one player j (and $j \neq i$, because $i \in \tilde{S}$). Thus, a transition to any other coalition than $\tilde{T}_i$ separates players i and j which violates condition (c) in definition of $\preceq$-operator (Definition 2) and implies zero value in game $c^{(S,P)}$. In $\tau_i^{\tilde{T}_i}(P)$ all conditions will be satisfied – (a) is obviously satisfied as $i \notin S$ and $\tilde{P}_{-\tilde{S}}$ was already a subset of P_{-S}; conditions (b) and (c) are satisfied because additional player i have analogical relations as j, who is already in the structure.

Again, we do not change the size of the partition. We have to check only one special case when $\tilde{S} = \{i\}$. But from $c^{(S,P)}(\tilde{S}, \tilde{P}) > 0$ we get $S \subseteq \tilde{S}$ and as we know that $i \notin S$ we get $S = \emptyset$ which means that the game $c^{(S,P)}$ is incorrect.

Thus, finally: $c^{(S,P)}(\tilde{S}_{-i}, \tau_i^{\tilde{T}_i}(\tilde{P})) > 0$. □

From Lemma 2 and Lemma 3 we have that every player $i \notin S$ is an α^{full}-null-player. Based on the α^{full}-Null-Player Axiom, $\varphi_i^{full}(d \cdot c^{(S,P)}) = 0$ and based on Symmetry and Efficiency we get:

$$\varphi_j^{\alpha^{full}}(d \cdot c^{(S,P)}) = \frac{1}{|S|} \cdot \sum_{j \in S} \varphi_j^{\alpha^{full}}(d \cdot c^{(S,P)}) = \frac{d}{|S|} \cdot c^{(S,P)}(N, \{N, \emptyset\}).$$

As our value $\varphi^{\alpha^{full}}$ clearly satisfies Efficiency, Additivity and Symmetry, the only observation we need to add is that it also satisfies the α^{full}-Null-Player Axiom. As players not from S are α^{full}-null-players and get nothing it is sufficient to show that players from S are not α^{full}-null-players. But every player j from S has a non-zero marginal contribution to (S, P): $c^{(S,P)}(S, P) = 1$ and $c^{(S,P)}(S_{-j}, \tau_j^T(P)) = 0$ for every $T \in P \setminus S$. This concludes the proof of Theorem 1. □

We end this chapter by showing that our unique value is indeed equal to the value proposed by McQuillin.

Theorem 2. *Let v be a game with externalities. Then $\varphi^{McQ}(v) = \varphi^{\alpha^{full}}(v)$.*

Proof. Again, based on Additivity and linear property $\varphi(c \cdot v) = c \cdot \varphi(v)$ of both values, we will show the adequacy on the constant-coalition games. In the proof of Theorem 1 we showed that $\varphi_i^{\alpha^{full}}(c^{(S,P)}) = 0$ for every $i \notin S$ and $\varphi_j^{\alpha^{full}}(c^{(S,P)}) = \frac{1}{|S|} \cdot c^{(S,P)}(N, \{N, \emptyset\})$ for every $j \in S$.

Let (S, P) be an embedded coalition. Now, assume that $|P|_\emptyset > 2$. As $|\{N, \emptyset\}|_\emptyset = 2 \neq |P|_\emptyset$, based on the definition of the constant-coalition games (Definition 3) we get $c^{(S,P)}(N, \{N, \emptyset\}) = 0$ and $\varphi_i^{\alpha^{full}}(c^{(S,P)}) = 0$ for every player $i \in N$. Also $\varphi_i^{McQ}(c^{(S,P)}) = 0$ for every player $i \in N$, because $\hat{v}^{McQ}(S) = 0$ for every $S \subseteq N$ as no embedded coalition of form $(\tilde{S}, \{\tilde{S}, N \setminus \tilde{S}\})$ has a non-zero value in $c^{(S,P)}$ (the reason here is the same – the partitions sizes do not match).

If $|P|_\emptyset = 2$, then P has the form $\{S, N \setminus S\}$ (and in the borderline case $P = \{N, \emptyset\}$) and game $c^{(S,\{S,N \setminus S\})}$ assigns a non-zero value (equal 1) only to an embedded coalition $(\tilde{S}, \{\tilde{S}, N \setminus \tilde{S}\})$ such that $S \subseteq \tilde{S}$. Hence, $\hat{v}^{McQ}(\tilde{S}) = 1$ when $S \subseteq \tilde{S}$ and $\hat{v}(\tilde{S}) = 0$ otherwise. Based on the basic Shapley's axioms for $\hat{v}^{McQ}$ we get that $\varphi_i^{McQ}(c^{(S,P)}) = 0$ for $i \notin S$ and $\varphi_j^{McQ}(e^{(S,P)}) = Sh_j(\hat{v}) = \frac{1}{|S|}$ for $j \in S$.

Finally, we check whether value $\varphi^{\alpha^{full}}$ gives the same results. As mentioned at the beginning of the proof, for $i \notin S$, $\varphi_i^{\alpha^{full}}(c^{(S,P)}) = 0$ and for $j \in S$ holds $\varphi_j^{\alpha^{full}}(c^{(S,P)}) = \frac{1}{|S|} \cdot c^{(S,P)}(N, \{N, \emptyset\}) = \frac{1}{|S|}$. This concludes the proof. □

Chapter 4

General Theorems

In this chapter we provide our two main results concerning the marginality approach. Theorem 3 states that Shapley's α-parametrized axiomatization, that is, Efficiency, Symmetry, Additivity, and α-Null-Player Axiom, is enough to obtain uniqueness for every α and provides a formula for the value. In Theorem 4, that follows, we show that all values that satisfy standard Shapley's axioms can be derived using marginality approach. Finally, we present the α-parametrized bargaining process that would produce the corresponding value as the expected outcome.

Note: This chapter is based on [51].

4.1 Uniqueness of the marginality approach

Before we proceed, let us introduce the notion of composition of weights $pr^{\alpha}_{\pi}(S,P)$. For all permutations $\pi \in \Omega(N \setminus S)$, let us define:

$$pr^{\alpha}_{\pi}(S,P) \stackrel{\text{def}}{=} \prod_{i \in N_{-S}} \alpha_i(S \cup C^{\pi}_i, P_{[S \cup C^{\pi}_i]}).$$

Thus, if players $N \setminus S$ leave the grand coalition in order π and form partition $P \setminus S$, then $pr^{\alpha}_{\pi}(S,P)$ is the product of weights associated with these transfers. Recursively, if i is the last element of π: $\pi = \pi' || i$, then $pr^{\alpha}_{\pi}(S,P) \stackrel{\text{def}}{=} pr^{\alpha}_{\pi'}(S_{+i}, \tau^{S}_{i}(P)) \cdot \alpha_i(S,P)$ with a borderline case $pr^{\alpha}_{\pi}(\{N\},\{N,\emptyset\}) = 1$.

For example, for permutation $\pi = (1,2,3)$ and coalition $\{4\}$ embedded in partition $\{\{1,2\},\{3\},\{4\}\}$, we have:

$$pr^{\alpha}_{\pi}(\{4\},\{\{4\},\{1,2\},\{3\},\emptyset\}) = \alpha_1(\{2,3,4\},\{\{2,3,4\},\{1\},\emptyset\}) \cdot \\ \cdot\alpha_2(\{3,4\},\{\{3,4\},\{1,2\},\emptyset\}) \cdot \alpha_3(\{4\},\{\{4\},\{1,2\},\{3\},\emptyset\}).$$

Theorem 3. *There exists a unique value (denoted α-value) that satisfies Efficiency, Symmetry, Additivity and the α-Null-Player Axiom for every α.*[1] *Moreover, it satisfies the following formula:*

$$\varphi^{\alpha}_i(v) \stackrel{\text{def}}{=} \frac{1}{|N|!}\sum_{\pi\in\Omega(N)}\sum_{P\in\mathcal{P}} pr^{\alpha}_{\pi}(\emptyset,P)\cdot[v(C^{\pi}_i\cup\{i\},P_{[C^{\pi}_i\cup\{i\}]})-v(C^{\pi}_i,P_{[C^{\pi}_i]})]. \quad (4.1)$$

Proof. First we prove that φ^{α} satisfies all four axioms. Then, we show that this is the only such value.

Part 1: We examine the axioms one by one. First, let us consider Efficiency. For any permutation π and partition P, the elementary marginal contributions add up to $v(N,\{N,\emptyset\})$; thus:

$$\sum_{i\in N}\varphi^{\alpha}_i(v) = \frac{1}{|N|!}\sum_{\pi\in\Omega(N)}\sum_{P\in\mathcal{P}} pr^{\alpha}_{\pi}(\emptyset,P)\sum_{i\in N}[v(C^{\pi}_i\cup\{i\},P_{[C^{\pi}_i\cup\{i\}]})-v(C^{\pi}_i,P_{[C^{\pi}_i]})]$$
$$= \frac{1}{|N|!}\sum_{\pi\in\Omega(N)}\sum_{P\in\mathcal{P}} pr^{\alpha}_{\pi}(\emptyset,P)\cdot v(N,\{N,\emptyset\}) = v(N,\{N,\emptyset\}),$$

where the last transformation comes from the fact that weights of all partitions sum up to one for every permutation: $\sum_{P\in\mathcal{P}} pr^{\alpha}_{\pi}(\emptyset,P) = 1$.

Formula (4.1) clearly shows that the value satisfies Symmetry and Additivity. Regarding Symmetry, it does not favor any player, hence a permutation of coalitions' values will permute payoffs accordingly. The value is additive as $\varphi^{\alpha}_i(v_1+v_2)$ can be split into two expressions representing $\varphi^{\alpha}_i(v_1)$ and $\varphi^{\alpha}_i(v_2)$.

To see that φ^{α} satisfies the α-Null-Player Axiom let us calculate the weight of a given elementary marginal contribution $v(S,P) - v(S_{-i},\tau^T_i(P))$ using formula (4.1). A transfer from (S,P) occurs only in permutations where players from $N \setminus S$ (and only them) appear before player i. Let us assume that they appear in order $\pi \in \Omega(N \setminus S)$. Then, regardless of the rest of the permutation, the

[1] Although we assumed a stronger definition of Additivity, our proof is based only on the weaker version that does not force linearity: $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$. Consequently, the theorem also holds for the weaker version.

elementary marginal contribution under consideration is multiplied by product of weights $pr^{\alpha}_{\pi}(S,P)$. The transfer to coalition T causes this product to be multiplied by $\alpha_i(S_{-i},\tau^T_i(P))$. Finally, we observe that the permutation and the arrangement of the remaining players does not have an impact on the value (there are $(|S|-1)!$ such permutations). This is because, for a given permutation, the sum of products of remaining transfers over all possible partitions sum up to one. Now, if we collect all transfers from a given embedded coalition (S,P) we get the following formula:

$$\varphi^{\alpha}_i(v) = \sum_{(S,P)\in EC, i\in S} \frac{(|S|-1)!}{|N|!} \sum_{\pi\in\Omega(N_{-S})} pr^{\alpha}_{\pi}(S,P)\cdot[mc^{\alpha}_i(v)](S,P). \qquad (4.2)$$

Part 2: Next, we will show that φ^{α} is the only value which satisfies all four Shapley's original axioms. To this end, let us recall the class of simple games $e^{(S,P)}$. This class forms the basis of the game space, *i.e.*, every game can be defined as a linear combination of games $e^{(S,P)}$: $v = \sum_{(S,P)\in EC} v(S,P)\cdot e^{(S,P)}$. Based on Additivity, we have $\varphi(v) = \sum_{(S,P)\in EC} \varphi(v(S,P)\cdot e^{(S,P)})$; thus, it is enough to prove that the axioms imply a unique value in simple game $e^{(S,P)}$ (multiplied by a scalar). For this purpose, we will use the reverse induction on the size of S: we will show that the value of game $e^{(S,P)}$ can be calculated from the values of simple games for bigger coalitions: $e^{(\tilde{S},\tilde{P})}$ where $|\tilde{S}| > |S|$. Our base case when $|S| = |N|$ comes from the Efficiency and Symmetry: $\varphi_i(c\cdot e^{(N,\{N,\emptyset\})}) = \frac{c}{|N|}$ for every i.

First, let (S,P) be any embedded coalition and assume that $i \notin S$. Let us consider game $\tilde{v}$ combined from two simple games:

$$\tilde{v} = c\cdot[\alpha_i(S,P)\cdot e^{(S_{+i},\tau^S_i(P))} + e^{(S,P)}].$$

It is easy to observe that player i's marginal contribution to $(S_{+i},\tau^S_i(P))$ equals zero, as with all other marginal contributions. Thus, from the Null-Player Axiom $\varphi_i(\tilde{v}) = 0$ and from Additivity:

$$\varphi_i(c\cdot e^{(S,P)}) = -\varphi_i(c\cdot\alpha_i(S,P)\cdot e^{(S_{+i},\tau^S_i(P))}), \qquad (4.3)$$

if $i \notin S$.

Now, let us assume otherwise, *i.e.*, that $i \in S$ and $|S| < |N|$ (we already considered simple game $e^{(N,\{N,\emptyset\})}$). We have that $v(N,\{N,\emptyset\}) = 0$. From Efficiency, we can evaluate the sum of payoffs of players from S as the opposite number to the sum of payoffs of outside players ($-\sum_{j\notin S}\varphi_j(c\cdot e^{(S,P)})$). This sum, in turn, can be calculated with formula (4.3). Now, based on Symmetry, all players from S divide their joint payoff equally:

$$\begin{aligned}\varphi_i(c \cdot e^{(S,P)}) &= \frac{1}{|S|}\sum_{k \in S} \varphi_k(c \cdot e^{(S,P)}) \\ &= -\frac{1}{|S|}\sum_{j \notin S} \varphi_j(c \cdot e^{(S,P)}) \\ &= \frac{1}{|S|}\sum_{j \notin S} \varphi_j(c \cdot \alpha_j(S,P) \cdot e^{(S_{+j}, \tau_j^S(P))}),\end{aligned}$$

if $i \in S$.
Thus, we provided two recursive equations for $\varphi_i(c \cdot e^{(S,P)})$ for both cases: $i \in S$ and $i \notin S$. This concludes our proof. □

4.2 Generality of the marginality approach

The marginality approach may seem arbitrary, that is, there may exist other values that satisfy Shapley's axiomatization but that cannot be uniquely derived from α-parametrized Shapley's axiomatization. This is, however, not the case. The next theorem states that the marginality approach encompasses *all values* that satisfy Shapley's axiomatization and exactly those.

Theorem 4. *The value φ can be obtained using the marginality approach if and only if it satisfies Efficiency, Symmetry, Additivity and the Null-Player Axiom.*

Proof. In the proof of Theorem 3 we showed that every value obtained using the marginality approach satisfies all four axioms. Thus, values that do not satisfy these axioms cannot be obtained using the marginality approach.

Assume that φ satisfies all four axioms. We will prove that there exists a weighting α such that φ also satisfies the α-Null-Player Axiom. Applying Theorem 3, this will conclude the proof, as there exists only one value which satisfies Efficiency, Symmetry, Additivity and the α-Null-Player Axiom. First, let us decompose game v into linear combination of simple games:

$$v = \sum_{(S,P) \in EC} v(S,P) \cdot e^{(S,P)}.$$

Based on Additivity, we have that:

$$\varphi_i(v) = \sum_{(S,P)\in EC} v(S,P)\cdot \varphi_i(e^{(S,P)}). \qquad (4.4)$$

Now, let $(\tilde{S},\tilde{P}) \in EC$ be any embedded coalition such that $i \in \tilde{S}$. Consider game $\tilde{v} = e^{(\tilde{S},\tilde{P})} + \sum_{T\in\tilde{P}_{-\tilde{S}}} e^{(\tilde{S}_{-i},\tau_i^T(\tilde{P}))}$ (*i.e.*, only $(\tilde{S},\tilde{P})$ and embedded coalitions obtained by transfer of i outside $\tilde{S}$ have non-zero values). Player i is a null-player in $\tilde{v}$; thus, from the Null-player Axiom we have that $\varphi_i(\tilde{v}) = 0$ and from Additivity that $\varphi_i(e^{(\tilde{S},\tilde{P})}) = -\sum_{T\in\tilde{P}_{-\tilde{S}}} \varphi_i(e^{(\tilde{S}_{-i},\tau_i^T(\tilde{P}))})$. As $(\tilde{S},\tilde{P}) \in EC$ has been chosen arbitrarily, it holds for every (S,P) such that $i \in S$; thus, we can transform equation (4.4) as follows:

$$\begin{aligned}\varphi_i(v) &= \sum_{(S,P)\in EC, i\in S} [v(S,P)\cdot\varphi_i(e^{(S,P)}) + \sum_{T\in P_{-S}} v(S_{-i},\tau_i^T(P))\varphi_i(e^{(S_{-i},\tau_i^T(P))})] \\ &= \sum_{(S,P)\in EC, i\in S}\sum_{T\in P_{-S}} -\varphi_i(e^{(S_{-i},\tau_i^T(P))})[v(S,P) - v(S_{-i},\tau_i^T(P))].\end{aligned}$$

Now, we can define $\alpha_i(S,P) = -\frac{\varphi_i(e^{(S_{-i},\tau_i^T(P))})}{\varphi_i(e^{(S,P)})}$:[2] these are proper weights, as they sum up to one for every (S,P) and – based on Symmetry – are symmetrical ($\varphi_i(e^{(S,P)}) = \varphi_{\pi(i)}(e^{(\pi(S),\pi(P))})$ for every $(S,P) \in EC$). The last transformation shows that φ satisfies the α-Null-Player Axiom:

$$\begin{aligned}\varphi_i(v) &= \sum_{(S,P)\in EC, i\in S} \varphi_i(e^{(S,P)}) \sum_{T\in P_{-S}} \alpha_i(S,P)[v(S,P) - v(S_{-i},\tau_i^T(P))] \\ &= \sum_{(S,P)\in EC, i\in S} \varphi_i(e^{(S,P)})[mc_i^\alpha(v)](S,P),\end{aligned}$$

which concludes the proof. □

4.3 BARGAINING PROCESS

To better understand how weights α affect the value assigned to a player, we present the bargaining process that would produce the value as its expected

[2]When $\varphi_i(e^{(S,P)}) = 0$ weights can be arbitrary, and do not have impact on the satisfiability of the α-Null-player Axiom.

outcome. To this end, we reverse the process presented by Shapley for the value for games with no externalities and, additionally, investigate partitions of players outside the grand coalition. For clarity of presentation, we limit ourselves to positive weights which can be interpreted as the *probability of a transfer* to occur.

Assume that players *leave* the grand coalition in a random order and divide themselves into groups outside. In each step, one player departs and, with the probability given by α, enters an existing group outside, or forms a new group. As the result of the leave, the player is granted his elementary marginal contribution, *i.e.*, with the loss of a coalition he left. Now, the value obtained using marginality approach with weights α is the expected outcome of the player's contribution.[3]

Chapter 6 contain the survey and the comparison of different weightings α proposed in the literature, but to illustrate above discussion we will now present the weights considered by Macho-Stadler *et al.*:

$$\alpha_i^{MSt}(S,P) = \frac{|P(i)_{-i}|}{|N| - |S| + 1}$$

under the convention that $|\emptyset| = 1$. According to this definition, the effects of transfers to bigger coalitions are taken with higher weights. Thus, if we look at the weights α as probabilities, this definition states that a player is more likely to transfer to a bigger coalition than to a smaller one. Interestingly, this means that the formation of a given partition in the bargaining process corresponds to the *Chinese restaurant process*, known in the field of probability theory.

We have just shown that the marginality approach restricts all payoff division schemes to those that satisfy Shapley's axioms, *i.e.*, α-values. In the next chapter, we will analyze how some desirable properties of an α-value translate into properties of weights α.

[3]To include negative weights in this process, one has to assign to players' transfers weights instead of probabilities. A composition of such weights for a given permutation will constitute a weight of the resulting partition. A marginal contribution in a given permutation should be calculated as a sum of all possible elementary marginal contributions multiplied by corresponding weights.

CHAPTER 5

PROPERTIES OF THE VALUE

MARGINALITY approach is build upon the weights associated with transfers. This leads to the natural question of how weights will affect the properties of the resulting value, and what conditions have to be met by the weights to obtain a value with the given properties.

In this chapter, we provide certain links between properties of an α-value and the properties of α-weights. We begin by identifying insignificant weights, that is, weights which do not have any impact on a value (Section 5.1). Then, we show how the axioms of *Weak/Strong Monotonicity*, *Strong Symmetry*, and the *Strong Null-Player Axiom* translate into properties of significant weights α (Sections 5.2, 5.3, 5.4, respectively). Chapter ends with a discussion how other axiomatizations – Young's monotonicity axiomatization and Myerson's axiomatization based on the concept of balanced contribution – can be translated to games with externalities and how they relate to the original axiomatization proposed by Shapley (Section 5.5).

Note: This chapter is based on [51].

5.1 SIGNIFICANT AND INSIGNIFICANT WEIGHTS

If we analyze formula for the α-value (formula (4.2)) carefully, we can see that most weights appear only in the products of weights and can be arbitrary when those products evaluate to zero. Moreover, weights of the form $\alpha_i(\emptyset, \_\_\_)$ appear only in marginal contributions of the form $[mc_i^{\alpha}(v)](\{i\}, P)$. These marginal contributions always evaluate to $v(\{i\}, P)$, i.e., they are independent of weights. This leads us to the notion of *significance*.

> *Significant weights*: weight $\alpha_i(S,P)$ is called *significant* if (S,P) is *probable* and non-empty, i.e., if $\sum_{\pi \in \Omega(N \setminus (S \cup \{i\}))} pr^{\alpha}_{\pi}(S_{+i}, \tau^S_i(P)) \neq 0$ and $S \neq \emptyset$.

If we limit ourselves to non-negative weights, then $\alpha_i(S,P)$, such that $S \neq \emptyset$, is significant if and only if there exists a permutation π such that $pr^{\alpha}_{\pi}(S,P) > 0$. In turn, if all weights are positive, then all $\alpha_i(S,P)$ such that $S \neq \emptyset$ are significant.

The following lemma states that only significant weights have an impact on the value.

Lemma 4. *Let α-marginality and $\hat{\alpha}$-marginality be two definitions of marginal contribution. Then, α-value and $\hat{\alpha}$-value differs if and only if there exists an embedded coalition (S,P) such that $\alpha(S,P) \neq \hat{\alpha}(S,P)$ and both weights are significant.*

Proof. First, we will prove that insignificant weights do not change the value. Based on equation (4.2), weight $\alpha_i(\emptyset, P)$ appears only in the marginal contribution $[mc^{\alpha}_i(v)](\{i\}, P_{-i} \cup \{i\}) = \sum_{T \in P} \alpha_i(\emptyset, \tau^T_i(P))[v(\{i\}, P_{-i} \cup \{i\}) - v(\emptyset, \tau^T_i(P))]$ which simplifies to $v(\{i\}, P_{-i} \cup \{i\})$ (therefore the particular weights do not matter as long as they sum up to one). Now, let us consider in which place insignificant weight $\alpha_i(S,P)$ with $S \neq \emptyset$ appears in equation (4.2). When we calculate the payoff of player i, weight $\alpha_i(S,P)$ appears only in marginal contribution $[mc^{\alpha}_i(v)](S_{+i}, \tau^S_i(P))$ preceded by $\sum_{\pi \in \Omega(N \setminus (S \cup \{i\}))} pr^{\alpha}_{\pi}(S_{+i}, \tau^S_i(P))$ which equals zero. In the payoff of other players, $\alpha_i(S,P)$ appears in the sum $\sum_{\tilde{\pi} \in N \setminus \tilde{S}} pr^{\alpha}_{\tilde{\pi}}(\tilde{S}, \tilde{P})$, where a given product is obtained by the sequence of transfers that transform $(N, \{N, \emptyset\})$ into $(\tilde{S}, \tilde{P})$. Thus, $\alpha_i(S,P)$ appears only if all other players from $N \setminus S$ have transferred before i (and only them):

$$\sum_{\tilde{\pi} \in N \setminus \tilde{S}} pr^{\alpha}_{\tilde{\pi}}(\tilde{S}, \tilde{P}) = \sum_{\pi_2 \in \Omega(S \setminus \tilde{S})} \sum_{\pi_1 \in \Omega(N \setminus (S \cup \{i\}))} pr^{\alpha}_{\pi_1 || i || \pi_2}(\tilde{S}, \tilde{P}).$$

Now, for a given permutation of players from $S \setminus \tilde{S}$: $\pi_2 \in \Omega(S \setminus \tilde{S})$, if we extract the product of last $|S| - |\tilde{S}|$ weights from the second sum, we see that the whole sum is multiplied by $\sum_{\pi \in \Omega(N \setminus (S \cup \{i\}))} pr^{\alpha}_{\pi}(S_{+i}, \tau^S_i(P)) \cdot \alpha_i(S,P)$. This equals zero regardless of weight $\alpha_i(S,P)$.

Now, we will prove that the significant weight has an impact on the value. Assume that α and $\tilde{\alpha}$ differ in at least one significant weight. Let (S, P) be an embedded coalition such that $\alpha_i(S, P) \neq \hat{\alpha}_i(S, P)$, both weights are significant and $\alpha_i(\tilde{S}, \tilde{P}) = \hat{\alpha}_i(\tilde{S}, \tilde{P})$ for every $(\tilde{S}, \tilde{P})$ with $S \subset \tilde{S}$. Now, consider simple game $e^{(S,P)}$. As $pr_\pi^{\alpha_1}(S_{+i}, \tau_i^S(P)) = pr_\pi^{\alpha_2}(S_{+i}, \tau_i^S(P))$ for every π and, from weights significance, we have that $\sum_{\pi \in \Omega(N \setminus (S \cup \{i\}))} pr_\pi^\alpha(S_{+i}, \tau_i^S(P)) \neq 0$, then from equation (4.2) we get $\varphi_i^\alpha(v^{(S,P)}) \neq \varphi_i^{\hat{\alpha}}(v^{(S,P)})$. □

This result indicates that we should only consider significant weight, which is what we will do from now on.

Before we go any further, let us discuss how weights associated with elementary marginal contributions can be looked upon as weights of partitions. To this end, let $(S, P) \in EC$ such that $i \notin S$ and let $\alpha_i(S, P)$ be significant. Then, there exists $\pi \in \Omega(N \setminus (S \cup \{i\}))$ such that $pr_\pi^\alpha(S_{+i}, \tau_i^S(P)) \neq 0$. Now, we have that:

$$\alpha_i(S, P) = \frac{pr_{\pi||i}^\alpha(S, P)}{pr_\pi^\alpha(S_{+i}, \tau_i^S(P))}, \tag{5.1}$$

that is, all significant weights α can be calculated from products pr^α. Moreover, if we use equation (5.1) for different embedded coalitions obtained by the transfer of i from $(S_{+i}, \tau_i^S(P))$, we get that all products pr^α for bigger coalitions can be obtained from products for smaller coalitions: $pr_\pi^\alpha(S_{+i}, \tau_i^S(P)) = \sum_{T \in P_{-S}} pr_{\pi||i}^\alpha(S, \tau_i^T(P))$. That leads us to the observation that defining values of $pr_\pi^\alpha(\emptyset, P)$ for every $\pi \in \Omega(N)$ and every partition P is equivalent to defining significant weights α. Based on the conditions imposed on weights α, all weight compositions must sum up to one (*i.e.*, $\sum_{P \in \mathcal{P}} pr_\pi^\alpha(\emptyset, P) = 1$ for every $\pi \in \Omega(N)$) and must be symmetrical (*i.e.*, $pr_{\pi_1}^\alpha(\emptyset, \pi_1(P)) = pr_{\pi_2}^\alpha(\emptyset, \pi_2(P))$ for every $\pi \in \Omega(N)$ and $P \in \mathcal{P}$).

This simple (but useful) observation allows us to focus on the weights of partitions (that may represent the probability that a given partition will form) instead of considering elementary transfers. To give an example, Hu and Yang argued that, independently of a permutation of players, all partitions should be equally likely to form [25]. Thus, $pr_\pi^{\alpha^{HY}}(\emptyset, P) = \frac{1}{\mathcal{P}(N)}$ for every $\pi \in \Omega(N)$ and $P \in \mathcal{P}(N)$. That immediately implies weights α:

$$pr_\pi^{\alpha^{HY}}(S, P) = \frac{|\{R \in \mathcal{P}(N) : R_{[S]} = P_{[S]}\}|}{|\mathcal{P}(N)|}$$

and

$$\alpha_i^{HY}(S,P) = \frac{|\{R \in \mathcal{P}(N) : R_{[S]} = P_{[S]}\}|}{|\{R \in \mathcal{P}(N) : R_{[S\cup\{i\}]} = P_{[S\cup\{i\}]}|}.$$

5.2 Weak and Strong Monotonicity

One of the most desirable properties of the division scheme is the *(Weak) Monotonicity*. It states, that if we increase the value of a particular coalition, then the payoffs, that is the shares of the grand coalition value assigned to its members, should not decrease. Analogously, the shares of non-members should not increase. The fact that Myerson's value violates Monotonicity is the main reason why it was criticized in the literature as unintuitive [29, 10, 33]. Formally, we have the following definition:

> *Weak Monotonicity* (increase of player's contributions does not decrease its payoff):
> *if* $v_1(S_{+i}, \tau_i^S(P)) - v_1(S,P) \geq v_2(S_{+i}, \tau_i^S(P)) - v_2(S,P)$ *holds for every* $(S,P) \in EC$*, such that* $i \notin S$*, then* $\varphi_i(v_1) \geq \varphi_i(v_2)$.

This formulation agrees with definitions proposed by Macho-Stadler *et al.* and De Clippel and Serrano, and differs from the one by McQuillin[1].

Now, we prove that the necessary and sufficient condition for Weak Monotonicity to be satisfied by an α-value is that weights α are non-negative.

Lemma 5. *An α-value satisfies Weak Monotonicity if and only if $\alpha_i(S,P) \geq 0$ for every significant weight.*

Proof. First, let us transform one more time the formula for α-value:

$$\varphi_i^\alpha(v) = \sum_{\substack{(S,P)\in EC \\ i\notin S}} \frac{|S|!}{|N|!} \sum_{\pi\in\Omega(N\setminus S_{+i})} pr^\alpha_{\pi||i}(S,P) \cdot [v(S_{+i}, \tau_i^S(P)) - v(S,P)]. \quad (5.2)$$

[1]McQuillin in his definition required only that the increase of a coalition's value causes no decrease of payoffs of the members. Formally, he defined Weak Monotonicity as follows: $\varphi_i(e^{(S,P)}) \geq 0$ if $i \in S$. For linear values, this definition is equivalent to the following one: if $v_1(S,P) \geq v_2(S,P)$ holds for every $(S,P) \in EC$, such that $i \in S$, then $\varphi_i(v_1) \geq \varphi_i(v_2)$. Indeed, in games with no externalities, this implies for symmetric values that other players' payoffs do not increase. However, this is not the case when externalities are present. For example, in a simple game $e^{(S,P)}$ for $(S,P) = (\{1\}, \{\{1\},\{2\},\{3,4\},\emptyset\})$ the following payoff scheme $\varphi_1(e^{(S,P)}) = \varphi_2(e^{(S,P)}) = a$ and $\varphi_3(e^{(S,P)}) = \varphi_4(e^{(S,P)}) = -a$ does not violate Symmetry, nor Weak Monotonicity.

Thus, we see that in the formula for φ_i^α the coefficient of marginal contribution $v(S_{+i}, \tau_i^S(P)) - v(S, P)$ equals $\frac{|S|!}{|N|!} \sum_{\pi \in \Omega(N \setminus S_{+i})} pr^\alpha_{\pi||i}(S, P)$ which is equivalent to

$$\frac{|S|!}{|N|!} \sum_{\pi \in \Omega(N \setminus S_{+i})} pr^\alpha_\pi(S_{+i}, \tau_i^S(P)) \cdot \alpha_i(S, P).$$

If this coefficient is negative, then its increase will decrease the payoff. Thus, α-value satisfies Weak Monotonicity if and only if all coefficients are non-negative. If all weights are non-negative then all products of weights are non-negative (thus, the condition is satisfied). In turn, using the reverse induction on $|S|$, we get that all significant weights $\alpha_i(S, P)$ must be non-negative: from inductive assumption, $pr^\alpha_\pi(S_{+i}, \tau_i^S(P)) \geq 0$ for every $\pi \in \Omega(N \setminus S_{+i})$, thus if the sum is greater than zero, then (S, P) is probable and must be non-negative. This concludes our proof. □

In Weak Monotonicity, we require that the increase of a player's contribution does not result in the decrease of their payoff. This, in particular, means that we allow for a hypothetical situation where a player's arbitrary big contributions to some coalitions, although not negatively affecting his payoff, may not affect his payoff at all. This would be discouraging for players. To address this issue, we propose a notion of Strong Monotonicity:

> *Strong Monotonicity* (increase of player's contributions increases its payoff): *if* $v_1(S_{+i}, \tau_i^S(P)) - v_1(S, P) \geq v_2(S_{+i}, \tau_i^S(P)) - v_2(S, P)$ *holds for every* $(S, P) \in EC$*, such that* $i \notin S$ *and this inequality is strict for at least one embedded coalition, then* $\varphi_i(v_1) > \varphi_i(v_2)$.

Lemma 6. *α-value satisfies Strong Monotonicity if and only if* $\alpha_i(S, P) > 0$ *for every significant weight.*

Proof. Proof is analogous to the proof of Lemma 5, but here all coefficients must be positive (otherwise, if $v(S_{+i}, \tau_i^S(P)) - v(S, P)$ has zero coefficients, then the payoff in game $-e^{(S,P)}$ does not increase the payoff for player i). □

The above lemma shows that Strong Monotonicity implies that, in the stochastic process (which is built upon the α value), every transfer is possible.

Next, we will analyze the axiom of Strong Symmetry.

5.3 Strong Symmetry

Macho-Stadler *et al.* proposed a strengthening of the axiom of Symmetry called *Strong Symmetry*. To look closer into this concept, let us consider simple game $e^{(S,P)}$ (where only particular S embedded in P has non-zero value). From Symmetry, all players from S have the same payoff. In turn, payoffs of players from $N \setminus S$ may differ between them. This may seem unfair, as they all have the same role in this game: they must form specific partition P for S to generate a value.

Let us then consider a bijection (*i.e.*, one-to-one mapping) $f : N \setminus S \to N \setminus S$. The axiom of *Strong Symmetry* states that, if $f_{(S,P)}(v)$ is a game obtained by exchanging the value of (S, P) and $(S, S \cup f(P \setminus S))$,[2] then all of the payoffs from game $f_{(S,P)}(v)$ are the same as payoffs from game v:

> *Strong Symmetry* (the value of a coalition affects the payoffs of outside players symmetrically):
>
> - $\varphi(f(v)) = f(\varphi)(v)$ for every game v and bijection $f : N \to N$;
> - $\varphi(f_{(S,P)}(v)) = \varphi(v)$ for every game v and every bijection $f : N \setminus S \to N \setminus S$.

This definition is equivalent to the condition $\varphi_i(e^{(S,P)}) = \varphi_j(e^{(S,P)})$ for every $i, j \notin S$ for linear values.

To translate this axiom to a property of the weight, we introduce the concept of the *interlace resistance*.

> *Interlace resistance* (product of weights should not depend on the order of corresponding transfers):
>
> $$pr^{\alpha}_{\pi_1}(S, P) = pr^{\alpha}_{\pi_2}(S, P)$$
>
> for every (S, P) such that $S \neq \emptyset$ and every $\pi_1, \pi_2 \in \Omega(N \setminus S)$.

For non-negative weights this condition simplifies to the equivalence of products of two weights.

[2] Formally, $[f_{(S,P)}(v)](S, P) = v(S, S \cup f(P \setminus S))$, $[f_{(S,P)}(v)](S, S \cup f(P \setminus S)) = v(S, P)$ and $[f_{(S,P)}(v)](\tilde{S}, \tilde{P}) = v(\tilde{S}, \tilde{P})$, otherwise. See Symmetry definition for a formal specification of $f(P)$.

Lemma 7. *If weights α are non-negative, then α is interlace resistance if and only if it satisfies $\alpha_i(S, P) \cdot \alpha_j(S_{+i}, \tau_i^S(P)) = \alpha_j(S, P) \cdot \alpha_i(S_{+j}, \tau_j^S(P))$ for all significant weights such that $i, j \notin S$.*

Proof. First, we prove that the above condition is necessary. Let (S, P) be an embedded coalition such that $i, j \notin S$, $S \neq \emptyset$ and (S, P) is probable (note that, for non-negative weights, if (S, P) is probable then $(S_{+i}, \tau_i^S(P)), (S_{+j}, \tau_j^S(P))$ and $(S_{+ij}, \tau_j^S(\tau_i^S(P)))$ are also probable). Let π be a permutation of $N \setminus (S \cup \{i, j\})$ such that $pr_\pi^\alpha(S_{+ij}, \tau_{ij}^S(P)) > 0$. Now, let us consider two different extensions of permutation π: $\pi||j||i, \pi||i||j \in \Omega(N \setminus S)$ (*i.e.*, the first one ends with (j, i), and the second one with (i, j)). Now, the condition $pr_{\pi||j||i}^\alpha(S, P) = pr_{\pi||i||j}^\alpha(S, P)$ simplifies to $\alpha_i(S, P) \cdot \alpha_j(S_{+i}, \tau_i^S(P)) = \alpha_j(S, P) \cdot \alpha_i(S_{+j}, \tau_j^S(P))$.

Next, we show that this condition is sufficient. This comes from the combinatorial fact that every permutation can be obtained from another using only transpositions of the adjacent elements. More formally, assume that (S, P) is an embedded coalition with $S \neq \emptyset$ and π_1, π_2 are some permutations of N_{-S}. We will prove that both products of weights $pr_{\pi_1}^\alpha(S, P), pr_{\pi_2}^\alpha(S, P)$ are equal if the condition holds. To this end, we observe that if in both products a zero weight appears then both are equal. If not, without loss of generality, all weights in $pr_{\pi_1}^\alpha(S, P)$ are nonzero and significant. Based on the condition $\alpha_i(S, P) \cdot \alpha_j(S_{+i}, \tau_i^S(P))$ equals $\alpha_j(S, P) \cdot \alpha_i(S_{+j}, \tau_j^S(P))$, transpositions of adjacent players in order π_1 do not change the product of weights. Thus, the proper sequence of transpositions will yield that $pr_{\pi_1}^\alpha(S, P) = pr_{\pi_2}^\alpha(S, P)$. □

The following theorem shows that an α-value satisfies Strong Symmetry if and only if weights are interlace resistant.

Theorem 5. *An α-value satisfies Strong Symmetry if and only if α-marginality is interlace resistant.*

Proof. Assume that α is interlace resistant. We will prove that for every (S, P) and $i, j \in N \setminus S$ holds $\varphi_i^\alpha(e^{(S,P)}) = \varphi_j^\alpha(e^{(S,P)})$. This fact, based on Additivity, will imply Strong Symmetry. Let $\pi_e \in \Omega(N \setminus S)$ be any permutation. Based on equation (5.2):

$$\varphi_i^\alpha(e^{(S,P)}) = -\frac{|S|!(|N| - |S| + 1)!}{|N|!} \cdot pr_{\pi_e}^\alpha(S, P) \tag{5.3}$$

do not depend on the player $i \notin S$. Thus, $\varphi_i^\alpha(e^{(S,P)}) = \varphi_j^\alpha(e^{(S,P)})$.

To prove that Strong Symmetry holds only for interlace resistant weights we use the reverse induction on the size of S. Of course $pr^{\alpha}_{\pi_1}(S,P) = pr^{\alpha}_{\pi_2}(S,P) = 1$ when $|S| = |N| - 1$ for every permutation $\pi_1, \pi_2 \in \Omega(N \setminus S)$. Let us assume that this equivalence holds when $|S| > k$. We will prove that it also holds when $|S| = k$. Let $\pi_1, \pi_2 \in \Omega(N \setminus S)$ be two permutations, and without the loss of generality assume i and j are the last players in π_1, π_2. Now, consider simple game $e^{(S,P)}$ in which (S,P) is the only embedded coalition with non-zero value. Based on Strong Symmetry, players i and j have equal payoffs: $\varphi^{\alpha}_i(e^{(S,P)}) = \varphi^{\alpha}_j(e^{(S,P)})$. From equation (5.2):

$$\varphi^{\alpha}_i(e^{(S,P)}) = -\frac{|S|!}{|N|!} \sum_{\pi \in \Omega(N \setminus S_{+i})} pr^{\alpha}_{\pi}(S_{+i}, \tau^S_i(P)) \cdot \alpha_i(S,P).$$

Based on the inductive assumption, product $pr^{\alpha}_{\pi}(S_{+i}, \tau^S_i(P))$ does not depend on the permutation $\pi \in \Omega(N \setminus S_{-i})$, thus $\varphi^{\alpha}_i(e^{(S,P)}) = -\frac{(|S|!)(|N|-|S|+1)!}{|N|!} pr^{\alpha}_{\pi_1}(S,P)$. Now, $\varphi^{\alpha}_j(e^{(S,P)}) = -\frac{(|S|!)(|N|-|S|+1)!}{|N|!} pr^{\alpha}_{\pi_2}(S,P)$ implies $pr^{\alpha}_{\pi_1}(S,P) = pr^{\alpha}_{\pi_2}(S,P)$. □

When we consider the values that satisfy Strong Symmetry, the general formula (4.2) can be simplified as follows:

$$\varphi^{\alpha}_i(v) = \sum_{(S,P) \in EC, i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} pr^{\alpha}(S,P)[mc^{\alpha}_i(v)](S,P), \quad (5.4)$$

where $pr^{\alpha}(S,P)$ denotes $pr^{\alpha}_{\pi}(S,P)$ for any permutation $\pi \in \Omega(N \setminus S)$, as this product is equal for every permutation. Observe that $v(S,P)$ appears in the above formula multiplied by $pr^{\alpha}(S,P)$ and $v(S_{-i}, \tau^T_i(P))$ by $pr^{\alpha}(S,P) \cdot \alpha_i(S_{-i}, \tau^T_i(P))$ which equals $pr^{\alpha}(S_{-i}, \tau^T_i(P))$. Thus, the value of a coalition in a given partition is always preceded with the probability that a given partition will form. This suggests the following, already mentioned at the end of Section 3.1, algorithm for evaluating a fair division in a game with externalities.

1. First, we create average game $\tilde{v}$ with no externalities from game v with externalities. This is done by calculating the value of every coalition as the average of its values in games with externalities:

$$\tilde{v}(S) = \sum_{P \ni S} a(S,P) \cdot v(S,P),$$

where $\sum_{P \ni S} a(S,P) = 1$ for every S.

2. Then, we calculate the Shapley value for average game $\tilde{v}$:

$$\varphi(v) = Sh(\tilde{v}).$$

This approach, called the *average approach*, was introduced by Macho-Stadler *et al.* They proved that the value that satisfies Shapley's axioms can be constructed using the average approach if and only if it satisfies Strong Symmetry (see Theorem 1 in [29]).

In our Theorem 4 we proved that marginality approach can produce every value that satisfies Shapley's axiom and Theorem 5 shows that the resulting value satisfies Strong Symmetry if and only if weights are interlace resistant. Thus, our two theorems combined with the result from Macho-Stadler *et al.* imply that a value can be obtained using the average approach if and only if it can be obtained using the marginality approach with interlace resistant weights.

Corollary 1. *The average approach is equivalent to the marginality approach with interlace resistant weights.*

Finally, in the next section we consider the Strong Null-Player Axiom.

5.4 Strong Null-Player Axiom

Another axiom proposed by Macho-Stadler *et al.* is the *Strong Null-Player Axiom*. Consider a game v in which i is a null-player in a strict sense, *i.e.*, i does not have an impact on the game whatsoever. In this case, the Null-Player Axiom requires that player i has zero payoff: $\varphi_i(v) = 0$. But it does not mean that he has no impact on the payoffs of others. In other words, removing a null-player from the game may affect the payoffs of the remaining players. Such a situation is infeasible if we rely on the Strong Null-Player Axiom proposed by Macho-Stadler *et al.*:

> *Strong Null-Player Axiom* (null-player does not have an impact on the payoffs of others): if i is a null-player then $\varphi_j(v) = \varphi_j(v_{-i})$ for every $j \in N$, where v_{-i} denotes the game without player i: $v_{-i}(S_{-i}, P_{-i}) \stackrel{\text{def}}{=} v(S, P)$ for every (S, P) such that $i \in S$.

When i is not a null-player, constructing the game v_{-i} can be challenging. This issue will be discussed in more detail in Section 5.5. However, the situation is much simpler when i is a null-player. This is because the value of embedded coalition $(S, P) \in EC(N \setminus \{i\})$ can be obtained by inserting i to an arbitrary coalition in partition P, as all possible values are equal.

Let us now analyze the constraints imposed by the Strong Null-Player Axiom on the values that satisfy Strong Symmetry. To this end, let us introduce the *expansion resistance* property.

Expansion resistance (weight does not depend on the size of S):

$$\alpha_i(S, P) = \alpha_i(S_{-j}, P_{-j})$$

for all significant weights such that $i \notin S$ and $j \in S$.

In terms of our bargaining process, this intuitive requirement says that the probability of joining a coalition by a player should depend only on the coalitions to choose from and not on the coalition that the player is leaving. The following theorem states that, for values satisfying Strong Symmetry, expansion resistance is necessary and sufficient to obtain the Strong Null-Player Axiom.

Theorem 6. *If an α-value satisfies Strong Symmetry then it satisfies the Strong Null-Player Axiom if and only if α-marginality is expansion resistant.*

Proof. In this proof we consider only α-values that satisfy Strong Symmetry. We will show that expansion resistance is equivalent to satisfying the Strong Null-Player Axiom by the sequence of equivalences.

PART 1: expansion resistance $\Leftrightarrow$ $pr^\alpha(S, P) = pr^\alpha(S_{-i}, P_{-i})$ for every (S, P) such that $i \in S$;

It is clear that expansion resistance implies the condition from the right-hand side. Also, if expansion resistance is not met, then products of weights must also differ: if (S, P) is the smallest coalition such that $\alpha_j(S, P) \neq \alpha_j(S_{-i}, P_{-i})$ then $pr^\alpha(S_{+j}, \tau_j^S(P)) = pr^\alpha((S_{-i})_{+j}, \tau_j^S(P_{-i}))$ and $pr^\alpha(S, P) \neq pr^\alpha(S_{-i}, P_{-i})$.

PART 2: $pr^\alpha(S, P) = pr^\alpha(S_{-i}, P_{-i}) \Leftrightarrow \varphi_j^\alpha(e^{(S,P)}) = \frac{|S|-1}{|N|} \cdot \varphi_j^\alpha(e^{(S_{-i},P_{-i})})$ *for* $i, j \in S$

This step, which translate weights characteristic to the value property is immediate from equation (5.4). Now, let us denote $\tilde{v}^{(S,P)} = e^{(S,P)} + \sum_{T \in P_{-S}} e^{(S_{-i}, \tau_i^T(P))}$. Thus, $\tilde{v}_{-i}^{(S,P)} = e^{(S_{-i},P_{-i})}$.

PART 3: $\varphi_j^\alpha(e^{(S,P)}) = \frac{|S|-1}{|N|} \cdot \varphi_j^\alpha(e^{(S_{-i},P_{-i})})$ *for* $i, j \in S \Leftrightarrow \varphi_j^\alpha(\tilde{v}^{(S,P)}) = \varphi_j^\alpha(\tilde{v}_{-i}^{(S,P)})$ *for* $j \in S$ *and null-player* $i \in S$

The right-hand side of the equivalence comes from the Strong Null-Player Axiom applied to game $\tilde{v}^{(S,P)}$. To prove equivalence, we will transform it using Additivity:

$$\varphi_j^\alpha(e^{(S,P)}) + \sum_{T\in P_{-S}} \varphi_j^\alpha(e^{(S_{-i},\tau_i^T(P))}) = \varphi_j^\alpha(e^{(S_{-i},P_{-i})}).$$

But based on Strong Symmetry all payoffs of players outside S_{-i} in $e^{(S_{-i},\tau_i^T(P))}$ are equal, thus from Efficiency:

$$\varphi_j^\alpha(e^{(S_{-i},\tau_i^T(P))}) \cdot (|S|-1) = -\varphi_i^\alpha(e^{(S_{-i},\tau_i^T(P))}) \cdot (|N|-|S|+1)$$

for every $T \in P_{-S}$. Thus,

$$\begin{aligned}
\varphi_j^\alpha(e^{(S_{-i},P_{-i})}) &= \varphi_j^\alpha(e^{(S,P)}) - \frac{|N|-|S|+1}{|S|-1} \sum_{T\in P_{-S}} \varphi_i^\alpha(e^{(S_{-i},\tau_i^T(P))}) \\
&= \varphi_j^\alpha(e^{(S,P)}) + \frac{|N|-|S|+1}{|S|-1} \cdot \varphi_i^\alpha(e^{(S,P)}) \\
&= \frac{|N|}{|S|-1} \cdot \varphi_j^\alpha(e^{(S,P)}),
\end{aligned}$$

where we used Symmetry $\varphi_i^\alpha(e^{(S,P)}) = \varphi_j^\alpha(e^{(S,P)})$ and the Null-Player Axiom for equality $\sum_{T\in P_{-S}} \varphi_i^\alpha(e^{(S_{-i},\tau_i^T(P))}) = -\varphi_i^\alpha(e^{(S,P)})$.

PART 4: $\varphi_j^\alpha(\tilde{v}^{(S,P)}) = \varphi_j^\alpha(\tilde{v}_{-i}^{(S,P)})$ for every (S,P) *such that* $j \in S$ *and null-player* $i \in S$ $\Leftrightarrow$ *Strong Null-Player Axiom*

Of course the Strong Null-Player Axiom implies the left-hand side. On the other hand, every game in which i is a null-player can be decomposed in the following way:

$$v = \sum_{(S,P),i\in S} v(S,P) \cdot (e^{(S,P)} + \sum_{T\in P_{-S}} e^{(S_{-i},\tau_i^T(P))}) = \sum_{(S,P),i\in S} v(S,P) \cdot \tilde{v}^{(S,P)}.$$

Thus, again based on Additivity, if value $\varphi_j^\alpha(\tilde{v}^{(S,P)}) = \varphi_j^\alpha(\tilde{v}_{-i}^{(S,P)})$ holds for every (S,P) and $j \in S$, then $\varphi_j^\alpha(v) = \varphi_j^\alpha(v_{-i})$. That implies also $\varphi_j^\alpha(v) = \varphi_j^\alpha(v_{-i})$ for $j \notin S$ from Strong Symmetry and concludes our proof. □

5.5 Relationship with Young's and Myerson's axiomatizations

In this section we discuss how Young's and Myerson's axiomatization of Shapley value can be translated to games with externalities.

Young argues that the concept of (Weak) Monotonicity can be restated as follows: if we consider two games such that in the first game all marginal contributions of a player are not smaller than the corresponding marginal contributions in the second game (*i.e.*, the difference of vectors of marginal contributions is non-negative in every coordinate), then the payoff in the former game should not be smaller than the payoff in the latter game. This yields another property, called the *Marginality Axiom*. This axiom says that if marginal contributions are equal, then the payoffs should also be equal. In other words, payoffs should depend only on the vector of marginal contributions. Young proved that the Shapley value is the only value which satisfies Efficiency, Symmetry and the Marginality Axiom.

In games with externalities, we have to specify which definition of marginal contribution we assume. That leads to the α-*Marginality Axiom*:

> α-*Marginality Axiom* (payoff of a player depends only on his marginal contributions): $mc_i^\alpha(v_1) = mc_i^\alpha(v_2) \Rightarrow \varphi_i(v_1) = \varphi_i(v_2)$ for every game v_1, v_2 and player $i \in N$.

Bolger [7] used Young's axiomatization to derive his value (with an additional Null-Player Axiom which is, in fact, redundant). Later, De Clippel and Serrano proved that externality-free value proposed by Pham Do and Norde (initially introduced using Shapley's standard axiomatization) can be also derived using this set of axioms [10]. Finally, Fujinaka provided a general theorem: for every definition of marginal contribution there exists a unique value which satisfies the Efficiency, Symmetry and α-Marginality Axioms [15]. For every α, the value proposed by Fujinaka based on Young's axiomatization is equal to our value (derived by Theorem 3). This means that both axiomatizations are equivalent.

Corollary 2. *Shapley's marginality-based axiomatization (Efficiency, Symmetry, Additivity and α-Null-Player Axiom) is equivalent to Young's axiomatization (Efficiency, Symmetry and α-Marginality Axiom). Moreover, both axiomatizations yield a unique value.*

Next, we will discuss an axiomatization proposed by Myerson [38] that is based on the concept of *Balanced Contributions*. We translate this axiom to games with externalities using our analysis of marginal contributions. It comes out that not every value obtained using the marginality approach satisfies the axiom of Balanced Contributions. Therefore, we characterize which values satisfy Myerson's concept using the properties of interlace and expansion resistance.

The Balanced Contributions principle guarantees a certain notion of stability. We say that mutual contributions of players i and j are balanced, if the withdrawal of player i from the game will result in the same loss to player j as the withdrawal of j to i. More formally, $\varphi_i(v) - \varphi_i(v_{-j}) = \varphi_j(v) - \varphi_j(v_{-i})$. Thus, the profit of cooperation is divided equally between players. It is important that this condition is met, as otherwise a player which gains less may threaten the other to leave the game. This is why the Balanced Contributions principle is usually the key piece in the mechanisms that implement the Shapley value (see, *e.g.*, Perez-Castrillo *et al.* [42]).

To translate this axiom to games with externalities we need to define how to calculate a game without a given player. In games without externalities, the value of a coalition S without a player i is uniquely defined. In games with externalities, different positions of i in the coalition structure may result in different values of S. Let us consider the following example:

Example 3. *Let us consider game v_{-1} created from game v in Example 1. We have that $v_{-1}(\{2\}, \{\{2\}, \{3\}, \emptyset\}) = b$ if we take value of $\{2\}$ from $\{\{1\}, \{2\}, \{3\}, \emptyset\}$ and $v_{-1}(\{2\}, \{\{2\}, \{3\}, \emptyset\}) = c$ if we take value from $\{\{1, 3\}, \{2\}, \emptyset\}$.*

This resembles the problem with defining the marginal contribution we faced before. Thus, we take a similar approach: from the value of coalition $S \cup \{i\}$, we subtract player's i marginal contribution:

$$v^{\alpha}_{-i}(S_{-i}, P_{-i}) \stackrel{\text{def}}{=} v(S, P) - mc^{\alpha}_i(S, P)$$

for every (S, P) such that $i \in S$.[3] Now, different definitions of marginal contributions (*i.e.* weights α) result in different values of the game without a player. Therefore, the corresponding axioms of Balanced Contributions will also be different:

[3]It is worth noting that under this definition of the game without a player α-values that satisfy Strong Symmetry and the Strong Null-Player Axiom satisfy also the Strong α-Null-Player Axiom that states that if i is an α-null-player then $\varphi_j(v) = \varphi_j(v^{\alpha}_{-i})$.

α-Balanced Contributions (profit of cooperation is divided equally between players): $\varphi_i(v) - \varphi_i(v^{\alpha}_{-j}) = \varphi_j(v) - \varphi_j(v^{\alpha}_{-i})$ for every game v and $i, j \in N$.

The principle of Balanced Contributions combined with Efficiency automatically yields a recursive formula for the unique value:

$$\varphi_i(v) = \frac{1}{|N|}(v(N, \{N, \emptyset\}) - v(\{N_{-i}, \{N_{-i}, \{i\}, \emptyset\}) + \sum_{j \neq i} \varphi_i(v^{\alpha}_{-j})). \qquad (5.5)$$

This comes from the sum of Balanced Contributions equations over all $j \in N$.

For games without externalities, Myerson proved that Efficiency and Balanced Contributions imply the Shapley value (thus it is equivalent, in particular, to Shapley's and Young's axiomatizations). But this is not the case in games with externalities – some α-values do not meet the corresponding axiom of Balanced Contributions parametrized with α.

To characterize which values meet Balanced Contributions, we will use the stronger versions of Symmetry and the Null-Player Axiom from Sections 5.3 and 5.4. To gain extra intuition behind it, note that if α-value still assigns zero to a null-player i, even if we remove player j from the game (thus, $\varphi^{\alpha}_i(v_{-j}) = 0$), then Balanced Contributions implies the Strong Null-Player Axiom: $\varphi^{\alpha}_j(v) - \varphi^{\alpha}_j(v_{-i}) = \varphi^{\alpha}_i(v) - \varphi^{\alpha}_i(v_{-j}) = 0 - 0$. On the other hand, the axiom of Balanced Contributions asks for the balance between the contributions of the two players contributions and ultimately implies Strong Symmetry.

Theorem 7. *Shapley's marginality-based axiomatization (Efficiency, Symmetry, Additivity and α-Null-Player Axiom) is equivalent to Myerson's axiomatization (Efficiency, α-Balanced Contributions) if and only if α is interlace and expansion resistant.*

Proof. Our proof is organized as follows: first we will argue that if α is interlace and expansion resistant then the α-value satisfies α-Balanced Contributions. Then, we will prove that α-Balanced Contributions implies the Strong Null-Player Axiom and Strong Symmetry. This result combined with Theorems 5 and 6 that link these axioms with properties of α will conclude the proof. Both parts of the proof will be based on the linear decomposition of the game v to simple games $e^{(S,P)}$. If α-Balanced Contributions is satisfied, then it must work for every simple game. On the other hand, if α-Balanced Contributions works for every simple game then based on Additivity it must be satisfied also for every linear combination, thus every possible game v. Thus, the α-value satisfies α-Balanced Contributions if and only if the following conditions are met:

(a) $\varphi_i(e^{(S,P)}) - 0 = \varphi_j(e^{(S,P)}) - 0$ for $i, j \in S$; this condition comes directly from Symmetry;

(b) $\varphi_i(e^{(S,P)}) - \varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S,P)) = \varphi_j(e^{(S,P)}) - 0$ for $i \in S, j \notin S$;

(c) $\varphi_i(e^{(S,P)}) - \varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S,P)) = \varphi_j(e^{(S,P)}) - \varphi_j(e^{(S,P_{-i})} \cdot \alpha_i(S,P))$ for $i, j \notin S$.

The zeros in the equations come from the fact that game $e^{(S,P)}$ without player from S is a zero game. A few times in our proofs we will use the following transformation:

$$\varphi_i(e^{(S_{+j},\tau_j^S(P))} \cdot \alpha_j(S,P)) = \varphi_j(e^{(S_{+j},\tau_j^S(P))} \cdot \alpha_j(S,P)) = \varphi_j(e^{(S,P)}), \qquad (5.6)$$

for $i \in S$, $j \notin S$, where first transformation comes from Symmetry of players i and j (both players are in the only embedded coalition with the value), and second – from the α-Null-Player Axiom.

First, assume α is interlace and expansion resistant (thus, α-value satisfies Strong Symmetry and the Strong Null-Player Axiom) and consider condition (b). From the Strong Null-Player Axiom and Strong Symmetry $\varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S,P)) = \frac{|N|}{|S|}\varphi_i(e^{(S_{+j},\tau_j^S(P))} \cdot \alpha_j(S,P))$ (this comes directly from formula (5.4)). From equation (5.6) this expression equals $\frac{|N|}{|S|}\varphi_j(e^{(S,P)})$, thus condition (b) simplifies to $\varphi_i(e^{(S,P)}) = -\frac{|N|-|S|}{|S|}\varphi_j(e^{(S,P)})$ which comes immediately from Strong Symmetry.

Let us focus on condition (c). Here, $\varphi_i(e^{(S,P)}) = \varphi_j(e^{(S,P)})$ (from Strong Symmetry) and condition simplifies to $\varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S,P)) = \varphi_j(e^{(S,P_{-i})} \cdot \alpha_i(S,P))$. We use the formula (5.4): $\varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S,P)) = \frac{|N|}{|S|+1}\varphi_i(e^{(S_{+j},\tau_j^S(P))} \cdot \alpha_j(S,P))$ which equals $\frac{|N|}{|S|+1}\varphi_i(e^{(S_{+ij},\tau_i^S(\tau_j^S(P)))} \cdot \alpha_j(S,P) \cdot \alpha_i(S_{+j}, \tau_j^S(P)))$ from the α-Null-Player Axiom. From Symmetry and the interlace resistant property we can replace all i with j which is equal to the analogous transformation of the right-hand side of the equation.

Secondly, we will prove that Efficiency and α-Balanced Contributions indeed imply Strong Symmetry and the Strong Null-Player Axiom. Let $\varrho(v)$ denote the number of players outside S in the smallest coalition with a non-zero value, *i.e.*, $\varrho(v) = max_{(S,P):v(S,P)\neq 0} \sum_{T\in P, T\neq S} |T|$. If v is a game of N players, then $\sum_{T\in P, T\neq S} |T| = |N| - |S|$, but as we will consider games of various numbers of players, we define this number by the size of partition P. We will use the induction by $\varrho(v)$. If $\varrho(v) = 0$ then $v = e^{(N,\{N,\emptyset\})}$ and both axioms are instantly satisfied. Now, assume that Strong Symmetry and the Strong Null-Player Axiom hold for games with $\varrho(v) < k$. We will prove that it holds also for games in which $\varrho(v) = k$. Let (S, P) be an embedded coalition such that $\sum_{T\in P, T\neq S} |T| = k$. Balanced Contributions implies $\varphi_i(e^{(S,P)}) - \varphi_i(e^{(S,P_{-j})} \cdot \alpha_j(S, P)) = \varphi_j(e^{(S,P)})$ for $i \in S, j \notin S$ (condition (b)). As $\varrho(e^{(S,P_{-j})} \cdot \alpha_j(S, P)) = k - 1$, then from Strong Symmetry and the Strong Null-Player Axiom $\varphi_i(e^{(S,P_{-j})}) = \frac{|N|}{|S|}\varphi_i(e^{(S_{+j},\tau_j^S(P))})$ and using equation (5.6) we have $\varphi_i(e^{(S,P)}) = -\frac{|N|-|S|}{|S|}\varphi_j(e^{(S,P)})$ which implies Strong Symmetry (payoff of player $j \notin S$ in game $e^{(S,P)}$ is equal for every j).

Now, we will prove that the Strong Null-Player Axiom is also satisfied. Assume that i is a null-player in a strict sense. As argued in the proof of Theorem 6, game v can be expressed as a linear combination of games of form $\tilde{v}^{(S,P)} = e^{(S,P)} + \sum_{T\in P_{-S}} e^{(S_{-i},\tau_i^T(P))}$. It is enough to show that for every such game, $\varphi_j(\tilde{v}^{(S,P)}) = \varphi_j(\tilde{v}_{-i}^{(S,P)})$ for every $j \in N$. If $j \in S$ this follows automatically from Balanced Contributions:

$$\varphi_j(\tilde{v}^{(S,P)}) - \varphi_j(\tilde{v}_{-i}^{(S,P)}) = \varphi_i(\tilde{v}^{(S,P)}) - \varphi_i(\tilde{v}_{-j}^{(S,P)}) = 0 - 0.$$

Based on Strong Symmetry this concludes the proof: for every $k \notin S$, it occurs that $\varphi_k(\tilde{v}^{(S,P)}) = \frac{|S|-1}{|N|-|S|}\varphi_j(\tilde{v}^{(S,P)})$ and $\varphi_k(\tilde{v}_{-i}^{(S,P)}) = \frac{|S|-1}{|N|-|S|}\varphi_j(\tilde{v}_i^{(S,P)})$, thus $\varphi_k(\tilde{v}^{(S,P)}) - \varphi_k(\tilde{v}_{-i}^{(S,P)}) = 0$. □

Chapter 6

Marginality Definitions and Other Axiomatizations

There are several definitions of marginal contribution (*i.e.*, weights α) proposed in the literature. All of them have already appeared as examples in different parts of our work. In this chapter, we analyze them in the context of properties introduced in the previous one (Section 6.1). Then, we describe axiomatic results from the literature, both marginality-based and other ones (Section 6.2).

Note: This chapter is based on [51].

6.1 Analysis of various marginality definitions

Before proceeding, let us recall that $P(i)$ denotes the coalition of player i in a partition P.

We start with Bolger's [7] definition of marginality, introduced after α-weights at the end of Chapter 2 as the most straightforward solution to the problem:

$$\alpha_i^B(S, P) = \frac{1}{|P_{-i}|}.$$

It is based on an idea that for a given embedded coalition all transfers have the same weight. All weights are positive and do not depend on the size of S; thus, they satisfy the expansion resistance. Conversely, the interlace resistance is not satisfied, as creating a new coalition affects the weights of joining an existing coalition. For example, if we consider two transfers from coalition $\{1,2,3\}$ embedded in $(\{1,2,3\},\{4\},\emptyset)$, one of 2 that forms a new coalition, and one of 3 that joins $\{4\}$, we have

$$\alpha_3^B(\{1,2\},\{\{1,2\},\{3,4\},\emptyset\}) \cdot \alpha_2^B(\{1\},\{\{1\},\{2\},\{3,4\},\emptyset\}) = \frac{1}{2}\frac{1}{2}.$$

But if the order were flipped we have:

$$\alpha_2^B(\{1,3\},\{\{1,3\},\{2\},\{4\},\emptyset\}) \cdot \alpha_3^B(\{1\},\{\{1\},\{2\},\{3,4\},\emptyset\}) = \frac{1}{2}\frac{1}{3}.$$

Thus,

$$pr^{\alpha^B}_{4||2||3}(\{1\},\{\{1\},\{2\},\{3,4\},\emptyset\}) \neq pr^{\alpha^B}_{4||3||2}(\{1\},\{\{1\},\{2\},\{3,4\},\emptyset\}).$$

The *externality-free marginality* proposed by Pham Do and Norde [11] was introduced in Section 3.1 when we discussed two values that limit the space of all extensions at two opposite extremes:

$$\alpha_i^{free}(S,P) = \begin{cases} 1 & \text{if } P(i) = \{i\} \\ 0 & \text{otherwise.} \end{cases}$$

According to this definition, there exists only one transfer – that is forming a new coalition – with non-zero weight. It is not difficult to notice that α^{free} is expansion resistant, because the weight of a transfer from (S,P) does not depend on S. As far as the interlace resistance property is concerned, a product of weights will evaluate to 1 (the only non-zero value) if and only if the corresponding transfers always create a new coalition. Thus, the order of transfers does not have an impact on the value. This means that weights α^{free} are interlace resistant.

In Chapter 3 we used weights which are dual to the previous ones to derive McQuillin's value [33]:

$$\alpha_i^{full}(S,P) = \begin{cases} \frac{1}{|P_{-i}|-1} & \text{if } P(i) \neq \{i\} \text{ or } P = \{N_{-i},\{i\},\emptyset\} \\ 0 & \text{otherwise.} \end{cases}$$

Here, the zero weight is associated with forming a new coalition (unless i leaves the grand coalition and there are no coalitions to join) and transfers to all the other coalitions have equal probability. Analogously to the externality-free marginality, α^{full} is expansion resistant and also interlace resistant. To see this, let us consider a product of weights that corresponds to a given sequence of transfers. If any player forms a new coalition, the product evaluates to zero; if not, the size of the partition does not change and all weights equal $\frac{1}{|P_{-i}|-1}$. Clearly, in both cases the power of weights does not depend on the order of transfers.

Another concept, introduced in Section 4.3 to illustrate bargaining process, was proposed by Macho-Stadler *et al.* [29]:

$$\alpha_i^{MS}(S,P) = \frac{|P(i)_{-i}|}{\sum_{T \in P_{-S}} |T|}$$

with the assumption that $|\emptyset| = 1$. The authors assumed that weight of forming a new coalition is relatively small, but when such a coalition becomes bigger, the player is more likely to join it. Here, expansion resistance is satisfied, as S is not counted in the denominator. Now, consider interlace resistance. We will prove that indeed $\alpha_i(S,P)\cdot\alpha_j(S_{+i},\tau_i^S(P)) = \alpha_j(S,P)\cdot\alpha_i(S_{+j},\tau_j^S(P))$ for every embedded coalition (S,P) such that $i,j \notin S$ (based on Lemma 7 this condition is sufficient). The product of the denominators (which simply increases by one after every transfer) appears in the formula on both sides. On the other hand, the numerator equals $(|P(i)|-1)(|P(j)|-1)$ (and $(|P(i)|-1)(|P(i)|-2)$ if both players are in the same coalition) regardless of the order of players.

The last marginality that results from the assumption that all partition will be formed with the same probability was proposed by Hu and Yang [25]. We already mentioned it in Section 5.1. Here, weights are defined as follows:[1]

$$\alpha_i^{HY}(S,P) = \frac{|\{R \in \mathcal{P}(N) : R_{[S]} = P_{[S]}\}|}{|\{R \in \mathcal{P}(N) : R_{[S \cup \{i\}]} = P_{[S \cup \{i\}]}|}$$

Intuitively, the numerator equals the number of partitions that contain the same partition of players $N \setminus S$. In the denominator, we assume that i has not left S yet – we count partitions that contain the same partition of players $N \setminus (S \cup \{i\})$. This marginality is not resistant to expansion, as the size of the S affects the proportion between the numerator and the denominator. E.g., $\alpha_3^{HY}(\{1\},\{\{1\},\{2\},\{3\},\emptyset\}) = \frac{3}{5}$, and $\alpha_3^{HY}(\{1,4\},\{\{1,4\},\{2\},\{3\},\emptyset\}) = \frac{10}{15}$.

[1]Note that here N is the set of all players in P. As we consider α for (S_{-j},P_{-j}) set N changes.

	non-negative	positive	interlace resistant	expansion resistant
Bolger	✓	✓	—	✓
Pham Do & Norde	✓	—	✓	✓
Skibski (Chapter 3)	✓	—	✓	✓
Hu & Yang	✓	✓	✓	—
Macho-Stadler *et al.*	✓	✓	✓	✓

Table 6.1: The properties of existing weights in the marginality approach.

On the other hand, Hu and Yang's marginality satisfies interlace resistance: product of two consecutive weights $\alpha_i(S,P) \cdot \alpha_j(S_{+i}, \tau_i^S(P))$ simplifies to $\frac{|\{R \in \mathcal{P}(N): R_{-S} = P_{-S}\}|}{|\{R \in \mathcal{P}(N): R_{-(S \cup \{i,j\})} = P_{-(S \cup \{i,j\})}\}|}$ regardless of order of i and j (again, this comes from Lemma 7).

All the above observations are summarized in Table 6.1. As we can see, most of the definitions of marginality satisfy our properties of the interlace and expansion resistance. The only value that meets all four properties – that is Weak and Strong Monotonicity, Strong Symmetry and the Strong Null-Player Axiom – is the value proposed by Macho-Stadler *et al.*

Finally, we address the value proposed by Myerson [37]. It also satisfies Shapley's axioms and can be derived using the marginality approach. However, Myerson's axiomatization based on the concept of carrier is far from the marginality analysis and results in a complex weights that do not meet any of the four properties. For instance, for $N = \{1,2,3\}$ and coalition $\{1,2\}$ embedded in partition $\{\{1,2\},\{3\},\emptyset\}$ the weight of transfer of 1 to a new coalitions equals $\alpha_1(\{2\},\{\{1\},\{2\},\{3\},\emptyset\}) = 2$, while joining player 3: $\alpha_1(\{2\},\{\{1,3\},\{2\},\emptyset\}) = -1$.

6.2 Axiomatic results

We can divide the work on extending the concept of the Shapley value to games with externalities into three bodies of literature. First, we discuss the marginality-based axiomatizations of values that satisfy Shapley's axioms. Next, we present other axiomatizations that yield values satisfying Shapley's axioms. Finally, we briefly address all the values that violate Shapley's axioms.

First, including our results from Chapter 3, three papers proposed new definitions of marginality and proved uniqueness based on Shapley's standard axiomatization [11, 25]. Pham Do and Norde, and Hu and Yang proposed new values, while we provided a marginal axiomatization for McQuillin's value (discussed in Chapter 3). These uniqueness results are the special cases of our Theorem 3. Some other authors used Young's axiomatization – Bolger modified it by adding an additional Null-Player Axiom to derive his value [7]; and De Clippel and Serrano in their analysis of externality-free value [10]. These results for Young's axiomatization were generalized by Fujinaka [15]. He was the first to propose a general formula for marginal contribution as the affine combination of elementary marginal contributions. Fujinaka proved that Young's axiomatization parametrized by any weights α implies a unique value. Our Theorem 3 is the equivalent of Fujinaka's result but for Shapley's axiomatization.

Macho-Stadler, Perez-Castrillo and Wettstein [29] proposed the average approach that was discussed in detail in Section 5.3, where we showed that it is equivalent to the marginality approach with interlace resistant weights (see Corollary 1)[2]. Using the average approach, the authors provided a value using Shapley's axioms together with Strong Symmetry (see Section 5.3) and Similar Influence. This latter axiom says that, if we exchange the values of two embedded coalitions in which players i and j appear in the first one together and, in the second one, as singletons, then their payoffs should not change. Although axiomatization departed from marginality, the authors introduce a definition of marginal contribution and note that value can be transformed as the weighted average of player's marginal contributions.

McQuillin [33] analyzed extending the Shapley value to games with externalities combined with Owen generalization [40]. If we specify how the payoffs of players should be generalized for payoffs of coalition (to this end, McQuillin provided a Rule of Generalization), then we can treat the payoffs as a game (as all coalitions have assigned payoffs). McQuillin argued that stability is reached when a given payoff is a fixed-point of this process (*i.e.* if we consider a value to be a game by itself, then the value computed for such a game should be the same). McQuillin called this requirement Recursion and proved that combined with Rules of Generalization, Weak Monotonicity and Shapley's axioms implies a unique value.

[2]We note that a one-way proof for positive weights was already presented by Hu and Yang [25]. More precisely, they proved that every value obtained using average approach with positive weights can be derived using marginality approach.

Myerson was the first to propose a new extension of the Shapley value to games in a partition-function form [37]. He based his value on the concept of Carrier. We say that a set C is a carrier if the value of any embedded coalition is determined by a partition of players from C. Now, Carrier implies that, if C is a carrier, then the payoff of the grand coalition is divided between players from C. Against this, Myerson showed that there exists a unique value that satisfies Symmetry, Additivity and Carrier. As the set of all players, N, is clearly a carrier and, if i is a null-player, then $N \setminus \{i\}$ is also a carrier, we have that Carrier implies both Efficiency and the Null-Player Axiom. This means that Myerson's value satisfies all four of Shapley's axioms.

Other authors proposed values that are rather far from Shapley's understanding of fairness.

Albizuri *et al.* [2] argued that, in a game with externalities, a coalition should be evaluated by the set of values it has, regardless of which partitions these values correspond to. The authors combined this principle, called Embedded Coalition Anonymity, with The Oligarchy Axiom (which can be understood as the weakened Myerson axiom) and three of Shapley's original axioms: Efficiency, Additivity and Symmetry. The resulting value can be derived as the Shapley value for a game without externalities calculated by assigning to every coalition an arithmetic average of all its values in games with externalities. Although, at first, it seems like a special case of the average approach, proposed weights violates the condition necessary to satisfy the Null-Player Axiom (see Theorem 1 in [29]).

In a stochastic process described by us in Section 4.3, players leave the grand coalition one by one. Grabisch and Funaki [19] formulate a different process. They take as a starting point the partition containing singletons of all players and consider all possible sequences of mergers which result in the grand coalition. That said, the contribution of a player is evaluated as the effect that the player merging with other coalitions makes on their values. If a player enters some coalition alone, he is rewarded with the whole change of its value, *i.e.*, with the marginal contribution; but if he is already a part of a coalition that merges with another one, Grabisch and Funaki argue that the change of the value of the coalition they merge with should be divided equally between him and other members of the coalition. This contradicts the Null-Player Axiom, as a null-player is rewarded with a payoff even though the coalition without him would cause the same impact on the merged coalition.

Finally, let us address the concepts proposed by Maskin [32] and Hafalir [21]. Here, the authors discarded the assumption that the grand coalition will form and proposed to divide the payoff of the optimal coalition structure. Maskin studied the coalition formation process and proposed an axiomatic characterization of a value expected in this process. Hafalir proposed a mechanism that implements a unique payoff division and provided axiomatization based on the idea of efficient-cover. These ideas, although interesting, result in an axiomatization significantly different to Shapley's.

We summarize the extensions of Shapley value to games with externalities in Table 6.2.

	MARGINAL VALUES - **ALL CAN BE DERIVED FROM SHAPLEY'S AXIOMS (THEOREM 3)**							OTHER VALUES				
	(A) AXIOMATIZATIONS BASED ON MARGINALITY					(B) NON-MARGINALITY AXIOMATIZATIONS						
	Pham Do & Norde 2007	Hu & Yang 2009	Skibski (Chapter 3) 2011	Bolger 1989	De Clippel & Serrano 2008	Macho-Stadler et al. 2007	McQuillin 2008	Myerson 1977	Albizuri et al. 2005	Hafalir 2007	Maskin 2007	Grabisch & Funaki 2012
MARKOVIAN-ERGODIC AXIOM												×
SCV SYMMETRY												×
SCV NULL-PLAYER AXIOM												×
CONSISTENCY											×	
OPPORTUNITY WAGES											×	
LIMITED EFFICIENCY											×	
COAL. PARETO OPTIMALITY											×	
EC NULL-PLAYER AXIOM										×		
EC ADDITIVITY										×		
EC ANONYMITY										×		
FULLY EFFICIENCY										×		
EMB. COALITION ANONYMITY									×			
OLIGARCHY AXIOM									×			
CARRIER								×				
RECURSION							×					
RULES OF GENERALIZATION							×					
WEAK MONOTONICITY							×					
SIMILAR INFLUENCE						×						
STRONG SYMMETRY						×						
WEAK NULL-PLAYER AXIOM				×		×	×					
α-MARGINALITY AXIOM				○	○							
α-NULL-PLAYER AXIOM	○	○	○									
ADDITIVITY	×	×	×	×		×	×	×	×			×
SYMMETRY	×	×	×	×	×	×	×	×	×			
EFFICIENCY	×	×	×	×	×	×	×		×			×

Table 6.2: Existing axiomatizations (that guarantee uniqueness) for various extensions of Shapley value to games with externalities (× denotes the axiom used and ○ denotes a special case of the axiom). EC stands for Efficient-Cover, SCV – for the Scenario-value.

Chapter 7

Approximation of the Shapley Value

This chapter presents the first – to our knowledge – approximation algorithm for evaluating extended Shapley values from the partition function form. We present the general scheme that works for every α-value (thus, based on Theorem 4, every value that satisfies direct translation of Shapley's axioms to games with externalities). The general scheme is based on the Monte Carlo sampling. In games without externalities the natural way is to sample over all possible permutation and for a given permutation gather players' marginal contributions. In games with externalities we randomly select not only permutation, but also partition. Here, the probability distribution is given by the weights α and strongly depends on the value that is approximated. Thus, for every existing α we specify how to choose a random partition with the corresponding probability. Especially, we propose a new method for selecting a random partition that is needed for Hu and Yang's value.

Note: This chapter is based on [50].

7.1 Approximation algorithm

To approximate the extended Shapley value for any weighting α we will use the following sampling process. Let the population be the set of pairs $(\pi, P) \in \Omega(N) \times \mathcal{P}$. In one sample, given permutation π and partition P, we will measure for each player i his elementary marginal contribution.

Algorithm 1: Approximation of α-value

1 **for all** $i \in N$ **do** $\hat{\varphi}_i^\alpha \leftarrow 0$;
2 **for** $i \leftarrow 1$ **to** m **do**
3 $\quad$ $\pi \leftarrow$ *random permutation from* $\Omega(N)$
4 $\quad$ $P \leftarrow$ *random partition of set* N *with distribution* pr_π^α *(see Algorithm 2)*
5 $\quad$ $S \leftarrow \emptyset$
6 $\quad$ **for** $j \leftarrow |N|$ ***downto*** 1 **do**
7 $\quad\quad$ $v_{before} \leftarrow v(S, P)$
8 $\quad\quad$ *transfer player* $\pi^{-1}(j)$ *in* P *to* S
9 $\quad\quad$ $v_{after} \leftarrow v(S, P)$
10 $\quad\quad$ $\hat{\varphi}_{\pi(j)}^\alpha \leftarrow \hat{\varphi}_{\pi(j)}^\alpha + v_{after} - v_{before}$
11 **for all** $i \in N$ **do** $\hat{\varphi}_i^\alpha \leftarrow \hat{\varphi}_i^\alpha / m$
12 **return** $\hat{\varphi}^\alpha$

As visible in the formula for α-value in Theorem 3, elementary marginal contributions do not occur with the same probability. Thus, to obtain an unbiased estimate we will use *probability sampling* with the odds of selecting a given sample (π, P) equal $pr_\pi^\alpha(\emptyset, P)/|N|!$. To this end, we will select a random permutation (each with equal probability: $1/|N|!$) and then select a partition with probability $pr_\pi^\alpha(\emptyset, P)$.[1] It is important to note that this probability depends on the definition of marginality (hence the α in the superscript) and the difficulty of the sampling process may vary depending on the definition adopted. We will address this issue later.

The pseudocode of this procedure is presented in Algorithm 1. Our procedure, which approximates α-value, is parametrized by the game v and number of samples m. We will discuss the required number of samples at the end of this section. The main for-loop sums samples elementary marginal contributions (variable $\hat{\varphi}^\alpha$). At the end, this sum is divided by the number of samples. To compute the players' contribution we reverse the process of creating partition P from the grand coalition (according to the intuition outlined before): we sequentially transfer players to the new (empty at start) coalition that represents a meeting point and measure the change of its value.

Now, let us focus on the randomized part of our algorithm (lines 3-4).

[1]Recall that, intuitively, $pr_\pi^\alpha(\emptyset, P)$ represents the probability that P will form if players leave the meeting point in order π.

Algorithm 2: Random partition of set N with distribution pr_π^α

switch α **do**
    case *externality-free weights*
        return $\{\{i\} \mid i \in N\}$
    case *full-of-externalities weights*
        return $\{N\}$
    case *Bolger's weights*
        $P \leftarrow \emptyset$
        for $i \leftarrow 1$ **to** $|N|$ **do**
            $m \leftarrow$ *random number from* 1 *to* $|P| + 1$
            if $m = |P| + 1$ **then** $P \leftarrow P \cup \{\pi^{-1}(i)\}$
            else *add* $\pi^{-1}(i)$ *to the coalition number* m
        return P
    case *Macho-Stadler's et al. weights*
        $P \leftarrow \emptyset$
        $\pi_2 \leftarrow$ *random permutation from* $\Omega(N)$
        for *every cycle* $i, \pi(i), \pi(\pi(i)), \ldots$ **do**
            add cycle as a coalition to P
        return P
    case *Hu and Yang's weights*
        return *random partition of set* N *(see Algorithm 3)*

We generate a random permutation using a well-known Knuth shuffle [12]. As mentioned before, the selection of a partition depends on the definition of α weights. That is why we are able to approximate all (marginality-based) extensions of the Shapley value and, in particular, those already proposed in the literature. We present the pseudocode in Algorithm 2.

Externality-free value

In the simplest concept of *externality-free value*, the whole probability is assigned to creation of a new coalition. Thus, the only partition with non-zero probability is the partition of singletons:

$$pr_\pi^{\alpha^{free}}(\emptyset, P) \stackrel{\text{def}}{=} 1 \text{ if } P = \{\{i\} \mid i \in N\},$$

and $pr_\pi^{\alpha^{free}}(\emptyset, P) \stackrel{\text{def}}{=} 0$ otherwise. Here, selection of P is straightforward.

FULL-OF-EXTERNALITIES VALUE

To obtain this value, proposed by McQuillin, in Chapter 3 we used the marginality that complements the previous one – the non-zero probabilities are assigned to transfers to the existing coalitions (with special case for the first player which must create a new coalition). If so, there will be only one coalition outside and the grand coalition will form at the end. Thus, the probability distribution simplifies to the following form:

$$pr_\pi^{\alpha^{full}}(\emptyset, P) \stackrel{\text{def}}{=} 1 \text{ if } P = \{N\},$$

and $pr_\pi^{\alpha^{full}}(\emptyset, P) \stackrel{\text{def}}{=} 0$ otherwise. Again, the random selection simplifies to generating one specific partition.

BOLGER VALUE

In Bolger's definition of marginal contribution every transfer is equally likely. The probability of partition $pr_\pi^{\alpha^B}(\emptyset, P)$ depends on the order in which the players leave:[2]

$$pr_\pi^{\alpha^B}(\emptyset, P) \stackrel{\text{def}}{=} \frac{1}{|P_{-\pi^{-1}(\{1,2,\ldots,|N|\})}| + 1} \cdot \frac{1}{|P_{-\pi^{-1}(\{2,\ldots,|N|\})}| + 1} \cdots \frac{1}{|P_{-\pi^{-1}(\{|N|\})}| + 1},$$

where $|P_{-\pi^{-1}(\{k,\ldots,|N|\})}|$ is the number of different coalitions of players from the first $k-1$ positions of permutation π. To select a partition with adequate probability, we simulate the process of leaving as follows: we take players from the permutation one by one and uniformly select one of the existing coalitions to join or a new one to create.

MACHO-STADLER ET AL. VALUE

In the value proposed by Macho-Stadler et al. [29], the weights of the transfer depend on the size of coalitions. Our approach to generate a random partition

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[2] For example, consider $pr_\pi^{\alpha^B}(\emptyset, \{N_{-i}, \{i\}\})$. If i is the last player in permutation π then $pr_\pi^{\alpha^B}(\emptyset, \{N_{-i}, \{i\}\}) = \frac{1}{1}\frac{1}{2}\frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^{|N|-1}}$. On the other hand, if it is the first one: $pr_\pi^{\alpha^B}(\emptyset, \{N_{-i}, \{i\}\}) = \frac{1}{1}\frac{1}{2}\frac{1}{3} \cdots \frac{1}{3} = \frac{1}{2 \cdot 3^{|N|-2}}$.

comes from the observation that probability

$$pr_{\pi}^{\alpha^{MSt}}(\emptyset, P) = \frac{\prod_{T \in P} |T-1|!}{|N|!}$$

equals the odds that a given partition will occur from the decomposition into disjoint cycles of a randomly selected permutation. Thus, we can again generate a random permutation with Knuth shuffle and divide players into coalitions according to the cycles of this permutation.

Hu and Yang value

Hu and Yang [25] designed their value in such a way that probabilities of every partition are equal:

$$pr_{\pi}^{\alpha^{HY}}(\emptyset, P) = \frac{1}{|\mathcal{P}(N)|}.$$

To generate a random partition we introduce the following technique (Algorithm 3).[3] We will create the partition successively (loop in lines 4-9), for each player selecting the coalition to join with probability that corresponds to the number of partitions of N that cover (i.e., respect) the obtained partial partition. For example, player 2 will join player 1 with probability $\frac{Bell(n-1)}{Bell(n)}$. This is because both players appear in $Bell(n-1)$ of $Bell(n)$ partitions together. Now, the number of partitions of N that cover given partial partition P_k of k players depends only on the number of coalitions in P_k. Thus, all the probabilities of transfers to the existing coalitions are the same. Based on this analysis, we will first randomly decide whether the player forms a new coalition (line 6) and if not, we will pick any existing coalition, all with the same probability (lines 8-9). We note here that the probability of creating a new coalition by player $k+1$ entering partition P_k can be precalculated, i.e., calculated once, before the sampling. This can be done in $\mathcal{O}(n^2)$ time using dynamic programing. The proper pseudocode is presented in function *precalculation*: $W[i][j]$ represents a number of partitions that cover a given partition P of i players with j coalitions and $PoN[i][j]$ – the percentage of these partitions in which player $i+1$ form a singleton coalition. What is also important from the computational point of view, this ratio is not less that $\frac{1}{|N|}$, thus we avoid precision problems that arises in other methods proposed in the literature [52].

[3] We thank our students for proposing this method.

Algorithm 3: Random partition of set N

```
function getRandomPartition begin
    if PoN is not initialized then PoN ← precalculation()
    P ← ∅
    for i ← 1 to |N| do
        r ← random double from (0, 1)
        if r < PoN[i][|P|] then P ← P ∪ {i}
        else
            m ← random number from 1 to |P|
            add i to the coalition number m in P
    return P
function precalculation begin
    n ← |N|
    PoN ← new array[n][n]; W ← new array[n][n]
    for i ← 1 to n do W[n][i] ← 1
    for i ← n − 1 downto 1 do
        for j ← 1 to i do
            W[i][j] ← W[i + 1][j + 1] + k · W[i + 1][j]
            PoN[i][j] = W[i + 1][j + 1]/W[i][j]
    return PoN
```

7.2 Error analysis

Let us briefly discuss the number of samples needed to obtain a required precision of the result. It is clear from the Theorem 3 that the estimator is unbiased: $E[\hat{\varphi}_i^\alpha] = \varphi_i^\alpha$. The variance equals $V[\hat{\varphi}_i^\alpha] = \frac{\sigma^2}{m}$ where m is the number of samples and σ^2 is the variance of the population:

$$\sigma^2 = \frac{1}{|N|!} \sum_{\pi \in \Omega(N)} \sum_{P \in \mathcal{P}} pr_\pi^\alpha(\emptyset, P) \cdot ([v(C_i^\pi \cup \{i\}, P_{[C_i^\pi \cup \{i\}]}) - v(C_i^\pi, P_{[C_i^\pi]})] - \varphi_i^\alpha(v))^2.$$

Now, based on the central limit theorem, $\hat{\varphi}_i^\alpha \sim N(\varphi_i^\alpha, \frac{\sigma^2}{m})$. Assume we want to obtain an error not bigger than ϵ with the probability not smaller than $1-\beta$, i.e., we need to satisfy the following inequality: $P(|\hat{\varphi}_i^\alpha - \varphi_i^\alpha| \leq \epsilon) \geq 1-\beta$. But $P(|\hat{\varphi}_i^\alpha - \varphi_i^\alpha| \leq \epsilon) = \Phi(\frac{\epsilon \cdot \sqrt{m}}{\sigma}) - \Phi(\frac{-\epsilon \cdot \sqrt{m}}{\sigma}) = 2 \cdot \Phi(\frac{\epsilon \cdot \sqrt{m}}{\sigma}) - 1$ where $\Phi(x)$

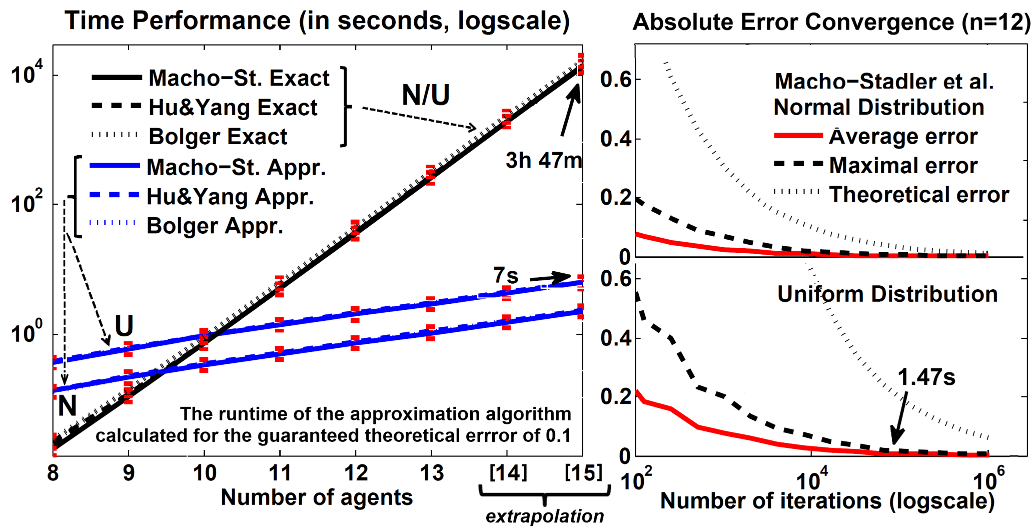

Figure 7.1: Performance evaluation of Algorithm 1.

is the cumulative distribution function of the standard normal distribution. Therefore, $\Phi(\frac{\epsilon \cdot \sqrt{m}}{\sigma}) \geq 1 - \frac{\beta}{2}$ and finally: $m \geq \frac{\sigma^2}{\epsilon^2} \cdot (\Phi^{-1}(1 - \frac{\beta}{2}))^2$, where $\Phi^{-1}(x)$ is the quantile function, i.e., $P(X \geq \Phi^{-1}(x)) = x$ for $X \sim N(0, 1)$. For example, for the uncertainty $\beta = 0.01$ holds $\Phi^{-1}(0.995) \approx 2.57$.

Next, we need to find an upper bound for σ^2. To this end, following Castro et al. [9], we will assume that we know some limits min_i, max_i on the player's marginal contribution, i.e., for every $\pi \in \Omega(N)$ and $P \in \mathcal{P}$ $min_i \leq v(C_i^\pi \cup \{i\}, P_{[C_i^\pi \cup \{i\}]}) - v(C_i^\pi, P_{[C_i^\pi]}) \leq max_i$. Then, the σ^2 is maximized when all marginal contributions equal min_i or max_i, and the average equals $\frac{min_i + max_i}{2}$ (to achieve this, the sum of probabilities of maximal marginal contributions must equal the sum of probabilities of minimal marginal contributions). Finally, $\sigma^2 \leq \frac{(max_i - min_i)^2}{4}$.

7.3 Performance evaluation

We test our algorithm on two distributions popular in the literature on coalitional games [27]:

- normal distribution: $v(S, P) = |S| \cdot N(1, 0.1)$; here, the bounds are: $min_i = -n \cdot 0.6 + 1.3$ and $max_i = n \cdot 0.6 + 0.7$.

- uniform distribution: $v(S,P) = |S| \cdot U(0,1)$; here, in a extreme case, $min_i = -n+1$ and $max_i = n$.

where, in the case of the normal distribution, we place the following additional limits: $0.7 \leq N(1, 0.1) \leq 1.3$.[4]

Figure 7.1(a) presents the time performance of our algorithm for $n = 8, 9, \ldots, 15$ and compares it to the exact brute-force approach – the only known alternative.[5] For $n = 11$ our approximation algorithm outperforms the exact brute-force at the first time. Already for $n = 15$, it would take almost 4 hours to compute the exact output (extrapolated result), whereas our algorithm returns the approximated solution in less than 7 seconds (with the guaranteed error of 0.1).

Figure 7.1(b) shows that the maximal error obtained from the random game is a few times lower than the theoretical error (for both distributions). For instance, for game of 12 players, which takes the brute-force algorithm more than 37 seconds to calculate, the maximal error of 0.018 is obtained after 1.47 seconds (65K samples). Moreover, the error clearly tends to zero, which shows that our estimator is indeed unbiased.

[4]These instances only happen with probability 0.14%.

[5]The simulations run on a PC-i7, 3.4 GHz and 8 GB of RAM.

PART II

GAMES ON GRAPHS

Chapter 8

Introduction to Part II

While the conventional model of a coalitional game assumes that any coalition can be created and may have an arbitrary value, there are many realistic settings where this assumption does not hold. Often, agents (or players) can communicate and cooperate only via some limited number of bilateral channels. If there is no direct channel between two agents, cooperation can be still possible indirectly, through an intermediary or a sequence of them. However, when no direct or indirect connection exists between agents, they cannot coordinate their activities. Such restrictions emerge in a variety of domains including: sensor networks, telecommunications, social networks analysis, trade agreements, political alliances, etc.

An influential approach for representing such scenarios was introduced by Myerson [36], who described a coalitional game over a graph in which nodes represent agents and edges represent communication channels between them. Such a game is often called a *graph-restricted game*. Some of the literature on graph-restricted games assumes that a coalition is feasible only if there exists a (direct or indirect) communication channel between any two of its members. Such a coalition is said to be connected, as it induces a connected subgraph of the underlying graph. Other work allows for the existence of both connected and disconnected coalitions. Here, disconnected coalitions are either all assumed to have a value of 0 (as in the model formalized by Amer and Gimenez [5]) or their values equal to the sum of values of the disjoint components they are composed of (as formalized by Myerson [36]).

In our work, we study the computational aspects of the two key solution concepts proposed for the above graph-restricted games, namely the *Shapley value* and the *Myerson value*. Specifically:

- In his seminal work, Shapley proposed a formula to quantify the contribution of an individual agent to the outcome achieved by all agent working together in one coalition. While this formula, known as the Shapley value, satisfies many desirable properties, it is defined only for settings where all (both connected and disconnected coalitions) are feasible.

- On the other hand, Myerson proposed a solution concept for the settings in which only connected coalitions are feasible [36]. This solution concept, known as the Myerson value, also has a number of attractive properties (see Section 8.1).

The computation of both of these values is challenging [34, 3]. As for the Shapley value, Michalak et al. recently proposed an algorithm to compute it for an arbitrary graph-restricted game [34]. As for the Myerson value, its computational aspects were considered for certain classes of graphs and/or games [3, 18, 13]. However, to date there is no algorithm for computing the Myerson value in arbitrary graphs.

Our aim in this work is to develop efficient algorithms for computing the Shapley value and the Myerson value. In particular, our contributions can be summarised as follows:

- In Chapter 9, we propose a new algorithm for the enumeration of all connected induced subgraphs of the graphs – one of the fundamental operations in graph theory. We show that our algorithm is faster than the state of the art, due to Moerkotte and Neumann [35]. We also show that, unlike the state of the art, our algorithm can easily be extended to capture extra information about each enumerated subgraph.

- In Chapter 10, building upon the above enumeration algorithm, we propose two new algorithm to compute the Shapley and Myerson value for graph-restricted games. We show that the algorithm for Shapley value is faster than the state of the art, due to Michalak et al. [34]. Algorithm for the Myerson value is, to our knowledge, the first one in the literature for arbitrary graph.

- In Chapter 11, we test both algorithms on an interesting application, recently proposed by [28], who used the Shapley value of graph-restricted games to measure importance of different members of a terrorist network. Our results suggest that the Myerson value-based measure is more suitable for this application.

- In Chapter 12, we address the problem of approximation of Shapley value for graph-based games – in many applications even the fastest exact algorithms fail to return the result in a reasonable amount of time. To this end, we propose an approximation algorithm for Shapley value in graph-restricted games and for two more complex definitions of a game used in the *gatekeepers metric* [44].

8.1 Preliminaries

A graph $G = (V, E)$ consists of vertices (or nodes) V and edges $E \subseteq V \times V$. Let $V' \subseteq V$ be a subset of vertices and let $E(V') \subseteq E$ denote the set of all edges between them in G. Now, the subgraph *induced* by V' is the pair $(V', E(V'))$. Since in this paper we do not consider non-induced subgraphs, we will often omit the word *induced* when there is no risk of confusion. A subgraph is *connected* if its edges form a path between any two of its nodes. We will denote the set of all connected induced subgraphs of G by $\mathcal{C}(G)$ (or simply $\mathcal{C}$ wherever G is clear from the context). We say that $v \in V$ is a *cut vertex* in connected graph (V, E) if its removal splits the graph, i.e., if $(V \setminus \{v\}, E(V \setminus \{v\}))$ is not connected. If the subgraph induced by V' is not connected, then it surely consists of several connected components, denoted $K(V') = \{K_1, K_2, \dots, K_m\}$. Finally, for any vertex $v \in V$, we denote by $\mathcal{N}(v)$ the set of neighbours of v.

Having discussed some fundamental notions of graph theory, let us turn now to cooperative game theory. The players in this part of our work are represented by nodes in graph G. In other words, V is interpreted as the set of players (or agents). Consequently, we will use the terms coalition and subgraph interchangeably. Especially, a coalition S is said to be connected if and only if the subgraph of G induced by S is connected. Otherwise, the coalition is said to be disconnected. All other definitions are consistent with the ones introduced in the overview of our work: ν, which denotes a game without externalities, assigns to every coalition of agents a real number and Shapley value, denoted by $SV(\nu)$ and defined with formulas (1) and (2), returns an evaluation (worth) of every player in game ν.

In graph-restricted games, which were first studied by Myerson [36], only connected coalitions could be assigned an arbitrary value (as the agents within are able to communicate and create value added). To formalise this class of games, let us first consider a new value function which corresponds to ν but is only defined over connected coalitions:

$$\nu_G : \mathcal{C}(G) \to \mathbb{R} \text{ and } \forall_{S \in \mathcal{C}(G)} \nu_G(S) = \nu(S)$$

This definition can be extended to incorporate disconnected coalitions; this has been done in two ways:

- Myerson argued that it is natural to consider a disconnected coalition as a set of disjoint, connected components. Each such component S' is, by definition, a coalition in $\mathcal{C}(G)$ whose members are able to attain a payoff of $\nu_G(S') = \nu(S')$. This leads to the following characteristic function, defined over both connected and disconnected coalitions [36]:

$$\nu_G^{\mathcal{M}}(S) = \begin{cases} \nu(S) & \text{if } S \in \mathcal{C}(G) \\ \sum_{K_i \in K(S)} \nu(K_i) & \text{otherwise,} \end{cases} \tag{8.1}$$

  where $\mathcal{M}$ stands for Myerson. In other words, the payoff available to a disconnected coalition is the sum of payoffs of its connected components.

- More recently, Amer and Gimenez [5] formalized an alternative approach to evaluate disconnected coalitions, where they assumed that all such coalitions have a value of 0. Under this assumption, they defined the following characteristic function of a simple game:[1]

$$\nu_G^{\mathcal{A}}(S) = \begin{cases} 1 & \text{if } S \in \mathcal{C}(G) \\ 0 & \text{otherwise,} \end{cases} \tag{8.2}$$

  where $\mathcal{A}$ stands for Amer and Gimenez. The game with the above function will be called a 0-1-*connectivity game*. This function was later on extended by Lindelauf et al. [28] to:

$$\nu_G^{f}(S) = \begin{cases} f(S,G) & \text{if } S \in \mathcal{C}(G) \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

  where f is an arbitrary function.

[1] Simple coalitional games are a popular class of games, where every coalition has a value of either 1 or 0.

Since $\nu_G^{\mathcal{M}}$ and $\nu_G^{\mathcal{A}}$ are defined over all $2^{|V|}$ coalitions, the Shapley value can be applied as a solution concept to both of these functions. However, this is not the case with ν_G. Now, a celebrated result of Myerson [36] is the solution concept he proposed for ν_G. In particular, Myerson showed that by allocating to agent $v_i \in V$ the payoff $MV_i(\nu_G)$, which is defined as follows:

$$MV_i(\nu_G) = SV_i(\nu_G^{\mathcal{M}}), \tag{8.4}$$

we obtain the unique payoff division scheme that is efficient and rewards any two connected agents equally from the bilateral connection between them. This payoff division scheme is known as the *Myerson value*.

Chapter 9

DFS Enumeration of Induced Connected Subgraphs

The enumeration of induced connected subgraphs is one of the fundamental algorithmic operations in many applications, e.g., cost-based query optimization [35], computing topological indices for molecular graphs [39], and searching for an optimal coalition structure in cooperative games on graphs [54]. A number of algorithms have been proposed to perform this operation. The early works include reverse search algorithms [6], and a breadth-first search algorithm [47]. Both of these algorithms, however, performed numerous redundant operations. This issue was later on resolved by Moerkotte and Neumann [35].[1]

In this chapter, we present our algorithm for enumerating all induced connected subgraphs, and then benchmark its performance against the state of the art by Moerkotte and Neumann. Our algorithm offers not only better performance, but also its depth-first search structure allows for gathering additional information on the structure of each enumerated subgraph if needed. This algorithm will be used in the subsequent chapters as the cornerstone upon which we build our algorithms for computing the Shapley value and the Myerson value for graph-restricted games.

Note: This chapter is based on [49].

[1] Recently, the same breadth-first search algorithm of Moerkotte and Neumann was rediscovered by Voice et al. [54].

9.1 Our algorithm

Broadly speaking, our enumeration algorithm traverses the graph in a depth-first manner, and uses a *divide-and-conquer* technique. We start with a single node and try to expand it to a bigger connected subgraph. Whenever a new node is analyzed, we explore all its edges one by one, and when we find a new – not yet discovered – node, we split the calculations into two parts: in the first one, we add a new node to our subgraph; in the second one, we mark this node as forbidden and never enter it again. Thus, the first part enumerates subgraphs with, and the second one without, the new node.

The pseudocode is presented in Algorithm 4. First, let us describe the recursive function $DFSEnumerateRec$. Whenever this function is called, nodes in the graph can be divided into three groups: S – elements of the subgraph; X – forbidden nodes (that we cannot include in the subgraph); and others – not yet discovered nodes. Moreover, nodes in S are either *partially-processed* or *fully-processed*. A node is fully-processed when all its edges have been explored. What is crucial, as in the classic DFS algorithm, is that *all partially-processed nodes form a path to the root of the subgraph tree* (parameter $path$), i.e., we do not process another edge until the node from the previous one is fully-processed. The last parameter of $DFSEnumerateRec$ is $startIt$, which – to avoid redundancy – indicates how many edges of the last node on $path$ have already been processed. This parameter is set to 1 as we enter a new node (lines 4 and 10) and can be deduced from the neighbour list whenever we backtrack from another node (line 14).[2]

Now, the goal of the function is to find all connected subgraphs that contain subgraph S and contain no forbidden nodes X. To this end, we start processing from the last node on the path (line 6), denoted v, and explore sequentially all its edges (lines 7-11). Whenever we find a new node u (that is neither forbidden nor included in S) we first enumerate all subgraphs with u: here, we call $DFSEnumerateRec$ with u added at the end of the path of partially-processed nodes (line 10). Then, to enumerate all subgraphs without u, we add it to the set of forbidden nodes (line 11) and proceed with a new edge. Finally, when all edges have already been explored, we remove v from the path and backtrack to the previous node (lines 12-15). When we have finished processing the last node on the path (root), the set

[2] As function $find(v)$ in line 14 is called multiple times, the proper values can be pre-calculated in the main function $DFSEnumerate$ and stored in the associative array to facilitate constant access time.

Algorithm 4: DFS Enumeration of Induced Connected Subgraphs

```
   Input: Graph G=(V, E)
   Output: List of all induced connected subgraphs of G
1  DFSEnumerate begin
2      sort nodes and list of neighbours by degree desc.;
3      for i ← 1 to |V| do
4          DFSEnumerateRec(G, (v_i), {v_i}, {v_1, ..., v_{i−1}}, 1);

5  DFSEnumerateRec(G, path, S, X, startIt) begin
6      v ← path.last();
7      for it ← startIt to |N(v)| do
8          u ← N(v).get(it); // it's neighbour of v
9          if u ∉ S ∧ u ∉ X then
10             DFSEnumerateRec(G, (path, u),  {u}, X, 1);
11             X ← X ∪ {u};

12     path.removeLast();
13     if path.length() > 0 then
14         startIt ← N(path.last()).find(v) + 1;
15         DFSEnumerateRec(G, path, S, X, startIt);
16     else print S;
```

S constitutes a final connected subgraph (line 16).

In the main function *DFSEnumerate* (lines 1-4), the i-th step of the loop enumerates all subgraphs in which the node with the smallest index is v_i (line 4). To this end, we simply mark previous nodes as forbidden and call the function *DFSEnumerateRec* with the node v_i as the initial subgraph.

The time complexity of our algorithm is linear in the number of connected subgraphs: $\mathcal{O}(|\mathcal{C}||E|)$. This follows from the fact that the number of steps performed for a given connected subgraph is $\mathcal{O}(|E|)$. To see how this is the case, consider a sequence of calls of *DFSEnumerateRec* that results in printing subgraph S in line 16. We consider a subgraph to be final if *path* is empty; thus, all nodes from S must be fully-processed. Moreover, all other nodes are either forbidden or not-discovered; thus, they are not added to *path* in this sequence (the recursive call in which we consider adding a forbidden node to the subgraph is calculated in our analysis for other connected subgraph). Therefore, the lines 9-11 from the single loop are entered once for every edge adjacent to a node in S, thus no more than $2|E|$ times. Moreover, every call of the function decreases the number of edges to discover, or decreases the number of nodes on the path; thus, other lines are called no more than $2|E| + |S|$ times. As $|S|$ is connected, $|S| \leq |E| + 1$ and the number of steps is $\mathcal{O}(|E|)$.

The running time of the algorithm depends on the order in which we process nodes (line 3) and nodes' neighbours (line 10). The optimal order of nodes is an open problem. In our experimental analysis we found that the order descending by the degree of the node can lead to a smaller number of steps. Therefore in line 2 we sort the nodes accordingly.

9.2 DFS vs. BFS enumeration of induced connected subgraphs

To date, the state-of-the-art algorithm for enumerating connected induced subgraphs was proposed by Moerkotte and Neumann [35]. As opposed to our algorithm, which traverses the graph in a depth-first manner, their algorithm uses breadth-first search. The pseudocode is presented in Algorithm 5. Specifically, in the i-th step of the main function, $EnumerateCSG$, the algorithm enumerates subgraphs with v_i and without previous nodes. The recursive function, named $EnumerateCSGRec$, is called with four parameters: graph G, an Old part of the subgraph, a New part of the subgraph, and the set of all nodes that we already considered, denoted X (the nodes from the subgraph and nodes we have considered but have not included). Now, $EnumerateCSGRec$ outputs the current subgraph ($Old \cup New$) and tries to enlarge it. In order to do that, it lists all not-yet considered neighbours (set N) and for every subset $S \subseteq N$ analyzes an adequate extension – it calls $EnumerateCSGRec$ with the subgraph enlarged by S and set of considered nodes expanded by all neighbours N.[3]

Our experiments show that the new algorithm outperforms BFS enumeration two or even three times. In particular, Figure 9.1 depicts the running

[3] Our pseudocode is more detailed than the original. If we merge both parts of the subgraph (Old and New) in the declaration (and calls) of $EnumerateCSGRec$, then to find neighbours in lines 7-11 we have to consider also nodes from the old part of the subgraph. This is clearly redundant, as all their neighbours are already in the set X.

Algorithm 5: BFS Enumeration of Induced Connected Subgraphs

Input: Graph $G{=}(V, E)$
Output: List of all Induced Connected Subgraphs of G

1 $EnumerateCSG$ **begin**
2   **for** $i \leftarrow 1$ **to** $|V|$ **do**
3         $EnumerateCSGRec(G, \emptyset, \{v_i\}, \{v_1, \ldots, v_i\})$;

4 $EnumerateCSGRec(G, Old, New, X)$ **begin**
5   **print**$\{Old \cup New\}$;
6     $N \leftarrow \emptyset$; `// not yet discovered neighbours of` New
7   **foreach** $v \in New$ **do**
8   **foreach** $u \in \mathcal{N}(v)$ **do**
9   **if** $u \notin X \cup N$ **then** $N \leftarrow N \cup \{u\}$;
10   **foreach** $S \subseteq N$ **do**
11         $EnumerateCSGRec(G, Old \cup New, S, X \cup N)$;

time for scale-free graphs, typically used to model contact networks. Graphs were generated using the preferential attachment generation model [1] with parameter $k = 4$ (we obtained analogous results for different values of k). For every $n = 20, \ldots, 30$, the run time and confidence intervals are calculated based on 500 random graphs (same for both algorithms). As can be seen, as n increases, the ratio of both algorithms does not change and oscillates at around 2.4. For instance, for $n = 30$, our algorithm takes on average 67 seconds, while it takes 161 seconds for BFS enumeration to finish.

To support our empirical results, we provide two lemmas, which show that for cliques our algorithm performs approximately two times fewer steps (examining edges is the key component of main loops in both algorithms).

Lemma 8. *$EnumerateCSG$ examines edges $2^{n-1}(n^2 - 3n + 2) + (n - 1)$ times for an n-clique.*

Proof. Let us calculate the number of steps needed to enumerate all connected subgraphs with v_k, but without previous nodes ($i = k$ in line 3). In the first call, $EnumerateCSGRec(G, \emptyset, \{v_k\}, \{v_1, \ldots, v_k\})$ checks all $n-1$ edges of v_k, acknowledge all nodes not from X as new neighbours and for every non-empty subset $S \subseteq \{v_{k+1}, \ldots, v_n\}$ call $EnumerateCSGRec(G, \{v_k\}, S, V)$ in line 11.

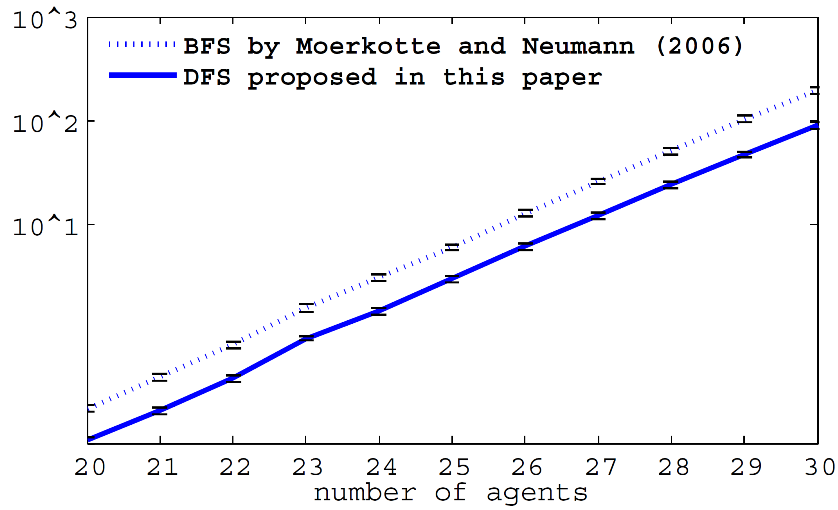

Figure 9.1: Comparison between algorithms for enumerating induced connected subgraphs: our new DFS-based algorithm and the state-of-the-art, BFS-based algorithm by Moerkotte and Neumann.

In every such call all edges of subset S are considered ($|S| \cdot (n-1)$), but no not-processed nodes are yet to found. Thus, no later calls take place and the total number of steps equals:

$$\begin{aligned}\sum_{k=1}^{n}((n-1) + \sum_{S \subseteq \{v_{k+1},\dots,v_n\}} |S|(n-1)) &= (n-1)\sum_{k=1}^{n}(n-k)2^{n-k-1} \\ &= 2^{n-1}(n^2-3n+2)+(n-1).\end{aligned}$$

□

Lemma 9. *$DFSEnumerate$ examines edges $2^{n-2}(n^2-n+4)-(n+1)$ times for an n-clique.*

Proof. We will count how many times an edge from v_k is processed. To do that, we will consider all calls of function $DFSEnumerateRec$ when v_k is the last node on the path. This happens in two situations: when we add v_k to our subgraph, thus we discover it for the first time (lines 4 and 10) and when we backtrack from other node (line 15). We will consider them separately.

The main observation of our proof is as follows:

Note 1. *Whenever $DFSEnumerateRec$ is called with v_k at the end of $path$ for the first time (with $startIt = 0$) all nodes with lower id must be either forbidden (in X) or partially-processed (in S and on $path$) and all nodes with higher id must not be discovered yet.*

This comes from the fact that we always consider a new node with the lowest id and the previous nodes cannot be fully-processed as long as they have not fully-processed neighbor. Now, all combination of the previous nodes must be possible and all can be achieved only once (as we always divide the space of connected subgraphs into disjoint parts), thus we discover node v_k 2^{k-1} times.

Now, let us consider how many times we enter the node v_k not for the first time (thus, when we backtrack from other node v_m). The following facts comes from the Note 1:

- $m > k$ (because we entered v_k before v_m),
- all nodes are discovered (otherwise, we wouldn't backtrack from v_m)
- all nodes with ids bigger than k are forbidden or fully-processed (partially-processed node would imply that we entered him before v_k);
- all nodes with id between k and m are forbidden (fully-processed node implies that we entered v_m before him).

And again, all nodes with ids lower than k are either forbidden or partially-processed. As all combinations of nodes appear exactly one in DFS algorithm, we backtrack from v_m to v_k exactly $2^{k-1} \cdot 2^{n-m}$ times.

That is enough to calculate how many times we consider an edge from node v_k – when we enter v_k for the first time (with $startIt = 0$) we process all $n-1$ edges; when we enter v_k backtracking from v_m last considered edge from v_k must have been (v_k, v_m), thus $startIt = m-1$ and we process the remaining $(n-1)-(m-1)$ edges:

$$\sum_{k=1}^{n} 2^{k-1}(n-1) + \sum_{m=k+1}^{n} 2^{k-1} 2^{n-m}(n-m) = 2^{n-2}(n^2 - n + 4) - (n+1).$$

□

Chapter 10

Algorithms for the Graph-restricted Games

In this chapter we present algorithms that calculate Shapley and Myerson value for the graph-restricted games. Both algorithms are based on the DFS-based enumeration of induced connected subgraphs presented in the previous chapter.

Note: This chapter is based on [49].

10.1 Shapley value for the graph-restricted games

In this section, we present a new algorithm for calculating the Shapley value for graph-restricted games based on formula (8.3) (i.e., connectivity games by Amer and Gimenez [5] with arbitrary function f). As mentioned earlier in the introduction, there already exists an algorithm designed for this purpose, due to [34], and we aim to develop a more efficient algorithm.

First, we show that to calculate the Shapley value it is suffices to traverse only the connected coalitions, because every non-zero marginal contribution involves the addition, or removal, of an agent from a connected coalition. In more detail, let S be an arbitrary connected coalition. Now, agents' contributions can be divided into three groups:

(a) a cut vertex (i.e., a node whose removal disconnects the subgraph S) contributes the entire value of the coalition, i.e., $f(S)$;

(b) any other member of S (whose removal does not disconnect S) contribute the following change in the value: $f(S) - f(S \setminus \{v\})$ for node v;

(c) finally, we have the nodes that are not members nor neighbours of S. The addition of any such node disconnects S, implying that it makes a negative contribution equal to $-f(S)$.

Note that we did not consider the contribution of neighbours of S. This is because, for every such neighbour, v, its contribution will be taken into account when dealing with $S \cup \{v\}$ instead of S. Above comments are summarized in the following theorem:

Theorem 8. *Shapley value for graph-restricted games based on formula (8.3) (i.e., connectivity games by Amer and Gimenez [5] with arbitrary function f) satisfies the following formula:*

$$SV_i(\nu_G^f) = \sum_{S \in \mathcal{C}} mc_i(S),$$

where $mc_i(S)$ stands for

$$mc_i(S) = \begin{cases} \xi_S \, f(S) & \text{if } v_i \in S \text{ and } S \setminus \{v_i\} \notin \mathcal{C}, \\ \xi_S \left(f(S) - f(S \setminus \{v_i\}) \right) & \text{if } v_i \in S \text{ and } S \setminus \{v_i\} \in \mathcal{C}, \\ -\xi_{S \cup \{x_i\}} \, f(S) & \text{if } v_i \notin S \text{ and } S \cup \{v_i\} \notin \mathcal{C}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will decompose the standard formula for the Shapley value (2) into four cases, where a coalition with or without an agent is either connected or disconnected:

$$\begin{aligned} SV_i(\nu_G^f) &= \sum_{S \subseteq N, v_i \in S} \xi_S \left(\nu_G^f(S) - \nu_G^f(S \setminus \{v_i\}) \right) \\ &= \sum_{\substack{S \subseteq N, v_i \in S \\ S \in \mathcal{C}, S \setminus \{v_i\} \in \mathcal{C}}} \xi_S \left(f(S) - f(S \setminus \{v_i\}) \right) + \sum_{\substack{S \subseteq N, v_i \in S \\ S \in \mathcal{C}, S \setminus \{v_i\} \notin \mathcal{C}}} \xi_S \left(f(S) - 0 \right) \\ &\quad + \sum_{\substack{S \subseteq N, v_i \in S \\ S \notin \mathcal{C}, S \setminus \{v_i\} \in \mathcal{C}}} \xi_S \left(0 - f(S \setminus \{v_i\}) \right) + \sum_{\substack{S \subseteq N, v_i \in S \\ S \notin \mathcal{C}, S \setminus \{v_i\} \notin \mathcal{C}}} \xi_S \left(0 - 0 \right), \end{aligned}$$

where zeros for disconnected coalitions follow directly from the definition of Amer and Gimenez connectivity game. Now, we remove the fourth case (which is 0), and in the third one we iterate through subsets S without v_i (instead through subsets with v_i):

$$\begin{aligned} \ldots \;=\; & \sum_{\substack{S \subseteq N, v_i \in S \\ S \in \mathcal{C}, S \setminus \{v_i\} \in \mathcal{C}}} \xi_S \left(f(S) - f(S \setminus \{v_i\}) \right) \\ & + \sum_{\substack{S \subseteq N, v_i \in S \\ S \in \mathcal{C}, S \setminus \{v_i\} \notin \mathcal{C}}} \xi_S \, f(S) - \sum_{\substack{S \subseteq N, v_i \notin S \\ S \cup \{v_i\} \notin \mathcal{C}, S \in \mathcal{C}}} \xi_{S \cup \{v_i\}} \, f(S). \end{aligned}$$

Thus, we can now traverse only connected coalitions as follows:

$$\ldots = \sum_{\substack{S \in \mathcal{C}, v_i \in S \\ S \setminus \{v_i\} \in \mathcal{C}}} \xi_S \left(f(S) - f(S \setminus \{v_i\}) \right) + \sum_{\substack{S \in \mathcal{C}, v_i \in S \\ S \setminus \{v_i\} \notin \mathcal{C}}} \xi_S \, f(S) - \sum_{\substack{S \in \mathcal{C}, v_i \notin S \\ S \cup \{v_i\} \notin \mathcal{C}}} \xi_{S \cup \{v_i\}} \, f(S).$$

This concludes the proof. □

Based on the above observations it is crucial to not only enumerate all connected subgraphs, but also identify the cut vertices, and the neighbours, of each enumerated subgraph. As for the identification of neighbours, it can easily be done. The harder part is to identify the cut vertices. To this end, in [34], the authors combined Moerkotte and Neumann's enumeration algorithm with the state-of-the-art algorithm for finding cut vertices, due to Hopcroft and Tarjan [23]. The way in Michalak et al. combined the two algorithms involved some additional improvements, see their paper for more details.

Against this background, we present the first dedicated algorithm that not only enumerates all connected subgraphs, but at the same time identifies cut vertices in each subgraph. To make this possible, our algorithm traverses all connected subgraphs in a depth-first-search (DFS) manner (as discussed in the previous section). Consequently, unlike the case with Moerkotte and Neumann's breadth-first-search (BFS) technique, our DFS techniques ensures that the edges which are used to enlarge the subgraph always form what is known as a Tremaux tree [23] – an important structure in graph theory. More specifically, a Tremaux tree of graph G is a rooted spanning tree – a subgraph consisting of all nodes and a subset of edges, which forms a tree, with one node selected as a the root. Importantly, for any Tremaux tree, T, and any two nodes that have an edge between them in G, it is guaranteed that one of those two nodes is an ancestor of the other in T. Now, let us show how this property of Tremaux trees helps identify cut vertices in a subgraph.

Algorithm 6: DFS-based algorithm for calculating Shapley value for the graph-restricted games

Input: Graph $G = (V, E)$, function $\nu : \mathcal{C} \to \mathbb{R}$
Output: Shapley value for game ν_G^f

$DFSConnSV(G)$ **begin**
    sort nodes and list of neighbours by degree desc.;
    for $i \leftarrow 1$ **to** $|V|$ **do** $SV_i(\nu_G^f) = 0$;
    for $i \leftarrow 1$ **to** $|V|$ **do**
        $DFSConnSVRec(G, (v_i), (\infty), \{v_i\}, \{v_1, \ldots, v_{i-1}\}, \emptyset, \emptyset, 1)$;

$DFSConnSVRec(G, path, low, S, X, SC, XN, startIt)$ **begin**
    $v \leftarrow path.last()$; $l \leftarrow low.last()$;
    for $it \leftarrow startIt$ **to** $|\mathcal{N}(v)|$ **do**
        $u \leftarrow \mathcal{N}(v).get(it)$; `// it's neighbour of v`
        if $u \notin S \land u \notin X$ **then**
            $DFSConnSVRec(G, (path, u), (low, \infty), S \cup \{u\}, X, SC, XN, 1)$;
            $X \leftarrow X \cup \{u\}$; $XN \leftarrow XN \cup \{u\}$;
        else if $u \in X$ **then** $XN \leftarrow XN \cup \{u\}$;
        else if $(path.find(u) < low.last())$ **then**
            $l \leftarrow path.find(u)$;
            $low.updateLast(l)$;
    $path.removeLast()$; $low.removeLast()$;
    if $path.length() > 0$ **then**
        if $l \geq path.length()$ **then** $SC.add(path.last())$;
        else if $l < low.last()$ **then** $low.updateLast(l)$;
        $startIt \leftarrow \mathcal{N}(path.last()).find(v) + 1$;
        $DFSConnSVRec(G, path, low, S, X, SC, XN, startIt)$;
    else
        if v *was added only once* SC **then** $SC.remove(v)$;
        foreach $v_i \in SC$ **do**
            $SV_i(\nu_G^f) \leftarrow SV_i(\nu_G^f) + \xi_S(f(S))$;
        foreach $v_i \in S \setminus SC$ **do**
            $SV_i(\nu_G^f) \leftarrow SV_i(\nu_G^f) + \xi_S(f(S) - f(S \setminus \{v_i\}))$;
        foreach $v_i \in V \setminus (S \cup XN)$ **do**
            $SV_i(\nu_G^f) \leftarrow SV_i(\nu_G^f) - \xi_{S \cup \{v_i\}}(f(S))$;

To this end, let v be an arbitrary node in G. Consider a subtree S rooted at a child of v. The removal of v from graph G disconnects nodes from S if and only if there is no edge in G that connects S to other parts of the graph. From the property of Tremaux tree, all such potential edges would go to the ancestors of v (we will call them *backedges*). Thus, to identify cut vertices, it suffices to know the node nearest to the root that can be reached from the children's subtrees. This information can be easily updated recursively when we backtrack in a depth-first search – for the subtree rooted at v, it is one of the nodes connected to v or one of the nearest nodes that can be reached from its subtrees.

The pseudocode is presented in Algorithm 6. To gather extra information, we expand the recursive function from Algorithm 4 by a few new parameters. Assume that a root is on level 1, and its children are on level 2, and so on. Now, for each node v from the *path*, list *low* stores the lowest level that can be reached from v (using already discovered edges) or its fully-processed children. The set SC contains identified cut vertices. Now, whenever we add a node v to a path (lines 5 and 11) we initialize its *low* to infinity. Then, we update this value in two situations. The first is when we find a backedge from v to the lower level (lines 14-16). The second is when we backtrack from a child with a lower value (line 20, *low* for child equals l, parent is the last node on the *path*). When we backtrack and child's value is not lower that the level of the parent (thus, subtree of a son does not have a backedge to any node closer to the root) we add the parent node to the set of cut vertices SC (line 19). Finally, in line 24, we remove root from the set of cut vertices if it has only one child in a tree, i.e., was added to this set just one time. The set of neighbours XN of the nodes in S can be easily updated as we consider all edges of nodes in the main loop (lines 8-16). Now, based on both sets, we calculate the Shapley value in lines 25-30 for every found connected coalition. The asymptotic time of the algorithm has not changed with respect to enumeration of connected subgraphs and equals $\mathcal{O}(|\mathcal{C}||E|)$.[1]

In Figure 10.1 we compare the performance of our new algorithm with the FasterSVCG proposed by Michalak et al. [34]. As in the comparison of DFS and BFS enumeration in Section 9.2, here we generated 500 random scale-free graphs for every size of graph n from 20 to 30 and computed average

[1] Note, that to obtain a constant time of $path.find(u)$ operation in line 14 list $path$ should be enriched with a associative array or alternatively additional array of nodes' levels can be passed along.

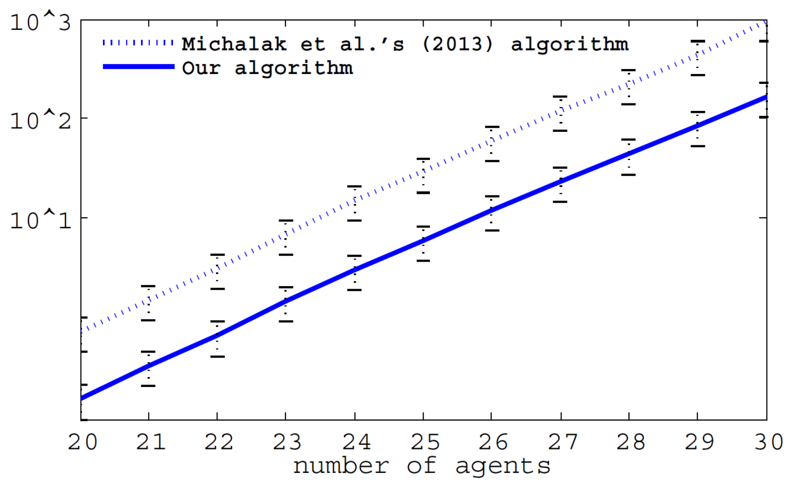

Figure 10.1: Comparison between algorithms for the Shapley value for connectivity games: our new DFS-based algorithm vs. the state-of-the-art BFS-based algorithm proposed by Michalak et al.

running time for both algorithms. As we can see, our DFS-based algorithm is more than 5 times faster for every size n. Importantly, only part of the advantage comes from the faster enumeration algorithm, which is 2-3 times faster than the alternative used in FasterSVCG. Moreover, the advantage increases very slightly with the number of nodes. We justify this as follows: although the pessimistic number of steps for every found connected subgraph in both enumeration algorithms equals $\mathcal{O}(E)$, for sparse graphs it does not exceed a small constant. Thus, separate searching for the cut vertices, which is a part of the FasterSVCG, is an additional action which takes linear time in respect to the number of edges – $\mathcal{O}(E)$ – while our new algorithm performs only $\mathcal{O}(V)$ steps updating Shapley value for all nodes.

10.2 MYERSON VALUE FOR THE GRAPH-RESTRICTED GAMES

In this section we present the first dedicated algorithm for computing the Myerson value for arbitrary graph-restricted games. Depending on the definition of function ν, the complexity of calculating the Myerson value may vary. In our paper we do not make any assumption for the form of function

ν: in our algorithm we treat ν as an oracle which gives the values, and for the complexity analysis we assume it does that in a constant time.

As in the previous section, we can argue that only traversing connected coalitions is necessary. Moreover, based on the oracle-assumption, we have to go through all of them (the size of the input is the number of values of ν). Thus, what is crucial is to minimize the number of steps performed for every connected coalition. In this section we prove that in the context of the Myerson value, identifying of cut vertices is not needed, and the number of steps is linear in the size of graph nodes.

To this end, we will use the permutation interpretation of the Shapley value (see formula (1)) and analyze the marginal contribution of a node, but this time more thoroughly. Let π be a permutation and assume agents that precede v_i form the components $K_1, K_2, \ldots, K_m$, the first j of which are connected to v_i. Now, as in Myerson's characteristic function value of a coalition equals sum of values of it's components, we can simplify marginal contribution to $\nu(K_1 \cup \ldots \cup K_j \cup \{v_i\}) - \nu(K_1) - \ldots - \nu(K_j)$: all components not connected with v_i contributes their values to the value of a coalition with and without v_i. Now, instead of considering marginal contribution as a whole, as in the previous section, we will decompose it into two parts: a positive and a negative one. Thus, we will calculate separately how many times (i.e., for which of the permutations) the value of a coalition $\nu(K_1 \cup \ldots \cup K_j \cup \{v_i\})$ with v_i is added, and how many times a value of a given coalition K_l without v_i is subtracted from his payoff.

Consider a connected coalition S and the Myerson value of v_i:

- if v_i is in S then the value of S is taken into account with a positive sign whenever two conditions are met: (1) all nodes from S appear in the permutation before v_i, and (2) all neighbours of S appear in the permutation after v_i. This happens with a probability of $\frac{(|S|-1)! \cdot |\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!}$, where $\mathcal{N}(S)$ is the set of neighbours of S.

- if v_i is not in S, then the value of S is taken into account with a negative sign, but only if all nodes from S appear in the permutation before v, and all neighbours of S appear after v_i, and v_i is a neighbour of S (otherwise, v_i contributes to some other coalition). This happens with a probability of $\frac{|S|! \cdot (|\mathcal{N}(S)|-1)!}{(|S|+|\mathcal{N}(S)|)!}$.

It is worth noting that in the second case we allow neighbours of v that are not connected to S to appear before v_i. Although they change the coalition resulting from the appearance of v_i and also the subtracted part of the marginal contribution, the value of S is *still* a component of this part. Above comments can be formalized into the following theorem.

Theorem 9. *The Myerson value for graph-restricted games satisfies the following formula:*

$$MV_i(\nu_G) = \sum_{S \in \mathcal{C}, v_i \in S} \zeta_1 \, \nu(S) - \sum_{\substack{S \in \mathcal{C}, v_i \notin S \\ S \cup \{v_i\} \in \mathcal{C}}} \zeta_2 \, \nu(S),$$

where

$$\zeta_1 = \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!}, \quad \zeta_2 = \frac{|S|!(|\mathcal{N}(S)|-1)!}{(|S|+|\mathcal{N}(S)|)!}$$

and $\mathcal{N}(S)$ denotes the set of neighbours of coalition S.

Proof. Based on the result by Myerson, we have that $MV_i(\nu_G) = SV_i(\nu_G^M)$. In other words, formula (1) for the Shapley value becomes:

$$MV_i(\nu_G) = \frac{1}{|V|!} \sum_{\pi \in \Pi} \left(\nu_G^M(S_i^\pi) - \nu_G^M(S_i^\pi \setminus \{v_i\}) \right). \tag{10.1}$$

Now, let us consider contribution of agent v_i to coalition S. Based on the definition of game ν_G^M, we have:

$$\nu_G^M(S) - \nu_G^M(S \setminus \{v_i\}) = \sum_{K_j \in K(S)} \nu(K_j) - \sum_{K_j \in K(S \setminus \{v_i\})} \nu(K_j).$$

As connected components of $S \setminus \{v_i\}$ which are not connected with v_i will appear in both sums, we can simplify the above formula to:

$$\nu_G^M(S) - \nu_G^M(S \setminus \{v_i\}) = \nu(K^i(S)) - \sum_{K_j \in K(K^i(S) \setminus \{v_i\})} \nu(K_j),$$

where $K^i(S)$ denotes component of v_i in S.

Next, we reconsider formula (10.1) for the Myerson value. We have:

$$\begin{aligned} MV_i(\nu_G) &= \frac{1}{|V|!} \sum_{\pi \in \Pi} \left(\nu(K^i(S_i^\pi)) - \sum_{K_j \in K(K^i(S_i^\pi) \setminus \{v_i\})} \nu(K_j) \right) \\ &= \sum_{S \in C, v_i \in S} \frac{|\{\pi \in \Pi \mid S = K^i(S_i^\pi)\}|}{|V|!} \nu(S) \\ &- \sum_{S \in C, v_i \notin S} \frac{|\{\pi \in \Pi | S \in K(K^i(S_i^\pi) \setminus \{v_i\})\}|}{|V|!} \nu(S) \end{aligned} \tag{10.2}$$

In the first sum above, the condition $S = K^i(S_i^\pi)$ means that S is the component of v_i in S_i^π. Thus, given that $v_i \in S$, this condition is equivalent to $S \in K(S_i^\pi)$. In the second sum, we count permutations in which S is one of the components that v_i merges into $K^i(S_i^\pi)$. Thus, v_i must be the neighbour of S, and S must be the component of $S_i^\pi \setminus \{v_i\}$:

$$\begin{aligned} MV_i(\nu_G) &= \sum_{S \in C, v_i \in S} \frac{|\{\pi \in \Pi \mid S \in K(S_i^\pi)\}|}{|V|!} \nu(S) \\ &\quad - \sum_{\substack{S \in C, v_i \notin S \\ S \cup \{v_i\} \in \mathcal{C}}} \frac{|\{\pi \in \Pi \mid S \in K(S_i^\pi \setminus \{v_i\})\}|}{|V|!} \nu(S) \end{aligned}$$

Now, the first fraction is the probability that, in a random permutation, S will be a component in a subgraph formed by v_i and agents that precede v_i. This will be the case if and only if: (i) all agents $S \setminus \{v_i\}$ are in π before v_i; and (ii) all neighbours of S are in π after v_i. Thus, this probability equals $\frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!}$ as for each $|S| + |\mathcal{N}(S)|$ positions of agents $S \cup \mathcal{N}(S)$ in π there are $(|S|-1)!|\mathcal{N}(S)|!$ out of $(|S| + |\mathcal{N}(S)|)!$ such orderings:

$$\frac{|\{\pi \in \Pi \mid S = K^i(S_i^\pi)\}|}{|V|!} = \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S| + |\mathcal{N}(S)|)!} = \zeta_1.$$

In the second sum, all agents from S must appear in permutation π before v_i and all other neighbours of S must be in π after v_i. Analogously, the probability equals $\frac{|S|!(|\mathcal{N}(S)|-1)!}{(|S|+|\mathcal{N}(S)|)!}$:

$$\frac{|\{\pi \in \Pi \mid S \in K(S_i^\pi \setminus \{v_i\})\}|}{|V|!} = \frac{|S|!(|\mathcal{N}(S)|-1)!}{(|S| + |\mathcal{N}(S)|)!} = \zeta_2.$$

□

Based on the above observations, we construct our algorithm for computing the Myerson value based on our fast DFS-based algorithm for enumerating connected subgraphs (Algorithm 7). As in our previous algorithm for computing the Shapley value, here the set XN gathers neighbours of the set S, and lines 19-22 update the Myerson value according to the aforementioned marginal contribution analysis.

Algorithm 7: DFS-based algorithm for calculating Myerson value for graph-restricted games

Input: Graph $G = (V, E)$, function $\nu : \mathcal{C} \to \mathbb{R}$
Output: Myerson value for game ν_G

$DFSMyersonV(G)$ **begin**
    sort nodes and list of neighbours by degree desc.;
    for $i \leftarrow 1$ **to** $|V|$ **do** $MV_i(\nu_f) = 0$;
    for $i \leftarrow 1$ **to** $|V|$ **do**
        $DFSMyersonVRec(G, (v_i), \{v_i\}, \{v_1, \ldots, v_{i-1}\}, \emptyset, 1)$;

$DFSMyersonVRec(G, path, S, X, XN, startIt)$ **begin**
    $v \leftarrow path.last()$;
    for $it \leftarrow startIt$ **to** $|\mathcal{N}(v)|$ **do**
        $u \leftarrow \mathcal{N}(v).get(it)$; // it's neighbour of v
        if $u \notin S \wedge u \notin X$ **then**
            $DFSMyersonVRec(G, (path, u), S \cup \{u\}, X, XN, 1)$;
            $X \leftarrow X \cup \{u\}$; $XN \leftarrow XN \cup \{u\}$;
        else if $u \in X$ **then** $XN \leftarrow XN \cup \{u\}$;
    $path.removeLast()$;
    if $path.length() > 0$ **then**
        $startIt \leftarrow \mathcal{N}(path.last()).find(v) + 1)$;
        $DFSMyersonVRec(G, path, S, X, XN, startIt)$;
    else
        foreach $v_i \in S$ **do**
            $MV_i(\nu_G) \leftarrow MV_i(\nu_G) + \frac{(|S|-1)!(|XN|)!}{(|S|+|XN|)!}\ \nu(S)$;
        foreach $v_i \in XN$ **do**
            $MV_i(\nu_G) \leftarrow MV_i(\nu_G) - \frac{(|S|)!(|XN|-1)!}{(|S|+|XN|)!}\ \nu(S)$;

Chapter 11

Application to Terrorist Networks

There is currently much interest in the possibility of applying social network analysis techniques to investigate terrorist organizations [45]. A particular attention is paid to the problem of identifying key terrorists. This not only helps to understand the hierarchy within these organizations but also allows for a more efficient deployment of scarce investigation resources [26]. One possible approach to this problem is to try and infer the importance of different individuals from the topology of the terrorist network. In graph theory, such an inference can be obtained in various ways, depending on the adopted *centrality measures*, i.e., the adopted way to measure the centrality, i.e., importance, of different nodes in a network, based on its topology.[1] A number of researchers have proposed to incorporate game-theoretic techniques into existing centrality measures [20, 17]. The basic idea behind such *game-theoretic centrality measures* is to define a coalitional game over the network and then to construct a ranking of nodes based on a chosen solution concept. Although such an approach is often computationally challenging, it has the following two advantages. Firstly, the combinatorial analysis of the cooperative game, which is embedded in the solution concept, becomes a combinatorial analysis of the network. Secondly, this approach is very flexible, as it can be changed along three dimensions: (i) the coalitional game can be of an arbitrary form (e.g., partition function form); (ii) the value function can also be arbitrary; and (iii) there are many available solution concepts,

[1] We refer the reader to [8, 14] for an overview of most commonly used centrality metrics.

each based on different prescriptive and normative considerations.[2]

Note: This chapter is based on [49].

11.1 LINDELAUF ET AL.'S MEASURE

Lindelauf et al. tried to develop a centrality measure that assesses the role played by individual terrorists in a way that accounts for the following two factors: the terrorists' role in connecting the network and additional intelligence available about the terrorists. To this end, Lindelauf et al. proposed to use the Shapley value for v_G^f as defined in formula (8.3). This function seems to be suitable to achieve Lindelauf et al.'s goal. As discussed at the beginning of Section 12.1, it assigns high marginal contributions to cut vertices. Such vertices, by definition, play an important role in connecting various parts of the network. As such, any agent who is a cut vertex will be called a pivotal agent.

Furthermore, one can manipulate $f(S,G)$ in formula (8.3) to incorporate available information and analytical needs. In particular, to incorporate additional intelligence on individual terrorist, the authors assigned weights to both edges and nodes in G. Let us denote such weights by ω_{ij} and ω_i, respectively. Based on this, Lindelauf et al. proposed to use the following alternative functions:

$$\begin{aligned}&\text{(a) } f(S) = |E(S)|/\textstyle\sum_{(v_i,v_j)\in E(S)} \omega_{ij},\\ &\text{(b) } f(S) = \textstyle\sum_{v_i\in S} \omega_i.\end{aligned}$$

In words, in (a) $f(S)$ equals the number of edges in the connected coalition divided by the sum of their weights and, in (b) by the sum of its nodes' weights. As an example, the rationale behind function (b) is that terrorists (nodes) with high weights "*play an important part in the operation. When such individuals team up, they have a significant effect on the potential success of the operation.*" [28, p. 237].

Summarising, the key idea of Lindelauf et al. was to develop a measure that evaluates the nodes based on two factors: their role in connecting the network; and additional intelligence. At first glance, v_G^f seems to be a

[2] We refer the reader to the recent book by Maschler et al. [31] for an excellent overview of solutions concepts in cooperative game theory.

good candidate for this purpose. However, in what follows we will show the disadvantages of this approach and propose an alternative.

We argue that, for sparse networks – and terrorist networks tend to be sparse [26] – the *connectivity* factor is over-represented in the measure based on $SV_i(v_G^f)$. As a result, we claim that the *additional intelligence* factor hardly ever affects the ranking. To see how this is the case, it is sufficient to examine the characteristic function ν_G^f which (for reasonable f) results in relatively very high marginal contributions assigned to pivotal agents, while other agents are assigned incremental values or relatively very big negative values.

To support our claim, let us perform a sensitivity analysis of Lindelauf et al.'s centrality by (i) evaluating different forms of the function $f(S,G)$ and (ii) considering different weights of nodes. Regarding (i), we consider the following general form of $f(S,G)$:

$$f(S,G)=|S|^{\alpha} \cdot |E(S)|^{\beta} \cdot (\sum_{v_i \in S} \omega_i/|S|)^{\gamma} \cdot (\sum_{(v_i,v_j)\in E(S)} \omega_{ij}/|E(S)|)^{\delta}, \qquad (11.1)$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are parameters for exponents. We set α and β to be integer values between -2 and 2 and we impose the additional condition that $\alpha + \beta \geq 0$ in order to preserve monotonicity. We performed our tests on the terrorist network responsible for the World Trade Center attacks (36 nodes, 64 edges) [26]. This is a bigger version of the network originally considered by Lindelauf et al., which only contained 19 nodes.

Table 11.1 presents the results of this sensitivity check. For each configuration of parameters we calculated the ranking of nodes and compared it with the ranking for the 0-1-connectivity game. Here, we concentrate on the top $\sqrt{|V|} = 6$ terrorists, as the main goal of this application is to identify key players. The top part of Table 11.1 presents the average size of the intersection of the top 6 terrorists in both rankings. In the lower part of Table 11.1, for the top 6 terrorists from the 0-1-connectivity game, we calculated the average distance between their positions in both rankings, i.e. the ranking from the 0-1-connectivity game and the ranking from $f(S,G)$ (each cell presents the maximum and minimum value of this average).

We observe that the top 6 members have changed in 15% of the tests. Furthermore, in only 2% of the tests, more than one new node has been identified as a key member. Also, within this group, changes of positions are minor; the average change of position rarely exceeded 1.3. The dedicated

$\alpha\backslash\beta$		2	1	0	-1	-2
	$\gamma\backslash\delta$	1|0|-1	1|0|-1	1|0|-1	1|0|-1	1|0|-1
2	1|0|-1	6.0	6.0	5.9	5.9	4.9
1	1|0|-1	6.0	6.0	6.0	5.1	—
0	1|0|-1	6.0	6.0	5.7	—	—
-1	1|0|-1	6.0	6.0	—	—	—
-2	1|0|-1	6.0	—	—	—	—

$\alpha\backslash\beta$		2	1	0	-1	-2
	$\gamma\backslash\delta$	1|0|-1	1|0|-1	1|0|-1	1|0|-1	1|0|-1
2	1|0|-1	1.3-1.3	1.3-1.3	1.3-1.3	0.7-1.3	1.3-2.5
1	1|0|-1	1.3-1.3	1.3-1.3	1.0-1.3	0.8-2.0	—
0	1|0|-1	1.3-1.3	1.0-1.3	0.0-1.0	—	—
-1	1|0|-1	1.3-1.3	0.3-1.3	—	—	—
-2	1|0|-1	0.7-1.3	—	—	—	—

Table 11.1: Comparison between top nodes based on games with the parametrized characteristic function from formula (11.1) and the 0-1-connectivity game.

characteristic function proposed by Lindelauf *et al.* for the WTC network obtained with parameters $\alpha = \gamma = 1$ and $\beta = \delta = 0$ yields the same group of 6 terrorists ranked with only minor rotations.

The above simulations show that, for sparse networks, the choice of $f(S,G)$ essentially does not matter. The main reason behind this phenomenon appears to be the fact that ν_G^f assigns relatively very high contributions to pivotal agents, and only incremental marginal contributions to non-pivotal agents. This is magnified by the fact that we deal here with a sparse network. Specifically in this network, out of all 6 billions induced subgraphs, only 0.6% are connected. Furthermore, the average number of pivotal agents in a connected subgraph was high (8 to be precise). Thus, nodes that are crucial from the connectivity point of view will have a high ranking because v_G^f favours pivotal agents.

As already mentioned, we also analysed the sensitivity of the characteristic function ν_G^f with respect to the weights ω_i (these weights were permuted at random in our experiments). We focused on the second function proposed by Lindelauf et al., i.e., $f(S) = \sum_{v_i \in S} \omega_i$. Here, only 3% out of 600

$\alpha\backslash\beta$		2	1	0	-1	-2
	$\gamma\backslash\delta$	1|0|-1	1|0|-1	1|0|-1	1|0|-1	1|0|-1
2	1|0|-1	4.8	4.7	4.3	1.1	0.0
1	1|0|-1	5.0	4.9	1.1	0.0	—
0	1|0|-1	4.4	4.2	0.0	—	—
-1	1|0|-1	3.3	0.8	—	—	—
-2	1|0|-1	1.1	—	—	—	—

$\alpha\backslash\beta$		2	1	0	-1	-2
	$\gamma\backslash\delta$	1|0|-1	1|0|-1	1|0|-1	1|0|-1	1|0|-1
2	1|0|-1	1.2-2.3	1.0-2.2	0.8-4.2	6.7-24.5	23.8-28.2
1	1|0|-1	1.2-2.8	1.0-2.8	8.8-28.0	21.3-28.3	—
0	1|0|-1	1.7-3.7	2.5-8.2	23.7-30.0	—	—
-1	1|0|-1	2.0-7.0	13.0-26.2	—	—	—
-2	1|0|-1	11.2-21.8	—	—	—	—

Table 11.2: Comparison between top nodes based on games with the parametrized characteristic function from formula (11.1) and the game with value of -1 for every connected component.

permutations resulted in a change within the top 6 terrorists in the ranking. Furthermore, all these changes concerned only one terrorist (who was replaced by another).

In the next section, we argue that the Myerson value is a better centrality measure for terrorist networks.

11.2 MYERSON VALUE FOR TERRORIST NETWORKS

We observe that the characteristic function $\nu_G^{\mathcal{M}}$ used to compute $MV_i(\nu_G)$ does not favour pivotal agents as much as ν_G^f. In particular, given an arbitrary (non-negative and monotonic) function, $f(S, G)$, the marginal contribution of a pivotal agent v to a coalition $S \cup \{v\}$ in the case of the connectivity game will be:

$$f(S \cup \{v\}, G) - 0 = f(S \cup \{v\}, G).$$

However, in the case of $\nu_G^{\mathcal{M}}$, it will be:

$$f(S \cup \{v\}, G) - \sum_{K_i \in K(S)} \nu(K_i) \leq f(S \cup \{v\}, G).$$

This means that the connectivity factor becomes relatively less dominant in the evaluation of the nodes.

In Table 11.2, we present results of a similar sensitivity check as before, but now for the Myerson value. As visible, this measure is more sensitive to changes of $f(S, G)$ than the Shapley value. This is also confirmed by the sensitivity check with respect to random permutations of nodes' weights. Specifically, in more than 80% of the cases, the top 6 terrorists changed, and, in most of those cases, the change was substantial. In particular, on average about 2.2 terrorists were repeated among the top 6.

Chapter 12

Approximation Algorithms

Shapley value-based network metrics are designed mainly for social networks, but also, as described in the previous chapter, can be used for terrorist network. Unfortunately – from the computational point of view – this networks can be arbitrary large. For example, the group of terrorists that was behind the Madrid train bombing in 2004 exceeded 80 people. Exact algorithms, even those presented in our work, fail to calculate the answer in a reasonable amount of time. To this end, in this chapter we present two ideas how approximation algorithms can be constructed for metrics in which the idea of connectivity plays a crucial role. In Section 12.1, we present our algorithm for the Shapley value in graph-restricted games. In Section 12.2, we introduce a more complicated definition of a game and present a dedicated algorithm to calculate the corresponding metric.

Note: This chapter is based on [34] (Section 12.1) and [44] (Section 12.2).

12.1 Approximation of the Shapley value for graph-restricted games

In this section we propose a dedicated application of Monte Carlo sampling to graph-restricted games. Unlike the existing algorithm to approximate the Shapley value for characteristic function games [9], in our algorithm we do not sample permutations, but coalitions. Since any marginal contribution of an agent, v_i, links two coalitions – one with this agent, i.e., $C \cup \{v_i\}$, and one without him, i.e., C – sampling of coalition C can be viewed as *sampling* *of v_i's marginal contribution*. Generally speaking, in our algorithm, we will randomly select a number of marginal contributions of v_i and approximate the Shapley value using the resulting average. Due to the fact that in the Shapley value marginal contributions are calculated with different weights, to obtain an unbiased estimator we have to sample marginal contributions with appropriate probabilities. To this end, we propose the following general process.

- **Step 1**: uniformly select $k \in \{0, \dots, |V|\}$.
- **Step 2**: choose a random coalition C of size k.
- **Step 3**: for every agent, compute the marginal contribution of this agent obtained by leaving/entering C.

To see this process works, let us transform the formula for the Shapley value as follows:

$$SV_i(v) = \frac{1}{|V|} \sum_{0 \le k < |V|} \frac{1}{\binom{|V|-1}{k}} \sum_{C \subseteq V \setminus \{i\}, |C|=k} v(C \cup \{i\}) - v(C)$$

From this formula we get two conditions required to get an unbiased estimator are:

(i) the probability that a randomly chosen marginal contribution is obtained from entering a coalition of size k is equal (to $1/|V|$) for every k,

(ii) marginal contributions to all coalitions of size k are chosen with the same probability $(1/|V| \cdot \binom{|V|-1}{k})$.

It is easy to see that both conditions imply the probability for selecting a given marginal contribution, thus they are also sufficient to obtain an unbiased estimator. Our process satisfies both of them. The second one is obviously met as we do not favor any coalition of a given size in the first or second step. The first condition needs more precise analysis. Note that a given marginal contribution $v(C \cup \{i\}) - v(C)$ may appear in our process in two ways – when we select coalition C or when we select $C \cup \{i\}$. More specifically, marginal contribution of agent v_i to coalition of size k will be selected with the probability

$$\frac{1}{n+1} \cdot \frac{n-k}{n} + \frac{1}{n+1} \cdot \frac{k+1}{n},$$

Algorithm 8: Approximation algorithm for calculating Shapley value for graph-restricted games

Input: Graph $G = (V, E)$, function $\nu : \mathcal{C} \to \mathbb{R}$
Output: Shapley value for game $\nu_G^{\mathcal{A}}$

1 $ApproximateSVCG(G, \nu)$ **begin**
2   **foreach** $v_i \in V$ **do** $SV_i(\nu_G^{\mathcal{A}}) \leftarrow 0$;
3   **for** $it = 1$ **to** $maxIter$ **do**
4     $k \leftarrow$ *random number from* $\{0, \ldots, |V|\}$;
5     $C \leftarrow$ *random coalition of size* k;
6   **if** *!CheckConnectedness(C)* **then** *continue*;
7     $P \leftarrow FindPivotals(C)$;
8   **foreach** $v_i \in C \setminus P$ **do**
9       $SV_i(\nu_G^{\mathcal{A}}) \leftarrow SV_i(\nu_G^{\mathcal{A}}) + \frac{|V|+1}{|C|} \cdot (f(C) - f(C \setminus \{v_i\}))$
10   **foreach** $v_i \in P$ **do**
11       $SV_i(\nu_G^{\mathcal{A}}) \leftarrow SV_i(\nu_G^{\mathcal{A}}) + \frac{|V|+1}{|C|} \cdot f(C)$
12   **foreach** $v_i \in (V \setminus C) \setminus \mathcal{N}(C)$ **do**
13       $SV_i(\nu_G^{\mathcal{A}}) \leftarrow SV_i(\nu_G^{\mathcal{A}}) - \frac{|V|+1}{|V|-|C|} \cdot f(C)$
14   **foreach** $v_i \in V$ **do** $SV_i(\nu_G^{\mathcal{A}}) \leftarrow SV_i(\nu_G^{\mathcal{A}})/maxIter$;

where $n = |V|$. Here, the first term represents the probability that we select a coalition of size k without agent v_i, while the second one – that we select the coalition of size $k + 1$, with agent v_i. The above expression simplifies to $\frac{1}{n}$ which proves that first condition is also satisfied.

This technique allows us to compute the marginal contributions of all agents for a randomly selected coalition, which in the graph-restricted games (and, potentially, in many more classes of games) can be performed much faster than estimating the Shapley value for each agent separately [30] or by sampling of a random permutation [9], where we have to calculate the marginal contributions for a sequence of coalitions growing in size.

To this end, in the algorithm we merge our Monte Carlo technique with the analysis of marginal contribution. The pseudocode is presented in Algorithm 8. Line 4 corresponds to **Step 1**, where we sample the size of a coalition. Now, we modify **Step 2** in order to select only *connected coali-*

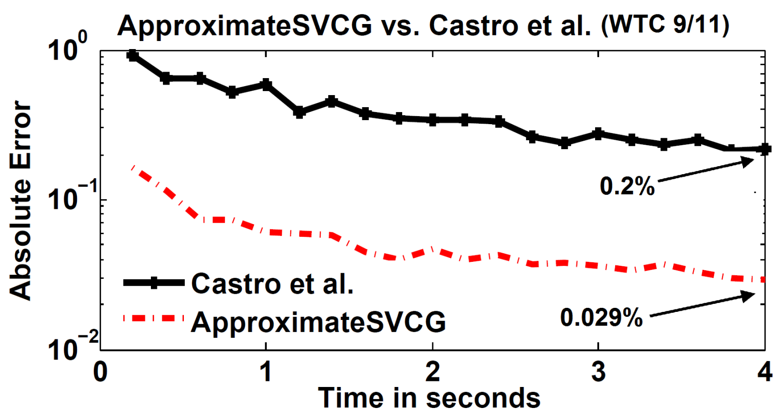

Figure 12.1: Error performance of ApproximateSVCG.

tions (lines 5 and 6).[1] We also modify **Step 3** where we consider cases (a), (b), and (c) from our analysis in Section 10.1 (lines 8-13). The modification of **Step 3** has to be done due to the following reason: since we no longer consider disconnected coalitions, any non-zero marginal contribution made to a disconnected coalition have to be transfered to a corresponding connected coalition (lines 9, 11, and 13). Furthermore, this should be done in a way that preserves appropriate probabilities (thus, in lines 9, 11, and 13 we multiply marginal contributions by adequate weights). The probabilities comes from the following analysis. Consider a pivotal agent i in coalition C of size $k+1$. Expected value from entering the disconnected coalition $C \setminus \{i\}$ equals $\frac{1}{n+1} \cdot \left(\frac{1}{\binom{n-1}{k-1}}\right) \cdot f(C)$. On the other hand, the same marginal contribution is obtained from leaving connected coalition S – here expected value equals $\frac{1}{n+1} \cdot \left(\frac{1}{\binom{n-1}{k}}\right) \cdot f(C)$. As we won't consider transfers from the disconnected coalitions, we have to ensure that expected value from this marginal contribution does not change, so if we obtain C with the probability $\frac{1}{n+1} \cdot \left(\frac{1}{\binom{n-1}{k}}\right)$ we add marginal contribution which equals $f(C) \cdot (1 + \frac{n-k+1}{k})$.

Finally, we divide the sum of the contributions by the number of iterations (line 14). For every sample, algorithm runs in time $O(|V| + |E|)$.

We end this section with performance evaluation of our approximate algorithm. Figure 12.1 presents the error convergence of our algorithm and compares the results to the random permutation sampling studied by Cas-

[1] It should be noted, that generating a random *connected coalition* uniformly will create a biased algorithm.

tro *et al.* [9]. Here, we focus on the maximum absolute error of the Shapley value, computed as a percentage of the value of the grand coalition. The results are calculated for Krebs' 9/11 WTC terrorist network with 36 nodes (as an average from over 30 iterations). Since connected coalitions constitute only a small subset of all coalitions (here, only 0.59%), our algorithm is likely to outperform Castro's method. Indeed, after 4 second, the error equals 0.029%, while for Castro et al. exceeds 0.2%. The fact that the absolute error ultimately converges to zero indicated that our sampling method is not biased.

12.2 GATE-KEEPERS METRIC

Game theoretic approaches have been used to work with central (or influential) nodes in the network in order to solve certain important problems associated with social network analytics. For instance, Hendrickx *et al.* [22] proposed a Shapley value based approach to identify key nodes to optimally allocate resources over the network. Alon *et al.* [4] proposed a game theoretic approach to determine k most popular or trusted users in the context of directed social networks. Ramasuri and Narahari [43] proposed a Shapley value based approach to measure the influential capabilities of individual nodes in the context of viral marketing. For the scenarios where teams of individuals come together to accomplish atomic tasks, Papapetrou *et al.* [41] presented a Shapley value based algorithm to attribute the team-wise scores to the individuals with application to the citation networks.

In this section we concentrate on a new centrality measure, called *gatekeeper centrality* introduced by Ramasuri *et al.* [44]. The goal of this metric is to determine a group of nodes that can disconnect the network into components with similar cardinality. This question arises in social network applications such as community detection or limiting spread of misinformation or a virus over the network.

In a attempt to address the above problem, gatekeeper centrality uses game theoretical approach: for every group of agents it assigns a value that represent its quality (in terms of disconnecting into similar components) and calculate agents' worth in this game with the Shapley value. Two variants of the characteristic function was proposed in the paper:

$$v_1^R(S) \stackrel{\text{def}}{=} \frac{1}{\sum_{K_i \in K(N \setminus S)} |K_i|^2}, \qquad v_2^R(S) \stackrel{\text{def}}{=} \frac{|K(N \setminus S)|}{\sum_{K_i \in K(N \setminus S)} |K_i|}.$$

Here, compared to both metrics based on the graph-restricted games, the structure of a game is much more complicated – knowing the values of two connected components we cannot argue anything about the value of the sum of them.

In this section we present a Monte Carlo algorithm that is dedicated to dealing efficiently with games used in gatekeeper metric. Our algorithm, unlike the one in the previous section, is based on the sampling of permutations. With a growing number of agents in the game, computing any reasonable approximation of the Shapley value may require sampling millions of permutations. Consequently, the time efficiency of Monte Carlo approach hinges upon the way in which $|N|$ marginal contributions are calculated in every permutation. To this end, we speed up our calculation using a dedicated data structure. We discuss it in a detail further.

Let us start our analysis with the following fundamental observation: in gatekeeper games, the value of the coalition S depends only on the structure formed by the outside agents $N \setminus S$. Thus, we can traverse the permutation backward and, as we sequentially add agents, assign the changes in the value of $N \setminus S$ (i.e., agents' marginal contributions) to adequate agents.

For this purpose, we propose a dedicated structure to store subgraph components (SGC) based on the idea of *FindUnion*, a disjoint-set data structure [16]. The main concept here is to store separate components of the graph as trees. Whenever we add a new edge between different components, we attach the root of one tree as a child of the second one. It is important that we do not store all graph edges, but maintain multiple statistics that allow us to calculate the value of the subgraph without traversing the whole structure.

SGC-structure allows for the following operations:

- *createEmpty()* - initializes the structure;
- *addNode(i)* - adds a new component ($parent[i] = i$) and updates statistics;
- *addEdge(i, j)* - finds the roots of the components of i and j (with *path compression*[2]); if roots differ, attach a root of the smaller tree to the bigger one (if $rank[i] < rank[j]$ then $parent[i] = j$; this technique is called *union rank*) and updates statistics; otherwise, only updates statistics if needed;

[2]As we traverse up the tree to the root we attach all passed nodes directly to the root to flatten the structure: *find(i) {if (parent[i] ≠ i) return parent[i] = find(parent[i]);}*.

Algorithm 9: Approximation algorithm for the Gate-keepers Metric

Input: Graph $G = (N, E)$ and a function $v : 2^N \to \mathbb{R}$
Output: Shapley value, SV_i, of each node $i \in N$

```
for all i ∈ N do SV_i ← 0;
for k = 1 to numberOfSamples do
    π ← random permutation of N;
    SGC.createEmpty();
    valueOfSGC ← v(SGC);
    foreach i ∈ π do
        SV_i = SV_i + valueOfSGC;
        SGC.addNode(i);
        foreach j ∈ neighbours(i) do
            if SGC.exist(j) then
                SGC.addEdge(i, j);
        valueOfSGC ← v(SGC);
        SV_i = SV_i − valueOfSGC;
for all i ∈ N do SV_i ← SV_i/numberOfSamples;
```

- *exist(i)* - return *true* if $parent[i]$ is set. Return *false* otherwise

This representation, based on the two improving techniques *union by rank* and the *path compression*, allows us to perform $|E|$ *addEdge()* and $|N|$ *addNode()* operations in time $O(|E| \cdot log^*(|N|))$ where $log^*(x)$ denotes the iterated logarithm and $log^*(x) \leq 5$ for $x \leq 2^{65536}$ [24].

Finally, let us address the statistics that we have to collect in order to calculate the value of the structure. To compute $v_2(S)$ we need to store the number of nodes (variable increased in *addNode()*) and number of components (variable increased in *addNode()* and decreased in *addEdge()* if edge links different components). The formula for $v_1(S)$ is based on the sum of squares of components' sizes (to this end, we store the size with every component, initialize it in *addNode()* and update it in *addEdge()*; in addition, we store the global sum of squares in $O(1)$ and update it whenever the size of a component changes).

The pseudocode is presented in Algorithm 9. In our procedure we aggregate agents' marginal contributions in variables SV_i, initialized to zero (line 1) and divided at the end by the number of samples considered (line 14). In the main loop (lines 2-13) after the initialization we traverse the random permutation π (lines 6-13) and sequentially add nodes and edges to the SGC-structure (lines 8-11). Based on the value of the structure before and after the addition of a given agent, we calculate its marginal contribution (line 7 and 13).

The time complexity of the algorithm depends on the number of samples chosen to calculate the Shapley value (and that depends on our target precision). Let us then comment on the complexity of single sample, i.e. the calculations needed to update Shapley value based on a randomly chosen permutation. Firstly, the selection of a permutation (line 3) is performed in a linear time using Knuth shuffle. Next, calculating value of a SGC-structure (lines 5 and 12) is done in a constant time. Finally, the loop over the permutation π (lines 6-13) performs $|N|$ operations $addNode()$, $|E|$ operations $addEdge()$ and $2|E|$ operations $exist()$. To summarize, the calculation of a single sample takes $O(|E| \cdot log^*(|N|))$. In other words, this is the time complexity of single iteration of the main loop.

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