Continuous separation of game languages

André Arnold
LaBRI
University of Bordeaux I

Damian Niwiński*
Institute of Informatics
Warsaw University

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Abstract

We show that a family of tree languages $W_{(\iota,\kappa)}$, previously used by J. Bradfield, and by the first author to show the strictness of the Mostowski index hierarchy of alternating tree automata, forms a hierarchy w.r.t. the Wadge reducibility. That is, $W_{(\iota,\kappa)} \leq_W W_{(\iota',\kappa')}$ if and only if the index (ι',κ') is above (ι,κ). This is one of the few separation results known so far, concerning the topological complexity of *non-deterministically* recognizable tree languages, and one of the few results about finite-state recognizable non-Borel sets of trees. The interest of the result is reinforced by the fact that a related family $M_{(\iota,\kappa)}$, witnessing the strictness of the index hierarchy of non-deterministic automata, does not have a similar property.

1 Introduction

Classical fields of computer science, like algorithmics or complexity theory, primarily deal with computable sets and functions over integers. If they benefit from numerous ideas of arithmetics and recursion theory, originally reaching beyond the computable world, it is by borrowing concepts and pursuing analogies, rather than by direct application. In particular, numerous hierarchies have been proposed to organize complexity classes, following the patterns traced back to the work of Borel, Tarski, Kuratowski, Kleene, Mostowski, Grzegorczyk, and others (see [1] for a short review of definability theory).

The situation appears to be different in those areas of computer science where infinite computations are relevant. For example, finite automata running in potentially infinite time constitute a useful tool in verification of non-deterministic, concurrent, open systems, whose behaviour is ideally non-terminating. These automata proceed over infinite words or trees, which can be encoded as reals, rather than integers. In contrast to the finitary case, finite automata then exhibit remarkable definability power in terms of classical hierarchies, going far beyonds computable (i.e., Δ^0_1) sets of reals. In particular, they recognize sets of infinite words complete in the bold-face classes $\mathbf{\Sigma^0_2}$ and $\mathbf{\Pi^0_2}$ (but no more than Δ^0_3 sets), and even non-Borel sets of *trees* (but no more than Δ^1_2 sets). Therefore, at least in principle, classical hierarchies are of direct interest in the studies of such automata, in particular of their definability and complexity properties.

A relevant problem is to find an automaton which is the simplest possible for a given task. In these context, several hierarchies of automata have been introduced and studied, depending on the input type (word or tree) and the computation mode of automaton: deterministic, non-deterministic, or alternating. Although the basic methods are automata-theoretic, sometimes a topological evidence is useful. For example, Rabin [19] separated (in today's terminology) Rabin tree automata from Büchi automata, by a direct proof that no Büchi automaton can recognize the set of 0-1-labeled trees, such that on each path, there is only finitely many 1's. But this fact follows also by a topological argument, since Rabin's counter-example can be easily seen $\mathbf{\Pi^1_1}$-complete, while Büchi automata recognize only Σ^1_1 tree languages. Another example concerns the game tree languages $W_{(\iota,\kappa)}$ (defined below), which are the subject of the present note. The first author [2] used the Banach Fixed Point Theorem to show that these languages require alternating automata

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of arbitrarily high indices—this solved an open problem and gave an alternative proof of the strictness of the μ-calculus hierarchy, previously shown by Bradfield [4] and independently Lenzi [11].

Topological aspect of finite-state recognizable sets of infinite words is by now well understood. Classification in terms of Borel classes was established by Landweber [10], and further refined by Wagner [23] in terms of the Wadge hierarchy (see also [17]). Let us recall the fundamental concept of continuous reducibility, studied by W. Wadge ([22], see also [9]).

Definition 1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topological spaces. A set $A \subseteq \mathcal{T}_1$ is *Wadge reducible* to a set $B \subseteq \mathcal{T}_2$, in symbols $A \leq_W B$, if there is a continuous function $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$, which *reduces* A to B, i.e., $A = \varphi^{-1}(B)$.

In this context we also say that A is reducible to B *via reduction* φ.

As mentioned above, the topological complexity of finite-state recognizable sets of infinite *trees* goes beyond the Borel hierarchy (and there are sets complete on any finite level [20]). Deterministically recognizable tree languages exhibit an interesting dichotomy: any such language is either $\mathbf{\Pi^1_1}$-complete, or in Π^0_3. Recently, Murlak [14] completely described the Wadge hierarchy of deterministic tree languages thus making an analogue to Wagner's classification of ω-regular sets of words. It is worth to note that all the results about deterministic languages are *effective*: given a deterministic automaton (on words or trees), one can effectively determine its exact level in the Borel/projective hierarchy, and in the Wadge hierarchy restricted to deterministic sets.

However not much is known about topological complexity of non-deterministic tree languages, except that they (obviously) comprise $\mathbf{\Sigma^1_1}$-complete sets, and by definition do not exceed the Δ^1_2 level of the analytical hierarchy.

The present paper makes some step in this direction: we show that the so-called game languages $W_{(\iota,\kappa)}$, which are non-Borel and non-deterministic (except for the first level), constitute a hierarchy w.r.t. the Wadge reducibility, i.e., $W_{(\iota,\kappa)} \leq_W W_{(\iota',\kappa')}$ if and only if the index (ι',κ') is above (ι,κ) (defined below). As in [2], the argument explores the following observation: no set $A \subseteq \mathcal{T}$ is reducible to its complement $\overline{A}$ *via* a *contracting* reduction f (where $\mathcal{T}$ is a complete metric space), because otherwise, by the Banach Fixed-Point Theorem, we would have

$$a \in A \iff f(a) \in \overline{A} \iff a \in \overline{A} \quad \text{(contradiction)}, \tag{1}$$

for the fixed point $a = f(a)$. Now, the main technical lemma of the present paper shows that any continuous reduction of $W_{(\iota,\kappa)}$ to L can be modified to a contracting one, for any set of trees L.

The knowledge of automata theory is not necessary for understanding the technical content of the paper, although it is a source of motivation, and the main context of this research. For an interested reader, we include an automata-theoretic primer in Appendix. Also, the recursion-theoretic and topological hierarchies, although frequently mentioned in comments, are not relevant in technical sense. On this matter, we refer the reader to monographs [8] or [9].

2 Preliminaries

Throughout the paper, ω stands for the set of natural numbers. The distance function in a metric space is usually denoted by d; the actual meaning is determined by the context.

Metrics on trees A full binary tree over a finite alphabet Σ (or shortly a tree, if confusion does not arise) is represented as a mapping $t : \{1,2\}^* \to \Sigma$.

We consider the classical topology *à la Cantor* on T_Σ induced by the metric

$$d(t_1,t_2) = \begin{cases} 0 & \text{if } t_1 = t_2 \\ 2^{-n} \text{ with } n = \min\{|w| : t_1(w) \neq t_2(w)\} & \text{otherwise} \end{cases} \tag{2}$$

It is well-known and easy to see that if Σ has at least two elements then T_Σ with this topology is homeomorphic to the Cantor discontinuum $\{0,1\}^\omega$. Indeed, it is enough to fix a bijection $\alpha : \omega \to \{1,2\}^*$, and a mapping (code) $C : \Sigma \to \{0,1\}^*$, such that $C(\Sigma)$ forms a maximal antichain w.r.t. the prefix ordering. Then $T_\Sigma \ni t \mapsto C \circ t \circ \alpha \in \{0,1\}^\omega$ is a desired homeomorphism. It is clear that, whenever $A \leq_W B$, for $A \subseteq T_{\Sigma_1}$, $B \subseteq T_{\Sigma_2}$ then the same relation holds if we identify A and B with their homeomorphic copies in $\{0,1\}^\omega$.

A mapping $f : T_{\Sigma_1} \to T_{\Sigma_2}$ is *contracting* if, for some constant $0 < c < 1$,

$$d_2(f(t_1), f(t_2)) \quad \leq \quad c \cdot d_1(t_1, t_2).$$

By the Banach Fixed-Point Theorem, any contracting mapping of T_Σ into itself has a (unique) fixed point.

As we have already remarked in Introduction, no $L \subseteq T_\Sigma$ is Wadge reducible (c.f. Definition 1) to its complement $\overline{L} = T_\Sigma - L$ *via* a contracting reduction, as it would yield contradiction (1).

Parity games A *parity game* is a perfect information game of possibly infinite duration played by two players, say Eve and Adam. We present it as a tuple $(V_\exists, V_\forall, Move, p_0, rank)$, where $V_\exists$ and $V_\forall$ are (disjoint) sets of positions of Eve and Adam, respectively, $Move \subseteq V \times V$ is the relation of possible moves, with $V = V_\exists \cup V_\forall$, $p_0 \in V$ is a designated initial position, and $rank : V \to \omega$ is the ranking function.

The players start a play in the position p_0 and then move the token according to relation *Move* (always to a successor of the current position), thus forming a path in the graph $(V, Move)$. The move is selected by Eve or Adam, depending on who is the owner of the current position. If a player cannot move, she/he looses. Otherwise, the result of the play is an infinite path in the graph, $v_0, v_1, v_2, \ldots$ Eve wins the play if $\limsup_{n\to\infty} rank(v_n)$, is even, otherwise Adam wins.

Parity games, introduced by Emerson and Jutla [6], and independently by A.W. Mostowski [12], have been recognized as a combinatorial essence of many verification problems. A crucial property is the *positional determinacy*: any position is winning for one of the players, and moreover a winning strategy of player θ can be made *positional*, i.e., represented by a (partial) function $\sigma : V_\theta \to V$.[1] We say simply that Eve *wins* the game $\mathcal{G}$ if she has a winning strategy, the similar for Adam.

More information on parity games can be found in [21, 7, 3].

Mostowski indices A *Mostowski index* is a pair (ι, κ), where $\iota \in \{0,1\}$ and $\iota \leq \kappa < \omega$. We may see it as the minimal and maximal value of the *rank* function in some parity game. We consider the following (partial) order on indices: $(\iota, \kappa) \sqsubseteq (\iota', \kappa')$ if either $\iota' \leq \iota$ and $\kappa \leq \kappa'$ (i.e., $\{\iota, \ldots, \kappa\} \subseteq \{\iota', \ldots, \kappa'\}$) or $\iota = 0$, $\iota' = 1$, and $\kappa + 2 \leq \kappa'$ (i.e., $\{\iota+2, \ldots, \kappa+2\} \subseteq \{\iota', \ldots, \kappa'\}$). We consider the indices $(1, n)$ and $(0, n-1)$ as *dual*, and let $\overline{(\iota, \kappa)}$ denote the index dual to (ι, κ). Note that $\overline{\overline{(\iota, \kappa)}} = (\iota, \kappa)$.

3 Game languages

We now recall the definition of *game languages* $W_{(\iota,\kappa)}$ ([5, 2, 3]). The alphabet of $W_{(\iota,\kappa)}$ is

$$\Sigma_{(\iota,\kappa)} = \{\exists, \forall\} \times \{\iota, \ldots, \kappa\}.$$

With each tree t in $T_{\Sigma_{(\iota,\kappa)}}$, we associate a parity game $G(t)$, as described in the previous section, with

- $V_\exists = \{v \in \{1,2\}^* : t(v) \downarrow_1 = \exists\}$,
- $V_\forall = \{v \in \{1,2\}^* : t(v) \downarrow_1 = \forall\}$,
- $Move = \{(w, wi) : w \in \{1,2\}^*, \, i \in \{1,2\}\}$,
- $p_0 = \varepsilon$ (the root of the tree),
- $rank(v) = t(v) \downarrow_2$, for $v \in \{1,2\}^*$.

The set $W_{(\iota,\kappa)}$ consists of those trees for which Eve wins the game $G(t)$.

The above family of languages was used in [5], to show the strictness of the index hierarchy of the alternating parity tree automata (see Appendix). Another proof has been given in [2], based on the following observation (we include its simple proof, for completeness).

[1] Positional strategy σ is *winning* for θ if every play $p_1 = q_1, q_2, \ldots$ is won by θ, provided that $q_\ell \in V_\exists$ follows $q_{\ell+1} = \sigma(q_\ell)$.

Lemma 1. *There is no contracting reduction of $W_{\overline{(\iota,\kappa)}}$ to $W_{(\iota,\kappa)}$.*

Proof. We show that $W_{\overline{(\iota,\kappa)}}$ coincides with $\overline{W_{(\iota,\kappa)}} = T_{\Sigma_{(\iota,\kappa)}} - W_{(\iota,\kappa)}$, up to renaming of symbols. More specifically, let $\iota = 1$; hence $\overline{(\iota,\kappa)} = (0, \kappa - 1)$ (the other case is symmetric). Let $\beta : \Sigma_{\overline{(\iota,\kappa)}} \to \Sigma_{(\iota,\kappa)}$ be a bijection of alphabets given by

$$\beta(Q,\ell) \quad = \quad (\overline{Q}, \ell + 1),$$

where $\overline{\exists} = \forall$, and $\overline{\forall} = \exists$. This induces a re-labeling of trees $B : t \mapsto \beta \circ t$, which is clearly an isometry of $T_{\Sigma_{(\iota,\kappa)}}$ and $T_{\Sigma_{\overline{(\iota,\kappa)}}}$, and moreover reduces $\overline{W_{(\iota,\kappa)}}$ to $W_{\overline{(\iota,\kappa)}}$. (A strategy for Eve on t amounts to a strategy of Adam on $\beta \circ t$, and *vice versa*.) If there were a contracting reduction γ of $W_{\overline{(\iota,\kappa)}}$ to $W_{(\iota,\kappa)}$, then $\gamma \circ B$ would be a contracting reduction of $\overline{W_{(\iota,\kappa)}}$ to $W_{(\iota,\kappa)}$. But this would yield a contradiction (1), noted in Introduction. $\square$

We will now see that the claim of the above Lemma can be made stronger. More generally, we have the following.

Theorem 1. *The game languages form a hierarchy w.r.t. the Wadge reducibility, i.e.,*

$$W_{(\iota,\kappa)} \leq_W W_{(\iota',\kappa')} \quad \textit{iff} \quad (\iota,\kappa) \sqsubseteq (\iota',\kappa').$$

The *if* implication follows easily from definitions; the required reduction can be simply the embedding of $T_{\Sigma_{(\iota,\kappa)}}$ into $T_{\Sigma_{(\iota',\kappa')}}$, if necessary (i.e., if $\iota < \iota'$), preceded by the re-labeling $(Q,\ell) \mapsto (Q, \ell+2)$.

The *only if* implication is based on the following.

Lemma 2 (Main). *Let $\mathcal{T}$ be a metric space and $L \subseteq \mathcal{T}$. If $W_{(\iota,\kappa)} \leq_W L$, then there is a* contracting *reduction of $W_{(\iota,\kappa)}$ to L.*

The remaining of this section is devoted to the proof of this lemma.

As (ι,κ) are fixed, let $T = T_{\Sigma_{(\iota,\kappa)}}$. Suppose a continuous mapping $f : T \to \mathcal{T}$ reduces $W_{(\iota,\kappa)}$ to L. Since T is a compact space, f is uniformly continuous. That is, we can fix some function $\delta_f : (0,\infty) \to (0,\infty)$, such that

$$d(t_1,t_2) \leq \delta_f(\varepsilon) \quad \Longrightarrow \quad d\left(f(t_1), f(t_2)\right) \leq \varepsilon. \tag{3}$$

Lemma 3. *Suppose that $g : T \to T$ satisfies*

$$d(g(t_1), g(t_2)) \quad \leq \quad \delta_f\left(\frac{d(t_1,t_2)}{2}\right), \tag{4}$$

for all $t_1, t_2 \in T$. Then $f \circ g$ is contracting with constant $c = \frac{1}{2}$.

Proof. >From (4) and (3), we have

$$d(f \circ g(t_1), f \circ g(t_2)) \quad \leq \quad \frac{d(t_1,t_2)}{2},$$

as required. $\square$

Thus, in order to establish the claim of Lemma 2, it is enough to construct g satisfying (4), which moreover reduces $W_{(\iota,\kappa)}$ to itself. The following definition is quite arbitrary, many other functions would do as well.

Definition 2 (stretching). Given a sequence of natural numbers $M = (m_0, m_1, m_2, \ldots)$, we define a function $\gamma : \{1,2\}^* \to \{1,2\}^*$, as follows.

- $\gamma(\varepsilon) = 1^{m_0}$.

- If $\gamma(v)$ is already defined, with $|v| = i - 1$, then

$$\begin{aligned} \gamma(v1) &= \gamma(v)1^{m_i} \\ \gamma(v2) &= \gamma(v)2^{m_i}. \end{aligned}$$

This further induces a mapping $\Gamma : T \to T$, which we call *stretching* parametrized by M, whose definition additionally depends on ι (recall that $T = T_{\Sigma_{(\iota,\kappa)}}$).

1. If $\iota = 0$, we let

$$\begin{aligned} \Gamma(t)(\gamma(v)) &= t(v) \\ \Gamma(t)(w) &= (\forall, 0), \quad \textit{otherwise.} \end{aligned}$$

2. If $\iota = 1$, we let

$$\begin{aligned} \Gamma(t)(\gamma(v)) &= t(v) \\ \Gamma(t)(w) &= (\exists, 1), \quad \textit{otherwise.} \end{aligned}$$

Obviously, the above mapping Γ is continuous.

Lemma 4. *The mapping Γ reduces $W_{(\iota,\kappa)}$ to itself.*

Proof. It is enough to show that whenever Eve or Adam wins the game $G(t)$, the same player wins the game $G(\Gamma(t))$. Suppose first Eve wins the game $G(t)$. Note that the only positions of Eve in $G(\Gamma(t))$ are those of the form $w = \gamma(v)$, where v is a position of Eve in $G(t)$. In that game, Eve has the strategy $v \mapsto v1$, or $v \mapsto v2$ (i.e., go left, or go right, respectively). Eve adopts exactly the same strategy in $G(\Gamma(t))$. Then it is easy to see that any infinite play consistent with this strategy has one of the two forms:

- $\alpha_0, \gamma(v_0), \alpha_1, \gamma(v_1), \alpha_2, \ldots, \alpha_i, \gamma(v_i), \alpha_{i+1}, \ldots$, where $v_0, v_1, v_2, \ldots$ is an infinite play in $G(t)$ won by Eve, and the α_i's consist of positions labeled by $(0, \forall)$.

- $\alpha_0, \gamma(v_0), \alpha_1, \gamma(v_1), \alpha_2, \ldots, \gamma(v_k)\alpha_\omega$,, where α_ω is an infinite sequence of positions labeled by $(0, \forall)$.

Clearly, in both cases, the play is won by Eve.

Now suppose that the winner of $G(t)$ is Adam. Clearly Adam can reach $\gamma(\varepsilon)$ from the initial position in $G(\Gamma(t))$ (which may be $\gamma(\varepsilon)$ itself if $m = 0$). In each position of his, which is of the form $\gamma(v)$, Adam uses exactly the same strategy (go left, or go right) as in $G(t)$ in position v, and then continues (if necessary) until the play reaches again a position in the range of γ. Again, it is easy to see that any infinite play consistent with this strategy has the form $\alpha_0, \gamma(v_0), \alpha_1, \gamma(v_1), \alpha_2, \ldots, \alpha_i, \gamma(v_i), \alpha_{i+1}, \ldots$, where the α_i's consist of positions labeled by $(0, \forall)$, but this time the play $v_0, v_1, v_2, \ldots$ is won by Adam in $G(t)$, and hence so is the play in $G(\Gamma(t))$. □

Lemma 5. *For some parameter M, the induced stretching Γ satisfies (4).*

Proof. It remains to choose $m_0, m_1, m_2, \ldots$.

Note that, by definition (2), the distance function on T_Σ takes values of the form 2^{-n} or 0.

Let m satisfy $\frac{1}{2^m} \leq \delta_f(\frac{1}{2})$. Then, for any stretching with $m_0 = m$, we have

$$d(\Gamma(t_1), \Gamma(t_2)) \leq \delta_f(\frac{1}{2}).$$

In particular, the inequality (4) is satisfied whenever $d(t_1, t_2) = 1$.

More generally, suppose $m_0, m_1, m_2, \ldots, m_k$, satisfy

$$\frac{1}{2^{m_0+m_1+\ldots+m_k}} \leq \delta_f(\frac{1}{2^{k+1}}). \tag{5}$$

Note that if $|w| = k$ then $|\gamma(w)| = m_0 + m_1 + \ldots + m_k$. Therefore, if $d(t_1, t_2) = \frac{1}{2^k}$ then $d(\Gamma(t_1), \Gamma(t_2)) = \frac{1}{2^{m_0+m_1+\ldots+m_k}}$. Hence

$$\begin{aligned} d(\Gamma(t_1), \Gamma(t_2)) &= \frac{1}{2^{m_0+m_1+\ldots+m_k}} \\ &\leq \delta_f(\frac{1}{2^{k+1}}) \\ &= \delta_f(\frac{d(t_1,t_2)}{2}). \end{aligned}$$

Therefore, if the parameter sequence satisfies (5) for every k, the induced stretching Γ satisfies (4), for all $t_1, t_2 \in T$, as required. □

We are ready to conclude the proof of Lemma 2. If a continues mapping f reduces $W_{(\iota,\kappa)}$ to L, we find a stretching Γ, satisfying (4), by Lemma 5. Then, by Lemma 4, $f \circ \Gamma$ is a desired contracting reduction of $W_{(\iota,\kappa)}$ to L.

Remark The game languages $W_{(\iota,\kappa)}$ arose as a "game variant" of simpler languages, previously considered in literature. At first, instead of trees, we can consider the set of infinite words $L_{(\iota,\kappa)} \subseteq \{\iota, \ldots, \kappa\}^\omega$, satisfying the parity condition. These languages also form a hierarchy w.r.t. the Wadge reducibility, as shown by Wagner [23]. The proof of the present note can be also adapted to this case.

Another interesting variant is the set of trees $M_{(\iota,\kappa)} \subseteq T_{\{\iota,\ldots,\kappa\}}$, such that each path satisfies the parity condition. The set $M_{(\iota,\kappa)}$ can be viewed as a restriction of $W_{(\iota,\kappa)}$ to those trees, where Adam is the only player. Tree languages $M_{(\iota,\kappa)}$ have been used in [15] to show the infinity of the index hierarchy for non-deterministic Rabin automata. Interestingly, these sets do not form a hierarchy in the sense of Wadge. Indeed, all these languages are in Π^1_1, while $M_{(0,1)}$ is complete even in the bold-face class $\mathbf{\Pi}^1_1$ (see, e.g., [16]). Hence, all $M_{(\iota,\kappa)}$ with $(\iota,\kappa) \sqsupseteq (0,1)$, are equivalent in the sense of Wadge.

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A Tree automata primer

Non-deterministic automata. A non-deterministic tree automaton over trees in T_Σ with a parity acceptance condition[2] is presented as $A = \langle \Sigma, Q, q_I, Tr, rank \rangle$, where Q is a finite set of *states* with an *initial state* q_I, $Tr \subseteq Q \times \Sigma \times Q \times Q$ is a set of *transitions*, and $rank : Q \to \omega$ is the *ranking* function. A transition (q, σ, p_1, p_2) is usually written $q \stackrel{\sigma}{\to} p_1, p_2$.

A *run* of A on a tree $t \in T_\Sigma$ is itself a Q–valued tree $\rho : \{1,2\}^* \to Q$ such that $\rho(\varepsilon) = q_I$, and, for each $w \in dom\,(\rho)$, $\rho(w) \stackrel{t(w)}{\to} \rho(w1), \rho(w2)$ is a transition in Tr. A *path* in ρ is *accepting* if the highest rank occurring infinitely often along it is even. More formally, for a path $P = p_0 p_1 \ldots \in \{1,2\}^\omega$, this means that $\limsup_{n\to\infty} rank(\rho(p_0 p_1 \ldots p_n))$ is even. A *run is accepting* if so are all its paths. The tree language $L(A)$ *recognized* by A consists of those trees in T_Σ which admit an accepting run.

[2] Currently most frequently used in the literature, these automata are well-known to be equivalent to historically previous automata with the Muller or Rabin conditions [21].

The *Mostowski index* of an automaton A is the pair $(\min(rank(Q)), \max(rank(Q)))$; without loss of generality, we may assume that $\min(rank(Q)) \in \{0,1\}$.

For example, an automaton recognizing the game language $W_{(\iota,\kappa)}$, may have states $\{\iota,\dots,\kappa\} \cup \{T\}$, with $rank(T) = 0$, and $rank(\ell) = \ell$, the initial state 0, and transitions

$$\ell \stackrel{(m,\forall)}{\to} m, m; \qquad \ell \stackrel{(m,\exists)}{\to} m, T; \qquad \ell \stackrel{(m,\exists)}{\to} T, m; \qquad T \stackrel{(m,Q)}{\to} T, T,$$

with $\iota \le \ell, m \le \kappa$, and $Q \in \{\exists, \forall\}$. Note that this automaton has Mostowski index (ι, κ).

An automaton A is *deterministic* if Tr is a *function* $Q \times \Sigma \to Q \times Q$. Deterministically recognizable tree languages form a proper subclass of the recognizable ones. In particular, the game languages $W_{(\iota,\kappa)}$ (except for $W_{(0,0)} = T_{\Sigma_{(0,0)}}$ and $W_{(1,1)} = \emptyset$) are not deterministic. The results on the Wadge hierarchy of recognizable tree languages known so far [14] concern only deterministic languages.

Alternating automata. In one possible setting, an alternating tree automaton over trees in T_Σ can be presented similarly as a nondeterministic one, except that $Tr \subseteq Q \times \Sigma \times Q^* \times Q^*$. A transition can be written by $q \stackrel{a}{\to} (q_1 \wedge \ldots \wedge q_k, p_1 \wedge \ldots \wedge p_m)$ which means that the automaton sends a copy in the state q_i to the left successor, and a copy in the state p_j to the right successor, for all $i = 1,\dots,k$, $j = 1,\dots,m$. Formally, a run of A on a tree $t \in T_\Sigma$ can be presented as a tree $\rho : dom\,(\rho) \to Q$ whose domain is a subset of $(\{l,r\} \times \omega)^*$, so that the projection $w \downarrow 1$ of a node $w \in dom\,(\rho)$ is a node of t. We require that $\rho(\varepsilon) = q_I$, and whenever w is a node in $dom\,(\rho)$ with $t(w \downarrow 1) = a$, there is a transition $\rho(w) \stackrel{a}{\to} (\rho(w(l,1)) \wedge \ldots \wedge \rho(w(l,k)), \rho(w(r,1)) \wedge \ldots \wedge \rho(w(r,m)))$, where $w(l,1),\dots,w(l,k),w(r,1),\dots,w(r,m)$, are all the successors of w in $dom\,(\rho)$. Again, a run is accepting if so are all its paths.

>From the results of Rabin [18], it follows that an alternating tree automaton can be always simulated by a non-deterministic one (a direct construction has been given in [13]), but the index can arbitrarily increase.

It was proved in [15] that the languages $M_{(\iota,\kappa)}$ (defined above) witness the Mostowski index hierarchy of non-deterministic automata. That is, each $M_{(\iota,\kappa)}$ is (obviously) recognized by a non-deterministic automaton of index (ι,κ) (in fact, even by a deterministic one), but *not* by any automaton with index $(\iota',\kappa') \not\sqsupseteq (\iota,\kappa)$. These languages do not, however, witness the Mostowski index hierarchy of alternating automata. Indeed, they can be all recognized by alternating automata of index $(0,1)$. In contrast, the game languages $W_{(\iota,\kappa)}$ do witness the hierarchy of alternating automata: each $W_{(\iota,\kappa)}$ can be recognized by an alternating automaton of index (ι,κ) (as we have seen, even by a non-deterministic one), but *not* by any alternating automaton with index $(\iota',\kappa') \not\sqsupseteq (\iota,\kappa)$ [5, 2] (see also [3]). A slightly stronger result can be claimed: if $W_{(\iota,\kappa)}$ is reduced to $L \subseteq T_{\Sigma_{(\iota,\kappa)}}$ by a *non-expansive* mapping then L cannot be recognized by any alternating automaton with index $(\iota',\kappa') \not\sqsupseteq (\iota,\kappa)$ (see [3], Theorem 8.2.13). Lemma 2 above allows us to further strengthen this claim.

Proposition 1. *If $W_{(\iota,\kappa)} \le_W L \subseteq T_{\Sigma_{(\iota,\kappa)}}$ then L cannot be recognized by any alternating automaton with index $(\iota',\kappa') \not\sqsupseteq (\iota,\kappa)$.*