Problem 1

We consider random variables A and B, taking their values in the set $\{0,1\}^n$, for some $n \geq 1$, where

$$\Pr(A \neq B) \quad \leq \quad \frac{1}{n}.$$

Prove that

$$H(A \mid B) \quad \leq \quad 2,$$

and indicate, for which n (if any) the equality holds.

Hint. For words $v, w \in \{0,1\}^*$, let

$$\mathit{diff}(v,w) \quad = \quad \begin{cases} 0 & \text{if} \quad v = w \\ 1 & \text{if} \quad v \neq w \end{cases}$$

It may be helpful to introduce a random variable D, defined by

$$D \quad = \quad \mathit{diff}(A,B),$$

and consider $H(A, D \mid B)$.

Problem 2

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of different words that are random in the sense of Kolmogorov, that is $C_U(w_n) \geq n$, for some universal Turing machine U. Prove that infinitely many words in this sequence contains a subword 111.

Hint. It may be helpful to first consider the case when the length of w_n is divisible by 3.

Bonus. Propose and prove a generalization of the task of this problem.