Exam in information theory 8.02.2024. Problems

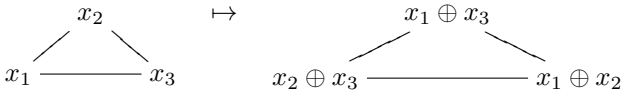
Problem 1

Let X_1, X_2, X_3 be **independent** random variables taking values in the set $\{0,1\}$ with the same probability distribution $\Pr(X_i = 0) = p$, where $0 < p < 1$. The symbol $\oplus$ means addition modulo 2 (XOR). Please compare the following values

$$H(X_1, X_2, X_3) \qquad H(X_2 \oplus X_3, X_1 \oplus X_3, X_1 \oplus X_2)$$

$$I(X_1; X_2|X_3) \qquad I(X_2 \oplus X_3, X_1 \oplus X_3, |X_1 \oplus X_2).$$

Remark. To help intuitions, the operation considered above can be illustrated on a triangle



that is, the value in each node is replaced by the $\oplus$ of its neighbours.

Note. In case of difficulties, please solve the problem for $p = \frac{1}{2}$. For the general case, use the Venn diagram and explore symmetry of the problem while avoiding long calculations.

Problem 2

We consider two channels whose input and output alphabet is $\{0,1\}^n$. Channel Γ_1 inputs a word w and with probability $\frac{1}{2}$ outputs it correctly, or outputs its mirror [1] image w^R.
Whereas channel Γ_2 inputs a word w and with probability $\frac{1}{2}$ outputs it correctly, or swaps its **first** bit. For example, with $n = 7$,

Compare the capacities of the two channels.

Notation.

Notation $\{0,1\}^n$ denotes the set of all words over alphabet $\{0,1\}$ of length n. For example,

$$\{0,1\} \quad = \quad \{000, 001, 010, 011, 100, 101, 110, 111\}.$$

Symbols I, H mean the same as I_2, H_2, respectively.

[1] For $w = w_1 w_2 \dots w_n$, $w^R = w_n w_{n-1} \dots w_1$.