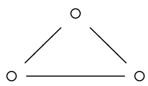
Exam in information theory 31.01.2023. Theoretical part

1. For a fixed triangle



we choose one of its edges with the uniform probability; this makes a random variable A. Next, again with the uniform probability, we choose one of the ends of the previously chosen edge, this makes a random variable B.

(a) $H(A|B) < H(B|A)$ **(b)** $H(A|B) = H(B|A)$ **(c)** $H(A|B) > H(B|A)$

Value: $I(A;B) = \log \mathbf{3} - \mathbf{1}$.

It follows from a straightforward calculation that $H(A \mid B) = H(B \mid A) = 1$.

2. Consider the set of binary representations of all prime numbers smaller than 2^{100} written with exactly 100 (one hundred) binary digits, with leading zeros added as needed. This code

(a) detects 4 errors, and corrects 2

(b) detects 0 errors

(c) it is not a code at all.

The binary representations of **2** and **3** differ only by one bit.

3. Suppose that, for a channel Γ with matrix $\begin{pmatrix} P & Q \\ Q & P \end{pmatrix}$, for all $\forall \varepsilon, \delta > 0$, there are infinitely many numbers $n \in \mathbb{N}$ and codes $\exists C \subseteq \{0,1\}^n$, such that $C_\Gamma - \varepsilon \leq R(C) \leq C_\Gamma$ and, for a random variable A uniformly distributed over C, it holds $\Pr_E(\Delta_o, A) \leq \delta$, where Δ_o is (attention!) the ideal observer rule. What can we claim **for sure** about the value P in the channel matrix (perhaps nothing of these)?

(a) $P = 0$ or $P = 1$ **(b)** $P \neq \frac{1}{2}$ **!** **(c)** $P > \frac{1}{2}$.

The Shannon channel theorem has been stated at the lecture for $P > \frac{1}{2}$ and the maximal likelihood rule Δ, with an even stronger claim: for any $\forall \varepsilon, \delta > 0$, the respective inequalities hold for **almost all** n's. If $P < \frac{1}{2}$, it is easy to see that $\Pr_E(\Delta_o, A)$ coincides with $\Pr_E(\Delta, A)$ for the dual channel $\begin{pmatrix} Q & P \\ P & Q \end{pmatrix}$ and the code $\bar{C} = \{\bar{w} : w \in C\}$. Hence we can use the Shannon theorem in this case as well. (In particular, we cannot claim **for sure** that $P > \frac{1}{2}$.) On the other hand, if $P = \frac{1}{2}$ then $\Pr_E(\Delta_o, A)$ is always $1 - \frac{1}{|C|}$. This contradicts the assumption as soon as the code C has at least 2 elements. This proviso has been usually implicitly present in our considerations, but not explicitly stated here. Indeed, for a trivial case of a **one element** code and $P = \frac{1}{2}$, we would have $\Pr_E(\Delta_o, A) = R(C) = C_\Gamma = 0$, satisfying (somewhat strangely) the assumption.

For this reason, **both answers: (b) or nothing were granted by 2 points**[1].

4. Two channels act over the alphabet $\{0, 1, \ldots, 8, 9\}$. The first one swaps 6 and 9 with probability $\frac{1}{2}$ (i.e., sends 6 to 9 or *vice-versa*), and sends the remaining digits correctly. The second channel sends 0 to some even digit (0,2,4,6 lub 8) with probability $\frac{1}{5}$, and sends the remaining digits correctly.

(a) the capacity of both channels is at least $\log 9$

(b) the capacity of exactly one of these channels is exactly $\log 10$

(c) the capacity of at least one of these channels is strictly smaller than $\log 9$.

Both channels can transmit 9 symbols without error and therefore can achieve $I(A;B) = \log 9$; on the other hand, no channel can transmit all bits without error and achieve $\log 10$. (Indeed, with

[1] The exceptional case of $|C| = 1$ has been commented by Mr. Wojda.

the formula for the sum of channels, it can be seen that the capacity of the first channel is exactly $\log 9$.)

5. Recall that the Chaitin constant Ω_U depends in general on a prefix-free universal Turing machine U. Does there exist a universal machine U, such that

   (a) $\Omega_U = \frac{1}{4}$

   (b) $\Omega_U = \frac{1}{\pi}$

   (c) every second digit in the binary representation of the fraction Ω_U is 1.

   None of these possibilities may hold true because of the two properties of the Chaitin constant shown at the lecture. Indeed, Ω_U cannot be a number whose bits can be effectively generated (as it is clearly the case for $\frac{1}{4}$ and $\frac{1}{\pi}$), as this would contradict the undecidability of the halting problem. Whereas the property **(c)** would contradict the **incompressibility** of the prefixes of Ω_U. Indeed, we can construct a (prefix-free) machine T, which, given k (in binary) and a word $\{0,1\}^k \ni v = v_1 \dots v_k$, produces the word $w = v_1 1 v_2 1 \dots v_{k-1} 1 v_k 1$. Thus (with $n = |w|$)

$$K_U(w) \quad \leq \quad \frac{n}{2} + 2 \log n + c.$$

   Hence, if Ω_U had the form of **(c)**, this would contradict the fact shown at the lecture that

$$K_U(\omega_1 \dots \omega_n) \quad \geq \quad n - c,$$

   for some constant c.