Analytic Tableau Systems and Interpolation for the Modal Logics *KB*, *KDB*, *K*5, *KD*5*

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Abstract. We give complete sequent-like tableau systems for the modal logics *KB*, *KDB*, *K*5, and *KD*5. Analytic cut rules are used to obtain the completeness. Our systems have the analytic superformula property and can thus give a decision procedure. Using the systems, we prove the Craig interpolation lemma for the mentioned logics.

1 Introduction

Tableau methods have been widely applied for modal logics, some of the best accounts of this are the works by Fitting [3] and Goré [5]. There are two kinds of tableau systems for modal logics: sequent-like, and labeled systems. In [11], Massacci successfully gives labeled tableau systems for all the basic normal modal logics obtainable from the logic K by the addition of any combination of the axioms T, D, 4, 5, and B in a modular way. There is a difficulty in developing sequent-like systems for symmetric modal logics (i.e. the ones containing the axiom B or/and 5) as in such logics "the future can affect the past", whereas in sequent-like systems "the past" cannot be changed. In [3], Fitting gives semi-analytic sequent-like tableau systems for the logics *KB*, *KDB*, *B*, and *S*5, but they do not have the analytic superformula property and thus cannot give a decision procedure. There are known sequent-like tableau systems with the analytic superformula property for the logics *B*, *KB*4, *K*45, *KD*45, and *S*5 (see [5] for the history), but such systems for the logics *KB*, *KDB*, *K*5, and *KD*5 are, as raised by Goré [5], open problems.

In this work, we present complete sequent-like tableau systems for the latter logics. These systems use analytic cuts and have the analytic superformula property. Our systems for the logics *KB* and *KDB* are based on the system $\mathcal{C}B$ of Rautenberg [14]. For the logics *K*5 and *KD*5, we use a special symbol to distinguish the "actual" world from the others. To obtain the analytic superformula property we use an extra connective as a "blocked" version of the modality $\Box$.

Our tableau formulation is based on the work by Goré [5]. We use a similar technique to prove completeness of the systems. To show completeness of $\mathcal{C}L$ we give an algorithm that, given a finite $\mathcal{C}L$-consistent formula set X, constructs

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a L-model graph that satisfies every one of its formulae at the corresponding world.

Using our tableau systems, we prove the Craig interpolation lemma for the logics KB, KDB, $K5$, and $KD5$. Craig [2] proved the lemma for the classical predicate logic in 1957. The lemma turned very influential in mathematical logic. Analogs for different modal logics have been studied (among others) in Gabbay [4], Fitting [3], Rautenberg [14], and Maksimova [10]. The proof of the Craig interpolation lemma for the propositional modal logics K, KD, T, $K4$, $KD4$, and $S4$ can be found in [3, 4], and for the logics KB, KDB, B, and $S5$ in [3]. Our proof for the logics KB and KDB has an advantage that it gives a constructive way to compute interpolation formulae, as it is based on analytic systems. We do not know whether the lemma has been previously proved for the logics $K5$ and $KD5$. The Craig interpolation lemma can be used to prove the Beth definability theorem for the logics considered in this paper, as done by Craig in [2].

2 Preliminaries

2.1 Syntax and Semantics Definition for Modal Logics

A modal formula, hereafter simply called a *formula*, is defined by the following rules: any primitive proposition p is a formula, $\bot$ is a formula, and if ϕ and ψ are formulae then so are $\neg\phi$, $\phi \wedge \psi$, and $\Box\phi$. The symbol $\bot$ stands for "false". We write $\top$, $\phi \vee \psi$, $\phi \to \psi$, and $\Diamond\phi$ to denote shortened forms of $\neg\bot$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg(\phi \wedge \neg\psi)$, and $\neg\Box\neg\phi$, respectively.

We use small letters p, q to denote primitive propositions, Greek letters like ϕ, ψ to denote formulae, and block letters like X, Y, Z to denote formula sets.

A Kripke *frame* is a triple $\langle W, \tau, R\rangle$, where W is a nonempty set of possible worlds, $\tau \in W$ is the actual world, and R is a binary relation on W called the accessibility relation. If $R(w, u)$ holds, then we say that the world u is accessible from the world w, or that u is reachable from w.

A Kripke *model* is a tuple $\langle W, \tau, R, h\rangle$, where $\langle W, \tau, R\rangle$ is a Kripke frame and h is a function mapping worlds to sets of primitive propositions. For $w \in W$, $h(w)$ is the set of primitive propositions which are "true" at w.

Given some Kripke model $M = \langle W, \tau, R, h\rangle$, and some $w \in W$, the satisfaction relation $M, w \vDash \phi$ is defined recursively as follows.

$$
\begin{array}{ll}
M, w \nvDash \bot; & \\
M, w \vDash p & \text{iff } p \in h(w); \\
M, w \vDash \neg\phi & \text{iff } M, w \nvDash \phi; \\
M, w \vDash \phi \wedge \psi & \text{iff } M, w \vDash \phi \text{ and } M, w \vDash \psi; \\
M, w \vDash \Box\phi & \text{iff for all } v \in W \text{ such that } R(w, v), M, v \vDash \phi.
\end{array}
$$

We say that ϕ *is satisfied at* w *in* M iff $M, w \vDash \phi$, and that ϕ *is satisfied in* M, and call M a *model of* ϕ, iff $M, \tau \vDash \phi$.

2.2 Modal Logic Correspondences

The smallest normal modal logic, called K, is axiomatized by the standard axioms for the classical propositional logic, the *modus ponens* inference rule, the *K-axiom* $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$, plus the *necessitation rule*

$$\frac{\vdash \phi}{\vdash \Box\phi}$$

It can be shown that a modal formula is provable in this axiomatization iff it is satisfied in every Kripke model (i.e. without any special R-properties) [9]. It is known that certain axiom schemata added to this axiomatization are mirrored by certain properties of the accessibility relation (see also [1, 8]).

Different modal logics are distinguished by their respective additional axiom schemata. The modal logics KB, KDB, $K5$, $KD5$ together with their axiom schemata are listed in Table 2. We refer to properties of the accessibility relation of a modal logic L as *L-frame restrictions*.

We call a model M a *L-model* if the accessibility relation of M satisfies all L-frame restrictions. We say that ϕ is L-satisfiable if there exists a L-model of ϕ. A formula ϕ is said to be *L-valid* if it is satisfied in every L-model.

Axiom	Schemata	First-Order Formula
D	$\Box\Phi \rightarrow \Diamond\Phi$	$\forall x \; \exists y \; R(x, y)$
B	$\Phi \rightarrow \Box\Diamond\Phi$	$\forall x, y \; R(x, y) \rightarrow R(y, x)$
5	$\Diamond\Phi \rightarrow \Box\Diamond\Phi$	$\forall x, y, z \; R(x, y) \land R(x, z) \rightarrow R(y, z)$

Table 1. Axioms and corresponding first-order conditions on R

Logic	Axiom	Frame Restriction
KB	KB	symmetric
KDB	KDB	serial and symmetric
$K5$	K5	euclidean
$KD5$	KD5	serial and euclidean

Table 2. Modal logics and frame restriction

2.3 Syntax, Soundness, and Completeness of Modal Tableau Systems

Our tableau formulation is adopted from the work by Goré [5], which in turn is related to the ones by Hintikka [6] and Rautenberg [14]. A number of terms and notations used in this work are borrowed from Goré [5].

A *tableau rule* δ consists of a numerator N above the line and a (finite) list of denominators D_1, D_2, ..., D_k (below the line) separated by vertical bars.

$$\frac{N}{D_1 \mid D_2 \mid \ \ldots \ \mid D_k}$$

The numerator is a finite formula set, and so is each denominator. As we shall see later, each rule is read downwards as "if the numerator is L-satisfiable, then so is one of the denominators". The numerator of each tableau rule contains one or more distinguished formulae called the *principal formulae*.

A *tableau system* (or *calculus*) $\mathcal{C}L$ for a logic L is a finite set of tableau rules.

A $\mathcal{C}L$-tableau for X is a tree with root X whose nodes carry finite formula sets. A tableau rule with numerator N is applicable to a node carrying a set Y if Y is an instance of N. The steps for extending a tableau are:

- choose a leaf node n carrying Y where n is not an end node (defined below), and choose a rule δ which is applicable to n;
- if δ has k denominators then create k successors for n, with successor i carrying an appropriate instance of denominator D_i;
- all with the proviso that if a successor s carries a set Z and Z has already appeared on the branch from the root to s then s is an *end node*.

Let Δ be a set of tableau rules. We say that Y is *obtainable from* X *by applications of rules from* Δ if there exists a tableau for X which uses only rules from Δ and has a node that carries Y.

A branch in a tableau is *closed* if its end node carries only $\perp$. A tableau is *closed* if every its branch is closed. A tableau is *open* if it is not closed. A finite formula set X is said to be $\mathcal{C}L$*-consistent* if every $\mathcal{C}L$-tableau for X is open. If there is a closed $\mathcal{C}L$-tableau for X then we say that X is $\mathcal{C}L$*-inconsistent*.

A tableau system $\mathcal{C}L$ is said to be *sound* if for any finite formula set X, if X is L-satisfiable then X is $\mathcal{C}L$-consistent. A tableau system $\mathcal{C}L$ is said to be *complete* if for any finite formula set X, if X is $\mathcal{C}L$-consistent then X is L-satisfiable.

Let δ be one of the rules of $\mathcal{C}L$. We say that δ is sound wrt. L if for any instance δ' of δ, if the numerator of δ' is L-satisfiable then so is one of the denominators of δ'. It is clear that if $\mathcal{C}L$ contains only rules sound wrt. L then $\mathcal{C}L$ is sound.

3 Tableau Systems for the Modal Logics *KB*, *KDB*, *K*5, and *KD*5

Tables 3 and 4 represent tableau rules and calculi for the modal logics KB, KDB, $K5$, and $KD5$. We write $X;Y$ for $X \bigcup Y$, and $X;\phi$ for $X \bigcup \{\phi\}$. We sometimes consider a formula set also as the conjunction of its formulae. The connective $\boxtimes$ has the same semantics as $\Box$, i.e. $M, w \models \boxtimes\phi$ iff $M, w \models \Box\phi$, but plays a different syntactical role. The symbol $*$ is a special formula with the following semantics: $M, w \models *$ iff there exists a world u such that $R(u, w)$ holds. By $\Box X$ we denote the set $\{\Box\phi \mid \phi \in X\}$. The sets $\boxtimes X$ and $\neg X$ are defined similarly.

Thinning is built into the rules of our systems, whereas in [5] Goré uses an explicit thinning rule. An explicit thinning rule is not desirable for our systems since the rule (5_*) is applicable only when the numerator contains $*$, and an explicit thinning rule would allow us to remove the $*$, leading to blocked proofs and wasted search.

Following Goré [5], we categorize each rule either as a *static rule* or as a *transitional rule*. The intuition behind this sorting is that in the static rules, the numerator and denominator represent the same world (in the same model), whereas in the transitional rules, the numerator and denominator represent different worlds (in the same model).

$$(\bot)\ \frac{X; \bot}{\bot} \qquad (\bot')\ \frac{X; \phi; \neg\phi}{\bot} \qquad (\neg)\ \frac{X; \neg\neg\phi}{X; \phi} \qquad (\wedge)\ \frac{X; \phi \wedge \psi}{X; \phi; \psi}$$

$$(K)\ \frac{X; \Box Y; \boxtimes Z; \neg\Box\phi}{Y; Z; \neg\phi} \qquad (KD)\ \frac{X; \Box Y; \boxtimes Z}{Y; Z}$$

$$(B_\Box)\ \frac{X; \Box\phi}{X; \Box\phi; \phi \mid X; \Box\phi; \neg\phi; \boxtimes\neg\Box\phi} \qquad (B_\Diamond)\ \frac{X; \neg\Box\phi}{X; \neg\Box\phi; \phi \mid X; \neg\Box\phi; \neg\phi; \boxtimes\neg\Box\phi}$$

$$(5)\ \frac{X; \Box Y; \boxtimes Z; \neg\Box U; \neg\Box\phi}{Y; Z; \neg\Box U; \neg\Box\phi; \neg\phi; *} \qquad (KD_*)\ \frac{X; \Box Y; \boxtimes Z}{Y; Z; *}$$

$$(5_*)\ \frac{X; \Box\phi; *}{X; \Box\phi; \boxtimes\Box\phi; *} \qquad (5_\Box)\ \frac{X; \Box\phi}{X; \Box\phi; \boxtimes\Box\phi \mid X; \Box\phi; \boxtimes\neg\Box\phi} \text{ where } * \notin X$$

$$(sfc_\Box)\ \frac{X; \Box\phi}{X; \Box\phi; \phi \mid X; \Box\phi; \neg\phi} \qquad (sfc_\Diamond)\ \frac{X; \neg\Box\phi}{X; \neg\Box\phi; \phi \mid X; \neg\Box\phi; \neg\phi}$$

$$(sfc_\vee)\ \frac{X; \neg(\phi \wedge \psi)}{X; \neg\phi; \neg\psi \mid X; \neg\phi; \psi \mid X; \phi; \neg\psi} \qquad (sfc)\ = \{(sfc_\Box), (sfc_\Diamond), (sfc_\vee)\}$$

Table 3. Tableau rules for KB, KDB, $K5$, and $KD5$

We write $Sf(\phi)$ to denote the set of all subformulae of ϕ. By $Sf(X)$ we denote the set $\bigcup_{\phi \in X} Sf(\phi)$. We say that X is *subformula-complete* if for every $\phi \in Sf(X)$, either $\phi \in X$ or $\neg\phi \in X$.

The rules $(B_\Box)$, $(B_\Diamond)$, $(sfc_\Box)$, $(sfc_\Diamond)$, $(sfc_\vee)$ are usually called *analytic cut* rules. They make $\mathcal{CL}$-saturations (defined in the next section) subformula-complete, which will be exploited to prove completeness of the given calculi.

A tableau system $\mathcal{CL}$ has the *analytic superformula* property iff to every finite set X we can assign a finite set $X^*_{\mathcal{CL}}$ such that $X^*_{\mathcal{CL}}$ contains all formulae that

$\mathcal{C}L$	Static Rules	Transitional Rules
$\mathcal{C}KB$	$(\bot), (\bot'), (\neg), (\wedge), (sfc_{\vee}), (B_{\Box}), (B_{\Diamond})$	(K)
$\mathcal{C}KDB$	$(\bot), (\bot'), (\neg), (\wedge), (sfc_{\vee}), (B_{\Box}), (B_{\Diamond})$	$(K), (KD)$
$\mathcal{C}K5$	$(\bot), (\bot'), (\neg), (\wedge), (sfc), (5_*), (5_{\Box})$	(5)
$\mathcal{C}KD5$	$(\bot), (\bot'), (\neg), (\wedge), (sfc), (5_*), (5_{\Box})$	$(5), (KD_*)$

Table 4. Tableau systems for KB, KDB, $K5$, and $KD5$

may appear in any tableau for X. In any rule δ of our systems, except $(\bot')$, any formula in the denominators of δ either belongs to $\neg Sf(N)$, where N is the numerator of δ, or is of one of the forms $\boxtimes\phi$, $\boxtimes\neg\phi$, where ϕ is a principal formula of δ. In our systems, there are no rules with a principal formula starting with $\boxtimes$. Therefore the systems have the analytic superformula property, with $X^*_{\mathcal{C}L} = Sf(\boxtimes\neg Sf(X))$.

The connective $\boxtimes$ is a blocked version of $\Box$. It behaves like $\Box$ in the transitional rules, but formulae starting with $\boxtimes$ play no roles in the static rules. It is $\boxtimes$ that guarantees the analytic superformula property of our systems. The technique of using an extra connective as a blocked version of $\Box$ has previously been used in the work by Hudelmaier [7].

Lemma 1. *The calculi $\mathcal{C}KB$, $\mathcal{C}KDB$, $\mathcal{C}K5$, and $\mathcal{C}KD5$ are sound.*

Proof. We show that $\mathcal{C}L$ contains only rules sound wrt. L, where L is KB, KDB, $K5$, or $KD5$. If the considered rule is static, then we show that if the numerator is satisfied at a world w, then so is one of the denominators. If the rule is transitional and its numerator is satisfied at w, then we show that the denominator is satisfied at some world reachable from w. Nontrivial cases are when the considered rule is one of $(B_{\Box})$, $(B_{\Diamond})$, (5), (5_*), $(5_{\Box})$.

For $(B_{\Box})$ and $(B_{\Diamond})$, just note that $\neg\phi \to \Box\neg\Box\phi$ is KB-valid.

For (5), suppose that $M, w \vDash X; \Box Y; \boxtimes Z; \neg\Box U; \neg\Box\phi$, where $M = \langle W, \tau, R, h\rangle$ is a $K5$-model and $w \in W$. There exists u such that $R(w,u)$ holds and $M, u \vDash \neg\phi$. Since $\neg\Box\psi \to \Box\neg\Box\psi$ is $K5$-valid, we have $M, u \vDash Y; Z; \neg\Box U; \neg\Box\phi; \neg\phi; *$. Therefore (5) is sound wrt. $K5$.

For (5_*), suppose that $M, w \vDash X; \Box\phi; *$, where $M = \langle W, \tau, R, h\rangle$ is a $K5$-model and $w \in W$. We show that $M, w \vDash \boxtimes\Box\phi$. It suffices to show that for any $u, v \in W$ such that $R(w,u)$ and $R(u,v)$ hold, $R(w,v)$ also holds. Since $M, w \vDash *$, there exists w_0 such that $R(w_0, w)$ holds. From the frame restriction $\forall x, y, z\ R(x,y) \wedge R(x,z) \to R(y,z)$, we derive that $R(w,w)$, $R(u,w)$, and $R(w,v)$ hold.

To show that $(5_{\Box})$ is sound wrt. $K5$ and $KD5$, it suffices to show that $\neg(\Box\neg\Box\phi) \to \Box\Box\phi$ is $K5$-valid. This assertion holds because $\neg\Box\neg\Box\phi \equiv \Diamond\Box\phi$, and $\Diamond(\Box\phi) \to \Box\Diamond(\Box\phi)$, $\Diamond\Box\phi \to \Box\phi$, and $\Box(\Diamond\Box\phi) \to \Box(\Box\phi)$ are $K5$-valid.

4 Completeness of the Calculi

From now on we use L to denote one of the logics KB, KDB, $K5$, $KD5$, and $\mathcal{C}L$ to denote the corresponding calculus. In order to prove completeness of the given calculi we first need some technical machinery.

4.1 Saturation

In the rules $(\neg)$, $(\wedge)$, $(sfc_\vee)$, the principal formula does not occur in the denominators. For δ being one of these rules, let δ' denote the rule obtained from δ by adding the principal formula to each of the denominators. Let $\mathcal{SC}L$ denote the set of static rules of $\mathcal{C}L$ with $(\neg)$, $(\wedge)$, $(sfc_\vee)$ replaced by $(\neg')$, $(\wedge')$, $(sfc'_\vee)$. Note that for any rule of $\mathcal{SC}L$, except $(\bot)$ and $(\bot')$, the numerator is included in each of the denominators.

For X being a finite $\mathcal{C}L$-consistent formula set, a formula set Y is called a *$\mathcal{C}L$-saturation of X* if Y is a maximal $\mathcal{C}L$-consistent set obtainable from X by applications of the rules of $\mathcal{SC}L$.

A set X is *closed wrt. a tableau rule* if, whenever the rule is applicable to X, one of the corresponding instances of the denominators is equal to X.

As stated by the following lemma, $\mathcal{C}L$-saturations have the same nature as "downward saturated sets" defined in the works by Hintikka [6] and Goré [5].

Lemma 2. *Let X be a finite $\mathcal{C}L$-consistent formula set, and Y a $\mathcal{C}L$-saturation of X. Then $X \subseteq Y \subseteq X^*_{\mathcal{C}L}$, Y is closed wrt. the rules of $\mathcal{SC}L$, and Y is subformula-complete.*

Proof. It is easily seen that the first assertion holds.

If a rule of $\mathcal{SC}L$ is applicable to Y, then one of the corresponding instances of the denominators is $\mathcal{C}L$-consistent. Since Y is a $\mathcal{C}L$-saturation (of X), Y is closed wrt. the rules of $\mathcal{SC}L$.

It is straightforward to prove by induction on the construction of ϕ that if ϕ belongs to Y, then for any subformula ψ of ϕ, either ψ or $\neg\psi$ belongs to Y. Therefore Y is subformula-complete.

Lemma 3. *There is an effective procedure that, given a finite $\mathcal{C}L$-consistent formula set X, constructs some $\mathcal{C}L$-saturation of X.*

Proof. We construct a $\mathcal{C}L$-saturation of X as follows: Let $Y = X$. While there is some rule δ of $\mathcal{SC}L$ applicable to Y with the property that one of the corresponding instances of the denominators, denoted by Z, is $\mathcal{C}L$-consistent and strictly contains Y, set $Y = Z$.

At each iteration, $Y \subset Z \subseteq X^*_{\mathcal{C}L}$. Hence the above process always terminates. It is clear that the resulting set Y is a $\mathcal{C}L$-saturation of X.

4.2 Proving Completeness Via Model Graphs

We prove completeness of our calculi via model graphs in a similar way as Rautenberg [14] and Goré [5] do for their systems.

A *model graph* is a tuple $\langle W, \tau, R, H \rangle$, where $\langle W, \tau, R \rangle$ is a Kripke frame and H is a function mapping worlds to sets of formulae. For $w \in W$, $H(w)$ is the set of formulae which should be "true" at the world w. We sometimes treat model graphs as models with h being H restricted to the set of primitive propositions. A model graph that satisfies all L-frame restrictions is called *L-model graph*.

Our definition of model graphs is not adequate with respect to Rautenberg's. A model graph $M = \langle W, \tau, R, H \rangle$ by his definition is accompanied by certain properties which guarantee that for any $\phi \in H(w)$, where $w \in W$, $M, w \vDash \phi$. Denote this condition by (*). We use the term "model graph" merely to denote a data structure, and leave the condition (*) as a criterion of *good* model graphs.

Given a finite $\mathcal{C}L$-consistent set X, as a L-model for X we construct a L-model graph $M = \langle W, \tau, R, H \rangle$ that satisfies the condition (*) and $X \subseteq H(\tau)$. We prove (*) for M by induction on the number of connectives occurring in ϕ. If for every $w \in W$, $H(w)$ is a $\mathcal{C}L$-saturation of some set, then (*) obviously holds (under inductive assumption) for the cases when ϕ is of the form p, $\neg p$, $\neg\neg\psi$, $\psi \wedge \zeta$, or $\neg(\psi \wedge \zeta)$. For the case when ϕ is of the form $\neg\Box\psi$, we show that there exists a world u reachable from w such that $\neg\psi \in H(u)$. For the case when ϕ is of the form $\Box\psi$ or $\boxtimes\psi$, in most of cases we show that for any world u reachable from w, $\psi \in H(u)$.

4.3 Completeness of $\mathcal{C}KB$ and $\mathcal{C}KDB$

In this subsection, let L denote one of the logics KB, KDB.

Algorithm 1
Input: A finite $\mathcal{C}L$-consistent set X of formulae not containing $\boxtimes$, $*$.
Output: A L-model graph $M = \langle W, \tau, R, H \rangle$ such that X is satisfied in the model corresponding to M.

1. Let $W = \{\tau\}$, $R_0 = \emptyset$, and $H(\tau)$ be a $\mathcal{C}L$-saturation of X. Mark τ as unresolved. (In this algorithm, we will mark the worlds of M either as *unresolved* or as *resolved*.)
2. While there are unresolved worlds, take one, denoted by w, and do the following:
   (a) For every formula $\neg\Box\phi$ in $H(w)$:
      i. Let Y be the result of the application of the rule (K) to $H(w)$, i.e.
      $$Y = \{\neg\phi\} \bigcup \{\psi \mid \Box\psi \in H(w) \text{ or } \boxtimes\psi \in H(w)\},$$
      and let Z be a $\mathcal{C}L$-saturation of Y.
      ii. If there exists a world $v \in W$ such that $H(v) = Z$, then add the edge (w, v) to R_0. Otherwise, add a new world w_ϕ with content Z to W, mark it as unresolved, and add the edge (w, w_ϕ) to R_0.
   (b) If $L = KDB$ and there is no x such that $R(w, x)$ holds, then

i. Let Y be the result of the application of the rule (KD) to $H(w)$, i.e.

$$Y = \{\psi \mid \Box\psi \in H(w) \text{ or } \boxtimes\psi \in H(w)\},$$

and let Z be a $\mathcal{C}L$-saturation of Y.

ii. Do the same thing as the step 2(a)ii.

(c) Mark w as resolved.

3. Let R be the symmetric closure of R_0.

This algorithm always terminates because H is one-to-one, and for any $w \in W$, $H(w) \subseteq X^*_{\mathcal{C}L}$.

Lemma 4. *Let X be a finite $\mathcal{C}L$-consistent set of formulae not containing $\boxtimes$, $*$. Let $M = \langle W, \tau, R, H\rangle$ be the model graph constructed by the above algorithm for X. Then for any $w \in W$ and any $\phi \in H(w)$, $M, w \vDash \phi$.*

Proof. We prove this lemma by induction on the number of connectives occurring in ϕ. The only nontrivial case is when ϕ is of the form $\Box\psi$ or $\boxtimes\psi$. For this case it suffices to show that if $R_0(u, w)$ holds, and $\Box\psi \in H(w)$ or $\boxtimes\psi \in H(w)$, then $M, u \vDash \psi$. Assume that $R_0(u, w)$ holds.

Suppose that $\boxtimes\psi \in H(w)$. The formula $\boxtimes\psi$ can be introduced only by the rule $(B_\Box)$ or $(B_\Diamond)$, hence ψ is of the form $\neg\Box\zeta$, and we have $\neg\zeta \in H(w)$. By inductive assumption, $M, w \vDash \neg\zeta$. Hence $M, u \vDash \neg\Box\zeta$, and $M, u \vDash \psi$. (Note that it is not necessary that $\psi \in H(u)$.)

Now assume that $\Box\psi \in H(w)$. We show that $\psi \in H(u)$. Suppose oppositely that $\psi \notin H(u)$. Note that w is created from u, and for any formula ζ, if $\Box\zeta \in Sf(H(w))$ then $\Box\zeta \in Sf(H(u))$. Since $\Box\psi \in H(w)$, it follows that $\Box\psi \in Sf(H(u))$. Hence $\Box\psi \in H(u)$ or $\neg\Box\psi \in H(u)$, since $H(u)$ is subformula-complete. By the rules $(B_\Box)$ and $(B_\Diamond)$, from $\psi \notin H(u)$ we derive that $\boxtimes\neg\Box\psi \in H(u)$. It follows that $\neg\Box\psi \in H(w)$, which contradicts the assumption that $\Box\psi \in H(w)$. Therefore $\psi \in H(u)$, and by inductive assumption, we have $M, u \vDash \psi$. This completes our proof.

Corollary 1. *Let X be a finite $\mathcal{C}L$-consistent set of formulae not containing $\boxtimes$, $*$. Then X is L-satisfiable.*

Proof. Let $M = \langle W, \tau, R, H\rangle$ be a model graph constructed by the above algorithm for X. It is clear that R satisfies all L-frame restrictions. By the above lemma, we have $M, \tau \vDash H(\tau)$. Since $H(\tau)$ is a $\mathcal{C}L$-saturation of X, we also have $M, \tau \vDash X$. Therefore X is L-satisfiable.

The following theorem immediately follows from the above corollary and Lemma 1.

Theorem 2. *The calculi $\mathcal{C}KB$ and $\mathcal{C}KDB$ are sound and complete.*

4.4 Completeness of $\mathcal{C}K5$ and $\mathcal{C}KD5$

In this subsection, let L denote one of the logics $K5$, $KD5$.

Algorithm 3
Input: A finite $\mathcal{C}L$-consistent set X of formulae not containing $\boxtimes$, $*$.
Output: A L-model graph $M = \langle W, \tau, R, H\rangle$ such that X is satisfied in the model corresponding to M.

1. Let $W_1 = W_2 = \emptyset$, and $H(\tau)$ be a $\mathcal{C}L$-saturation of X.
2. For every formula $\neg\Box\phi \in H(\tau)$:
   - Let Y be the result of the application of the rule (5) to $H(\tau)$, i.e.

$$Y = \{\neg\phi, *\} \bigcup \{\psi \mid \Box\psi \in H(\tau) \text{ or } \boxtimes\psi \in H(\tau)\} \\ \bigcup \{\neg\Box\psi \mid \neg\Box\psi \in H(\tau)\}.$$

   - Add a new world w with $H(w)$ being a $\mathcal{C}L$-saturation of Y to W_1.
3. If $L = KD5$ and there is no $\neg\Box\phi \in H(\tau)$, then:
   - Let Y be the result of the application of the rule (KD_*) to $H(\tau)$, i.e. $Y = \{*\} \bigcup \{\psi \mid \Box\psi \in H(\tau) \text{ or } \boxtimes\psi \in H(\tau)\}$.
   - Add a new world w with $H(w)$ being a $\mathcal{C}L$-saturation of Y to W_1.
4. Let $R_0 = \{\tau\} \times W_1$.
5. For every $w \in W_1$, and for every formula $\neg\Box\phi \in H(w)$:
   - Let Y be the result of the application of the rule (5) to H(w), i.e.

$$Y = \{\neg\phi, *\} \bigcup \{\psi \mid \Box\psi \in H(w) \text{ or } \boxtimes\psi \in H(w)\} \\ \bigcup \{\neg\Box\psi \mid \neg\Box\psi \in H(w)\}.$$

   - Add a new world w_ϕ with $H(w_\phi)$ being a $\mathcal{C}L$-saturation of Y to W_2, and add the edge (w, w_ϕ) to R_0.
6. Let $W = \{\tau\} \bigcup W_1 \bigcup W_2$, and R be the euclidean closure of R_0.

It is clear that this algorithm always terminates. The following lemma has the same content as Lemma 4, but L now refers to $K5$, $KD5$.

Lemma 5. *Let X be a finite $\mathcal{C}L$-consistent set of formulae not containing $\boxtimes$, $*$. Let $M = \langle W, \tau, R, H\rangle$ be the model graph constructed by the above algorithm for X. Then for any $w \in W$ and any $\phi \in H(w)$, $M, w \vDash \phi$.*

Proof. We prove this lemma by induction on the number of connectives occurring in ϕ. It suffices to show that

1. For any $x \in W_2$ and any $\neg\Box\phi \in H(x)$, there exists $y \in W_2$ such that $\neg\phi \in H(y)$.
2. For any $x, y \in W_1 \bigcup W_2$ and any formula ϕ, if $\Box\phi$ or $\boxtimes\phi$ belongs to $H(x)$, then $\phi \in H(y)$.

Proof of 1:
Suppose that $x \in W_2$ and $\neg\Box\phi \in H(x)$. There exists $u \in W_1$ such that $R_0(u, x)$ holds. We have $* \in H(u)$. Since x is created from u and $\neg\Box\phi \in H(x)$, it follows that $\Box\phi \in Sf(H(u))$. Hence either $\Box\phi \in H(u)$ or $\neg\Box\phi \in H(u)$, since $H(u)$ is subformula-complete. If $\Box\phi \in H(u)$, then, by the rule (5_*), $\boxtimes\Box\phi \in H(u)$,

and hence $\Box\phi \in H(x)$, which contradicts the fact that $\neg\Box\phi \in H(x)$. Therefore $\neg\Box\phi \in H(u)$, and there exists $y \in W_2$ such that $\neg\phi \in H(y)$.
Proof of 2:

Suppose that $x \in W_1$ and $\Box\phi \in H(x)$. Since x is created from τ and $\Box\phi \in H(x)$, we have $\Box\phi \in Sf(H(\tau))$. Hence either $\Box\phi \in H(\tau)$ or $\neg\Box\phi \in H(\tau)$. Since $\Box\phi \in H(x)$, $\neg\Box\phi$ and $\boxtimes\neg\Box\phi$ cannot belong to $H(\tau)$, otherwise, by the rule (5), we would have $\neg\Box\phi \in H(x)$ (a contradiction). Hence $\Box\phi \in H(\tau)$, and by the rule ($5_\Box$), $\boxtimes\Box\phi \in H(\tau)$. It follows that for every $y \in W_1 \bigcup W_2$ we have $\phi \in H(y)$.

Now suppose that $x \in W_2$ and $\Box\phi \in H(x)$. Let $u \in W_1$ be the world such that $R_0(u, x)$ holds. Since x is created from u and $\Box\phi \in H(x)$, it follows that $\Box\phi \in Sf(H(u))$. Hence either $\Box\phi \in H(u)$ or $\neg\Box\phi \in H(u)$. If $\neg\Box\phi \in H(u)$, then, by the rule (5), $\neg\Box\phi \in H(x)$, which contradicts the assumption that $\Box\phi \in H(x)$. Hence $\Box\phi \in H(u)$. Reasoning similarly as for the above case, we derive that for every $y \in W_1 \bigcup W_2$ it holds that $\phi \in H(y)$.

For the last case, assume that $x \in W_1 \bigcup W_2$ and $\boxtimes\phi \in H(x)$. Since $* \in H(x)$, the formula $\boxtimes\phi$ in $H(x)$ must be introduced by the rule (5_*). Hence ϕ is of the form $\Box\psi$ and $\Box\psi \in H(x)$. Reasoning similarly as for the above cases, we derive that $\boxtimes\Box\psi \in H(\tau)$. It is clear that for every $y \in W_1$, $\Box\psi \in H(y)$. Suppose that $y \in W_2$. There exists $z \in W_1$ such that $R_0(z, y)$ holds. Since $\{\Box\psi, *\} \subseteq H(z)$, by the rule ($5_*$), it follows that $\boxtimes\Box\psi \in H(z)$, and hence $\Box\psi \in H(y)$. Therefore, for every $y \in W_1 \bigcup W_2$, $\phi \in H(y)$.

Corollary 2. *Let X be a finite $\mathcal{C}L$-consistent set of formulae not containing $\boxtimes$, $*$. Then X is L-satisfiable.*

The proof of this corollary is similar to the proof of Corollary 1. The following theorem immediately follows from this corollary and Lemma 1.

Theorem 4. *The calculi $\mathcal{C}K5$ and $\mathcal{C}KD5$ are sound and complete.*

Our proofs give refined L-models for these logics, as done in Goré's article [5]. That is, for any L-satisfiable formula ϕ, there exists a finite L-model of ϕ with a frame $\langle W, \tau, R\rangle$ such that:

- if $L = KD5$, then, for $U = W - \{\tau\}$, there exists $V \subseteq U$ such that $R = \{\tau\} \times V \bigcup U \times U$;
- if $L = K5$, then $W = \{\tau\}$ and $R = \emptyset$, or the frame is a $KD5$-frame.

5 Interpolation for the Modal Logics *KB*, *KDB*, *K*5, *KD*5

We still use L to denote one of the logics KB, KDB, $K5$, $KD5$, and $\mathcal{C}L$ to denote the corresponding calculus. We say that ζ is an *interpolation formula in L for the formula $\phi \to \psi$* if all primitive propositions of ζ are common to ϕ and ψ, and $\phi \to \zeta$ and $\zeta \to \psi$ are both L-valid. The Craig interpolation lemma for L states that if $\phi \to \psi$ is L-valid, then there exists an interpolation formula in L for $\phi \to \psi$. In this section, using given tableau systems we show that this lemma holds for the logics KB, KDB, $K5$, and $KD5$. The lemma has been previously proved for the logics KB and KDB by Fitting [3].

Our tableau systems are refutation systems, so we use an indirect formulation of interpolation. We say that ϕ is an interpolation formula wrt. $\mathcal{C}L$ for the pair $\langle X, Y\rangle$, and also that $X \xrightarrow{\phi} Y$ *is $\mathcal{C}L$-valid*, if all primitive propositions of ϕ are common to X and Y, ϕ does not contain $\boxtimes$ or $*$, and $X; \neg\phi$ and $\phi; Y$ are both $\mathcal{C}L$-inconsistent. Since $\mathcal{C}L$ is sound and complete, it follows that if $\phi \xrightarrow{\zeta} \neg\psi$ is $\mathcal{C}L$-valid, then ζ is an interpolation formula in L for $\phi \to \psi$.

In this section, we show that for any finite formula sets X and Y not containing $*$, if $X; Y$ is $\mathcal{C}L$-inconsistent, then there exists an interpolation formula wrt. $\mathcal{C}L$ for the pair $\langle X, Y\rangle$. It follows that the Craig interpolation lemma holds for L.

Since $\mathcal{C}L$ contains the rule $(\neg)$, $Y \xrightarrow{\neg\phi} X$ is $\mathcal{C}L$-valid iff $X \xrightarrow{\phi} Y$ is $\mathcal{C}L$-valid. We call $Y \xrightarrow{\neg\phi} X$ the *reverse form* of $X \xrightarrow{\phi} Y$.

We call the following an *interpolation rule*

$$(\delta)\ \frac{N \xrightarrow{\phi} N'}{D_1 \xrightarrow{\phi_1} D'_1 \mid \ldots \mid D_k \xrightarrow{\phi_k} D'_k}$$

where δ is the name of a tableau rule, and

$$\frac{N; N'}{D_1; D'_1 \mid \ldots \mid D_k; D'_k}$$

is an instance of the tableau rule δ. This interpolation rule is said to be *$\mathcal{C}L$-sound* if whenever $D_1 \xrightarrow{\phi_1} D'_1, \ldots, D_k \xrightarrow{\phi_k} D'_k$ are $\mathcal{C}L$-valid, $N \xrightarrow{\phi} N'$ is also $\mathcal{C}L$-valid, provided that δ is a $\mathcal{C}L$-tableau rule.

In Table 5 we present interpolation rules for the considered logics. For the interpolation rules $(\bot)$ and $(\bot')$, we discard the parts of denominators because they are not necessary. For each tableau rule given in Table 3, except $(\bot')$, there is one corresponding interpolation rule. There are two interpolation rules for $(\bot')$ because the tableau rule $(\bot')$ has more than one principal formulae.

Lemma 6. *Let δ be a $\mathcal{C}L$-tableau rule. Then the interpolation rules corresponding to δ given in Table 5 are $\mathcal{C}L$-sound.*

Proof. The proof of this lemma is straightforward. We show here, for example, the proof for the rule (K).

Suppose that $Y; Z \xrightarrow{\psi} Y'; Z'; \neg\phi$ is $\mathcal{C}L$-valid, where L is KB or KDB. It follows that $Y; Z; \neg\psi$ and $\psi; Y'; Z'; \neg\phi$ are $\mathcal{C}L$-inconsistent, and there are closed $\mathcal{C}L$-tableaux for these sets. By applying the tableau rule (K) to $X; \Box Y; \boxtimes Z; \neg\Box\psi$ and $\Box\psi; X'; \Box Y'; \boxtimes Z'; \neg\Box\phi$ we obtain $Y; Z; \neg\psi$ and $\psi; Y'; Z'; \neg\phi$, respectively. Hence there are closed $\mathcal{C}L$-tableaux for $X; \Box Y; \boxtimes Z; \neg\Box\psi$ and $\Box\psi; X'; \Box Y'; \boxtimes Z'; \neg\Box\phi$. Thus $X; \Box Y; \boxtimes Z \xrightarrow{\Box\psi} X'; \Box Y'; \boxtimes Z'; \neg\Box\phi$ is $\mathcal{C}L$-valid.

Lemma 7. *Let T be a closed $\mathcal{C}L$-tableau and n be a node but not a leaf node of T. Let N be the set carried by n, and let $N = N' \bigcup N''$, where $* \in N'$ iff $* \in N''$. Then there exists a formula ϕ such that $N' \xrightarrow{\phi} N''$ is $\mathcal{C}L$-valid.*

Table 5. Interpolation rules

$$(\bot)\ X \stackrel{\top}{\rule{2em}{0.4pt}} X';\bot \qquad (\bot')\ X \stackrel{\top}{\rule{2em}{0.4pt}} X';\phi;\neg\phi \qquad (\bot')\ X;\phi \stackrel{\phi}{\rule{2em}{0.4pt}} X';\neg\phi$$

$$(\neg)\ \frac{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\neg\neg\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\phi} \qquad (\wedge)\ \frac{X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\phi\wedge\psi}{X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\phi;\psi} \qquad (5_*)\ \frac{X;* \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\Box\phi;*}{X;* \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\Box\phi;\boxtimes\Box\phi;*}$$

$$(K)\ \frac{X;\Box Y;\boxtimes Z \stackrel{\Box\psi}{\rule{2em}{0.4pt}} X';\Box Y';\boxtimes Z';\neg\Box\phi}{Y;Z \stackrel{\psi}{\rule{1.5em}{0.4pt}} Y';Z';\neg\phi}$$

$$(KD)\ \frac{X;\Box Y;\boxtimes Z \stackrel{\Box\psi}{\rule{2em}{0.4pt}} X';\Box Y';\boxtimes Z'}{Y;Z \stackrel{\psi}{\rule{1.5em}{0.4pt}} Y';Z'} \qquad (KD_*)\ \frac{X;\Box Y;\boxtimes Z \stackrel{\Box\psi}{\rule{2em}{0.4pt}} X';\Box Y';\boxtimes Z'}{Y;Z;* \stackrel{\psi}{\rule{1.5em}{0.4pt}} Y';Z';*}$$

$$(5)\ \frac{X;\Box Y;\boxtimes Z;\neg\Box U \stackrel{\Box\psi}{\rule{2em}{0.4pt}} X';\Box Y';\boxtimes Z';\neg\Box U';\neg\Box\phi}{Y;Z;\neg\Box U;* \stackrel{\psi}{\rule{1.5em}{0.4pt}} Y';Z';\neg\Box U';\neg\Box\phi;\neg\phi;*}$$

$$(5_\Box)\ \frac{X \stackrel{\psi\wedge\zeta}{\rule{2em}{0.4pt}} X';\Box\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\Box\phi;\boxtimes\Box\phi \mid X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\Box\phi;\boxtimes\neg\Box\phi} \text{ where } * \notin X \bigcup X'$$

$$(B_\Box)\ \frac{X \stackrel{\psi\wedge\zeta}{\rule{2em}{0.4pt}} X';\Box\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\Box\phi;\phi \mid X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\Box\phi;\neg\phi;\boxtimes\neg\Box\phi}$$

$$(B_\Diamond)\ \frac{X \stackrel{\psi\wedge\zeta}{\rule{2em}{0.4pt}} X';\neg\Box\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\neg\Box\phi;\phi \mid X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\neg\Box\phi;\neg\phi;\boxtimes\neg\Box\phi}$$

$$(sfc_\Box)\ \frac{X \stackrel{\psi\wedge\zeta}{\rule{2em}{0.4pt}} X';\Box\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\Box\phi;\phi \mid X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\Box\phi\ \neg\phi}$$

$$(sfc_\Diamond)\ \frac{X \stackrel{\psi\wedge\zeta}{\rule{2em}{0.4pt}} X';\neg\Box\phi}{X \stackrel{\psi}{\rule{1.5em}{0.4pt}} X';\neg\Box\phi;\phi \mid X \stackrel{\zeta}{\rule{1.5em}{0.4pt}} X';\neg\Box\phi\ \neg\phi}$$

$$(sfc_\vee)\ \frac{X \stackrel{\zeta_1\wedge\zeta_2\wedge\zeta_3}{\rule{4em}{0.4pt}} X';\neg(\phi\wedge\psi)}{X \stackrel{\zeta_1}{\rule{1.5em}{0.4pt}} X';\neg\phi;\neg\psi \mid X \stackrel{\zeta_2}{\rule{1.5em}{0.4pt}} X';\neg\phi;\psi \mid X \stackrel{\zeta_3}{\rule{1.5em}{0.4pt}} X';\phi;\neg\psi}$$

Proof. We prove this lemma by induction on the depth of n.

Consider the case when n is a predecessor of a leaf node. N must be of the form $X; \bot$ or $X; \phi; \neg\phi$. By the interpolation rules $(\bot)$, $(\bot')$, and their reverse forms, the assertion of the lemma holds.

Now suppose that n is not a predecessor of any leaf node and n has k child nodes $d_1, \ldots, d_k$ that carry $D_1, \ldots, D_k$, respectively. Assume that the assertion of this lemma holds for all $d_1, \ldots, d_k$. We show that the assertion also holds for n.

Suppose that $D_1, \ldots, D_k$ are created from N by applying a $\mathcal{C}L$-tableau rule δ. Consider the case when the principal formula belongs to N''. There exist D'_1, $D''_1, \ldots, D'_k, D''_k$ such that $D_i = D'_i \bigcup D''_i$, for $1 \le i \le k$, and that

$$\frac{N' \overset{?}{-} N''}{D'_1 \overset{?}{-} D''_1 \mid \ldots \mid D'_k \overset{?}{-} D''_k}$$

is an instance of the interpolation rule δ, where the symbol ? can match any formula. By inductive assumption, there exist formulae $\phi_1, \ldots, \phi_k$ such that $D'_i \overset{\phi_i}{-} D''_i$ is $\mathcal{C}L$-valid, for $1 \le i \le k$. Choose ϕ to be the formula built from ϕ_1, $\ldots, \phi_k$, as specified by the interpolation rule δ, such that $N' \overset{\phi}{-} N''$ is $\mathcal{C}L$-valid.

If the principal formula (of the application of the tableau rule δ to N) does not belongs to N'', then it must belong to N'. Suppose that it is the case. Let ψ be a formula such that $N'' \overset{\psi}{-} N'$ is $\mathcal{C}L$-valid, and let $\phi = \neg\psi$. Thus $N' \overset{\phi}{-} N''$ is $\mathcal{C}L$-valid.

We now turn to the main lemma of this section.

Lemma 8 (Craig Interpolation Lemma for *KB*, *KDB*, *K*5, *KD*5). *Let L be one of the modal logics KB, KDB, K5, and KD5, and $\phi \to \psi$ be L-valid. Then there exists an interpolation formula in L for $\phi \to \psi$.*

Proof. Since $\phi \to \psi$ is L-valid, the set $\phi; \neg\psi$ is $\mathcal{C}L$-inconsistent. Hence there exists a closed $\mathcal{C}L$-tableau for $\phi; \neg\psi$. By Lemma 7, there exists a formula ζ such that $\phi \overset{\zeta}{-} \neg\psi$ is $\mathcal{C}L$-valid. Thus ζ is an interpolation formula in L for $\phi \to \psi$.

6 Conclusions

We have given complete sequent-like tableau systems for the modal logics *KB*, *KDB*, *K*5, and *KD*5. Our systems have the analytic superformula property and can thus give a decision procedure. Our presentation fulfills the picture of sequent-like tableau systems for the basic normal modal logics (i.e. the ones obtainable from the logic *K* by the addition of any combination of the axioms T, D, 4, 5, and B).

Using the given systems, we have presented a proof of the Craig interpolation lemma for the considered logics. (The lemma has been previously proved for the logics *KB* and *KDB* by Fitting [3].) Our proof has an advantage that it gives a constructive way to compute interpolation formulae (in the considered logics), as it is based on analytic systems.

Our tableau systems are not efficient since they use cuts. In our recent work [12], efficient clausal tableau systems for the modal logic KB, KDB, B, among others, are presented. The systems are sequent-like, cut-free, and give a decision procedure that runs in $O(n^2)$-space. They require, however, inputs in clausal form.

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