2023 / EGZAMIN / 3

$X_1, \dots, X_m \quad m>1 \qquad F_X, f_X$

$Y_1 = \max X_i \qquad Y_2 = \max_2 X_i$

·) $P(Y_1 \le y) = P(\max_i X_i \le y) = P(X_1 < y \wedge \dots \wedge X_m < y) =$

$= \prod_{i=1}^{m} P(X_i \le y) = \prod_{i=1}^{n} F_X(y) = [F_X(y)]^n$

·) $P(Y_2 \le x) = P(\{\exists! i : X_i > x\}) = \sum_i P(X_1 \le x, \dots, X_i > x, \dots, X_n \le x)$

$= \sum_i P(X_i > x) \prod_{j \ne i} P(X_j \le x) = \sum_i (1 - F_X(x)) (F_X(x))^{n-1} =$

$= n (1 - F_X(x)) (F_X(x))^{n-1}$

$f_{X|Y=y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}$

$P(X \in A \mid Y = y) = \int_A f_{X|Y=y}(x)\,dx$

$P(Y_1 \le z \mid X_1 = x) = \int_0^z \frac{f_{Y_1 X_1}(t, x)}{f_{X_1}(x)}\,dt$

$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$

$F_{Y_1 X_1}(t, x) = P(Y_1 < t \wedge X_1 < x) = P(X_1 < t \wedge \dots \wedge X_m < t \wedge X_1 < x)$

$\prod_{i=2}^{n} P(X_i < t) \cdot P(X_1 < t \wedge X_1 < x) = \begin{cases} [F_X(t)]^{n-1} \cdot F_X(t) & t > x \\ [F_X(t)]^{n-1} \cdot F_X(x) & t < x \end{cases}$

itd...

EGZAMIN 2022 / 3 $\quad f_x = \theta e^{-\theta x} \quad f_y = 1$

$X \sim Exp(\theta) \quad Y \sim Unif(0,1) \quad Z = X+Y-1$

$$P(Z \le x \mid Z>0) = \frac{P(Z \le x \wedge Z>0)}{P(Z>0)} = \frac{F_z(x) - F_z(0)}{1 - F_z(0)}$$

$$F_z(t) = P(Z \le t) = P(X+Y-1 \le t) = P(X+Y \le 1+t) =$$

$$\int P(X+y \le 1+t) \cdot f_Y(y)\, dy = \int_0^1 \int_0^{1+t-y} P(x+y \le 1+t) \cdot f_x(x) f_y(y)\, dx\, dy$$

$$\int_0^1 \left[\int_0^{1+t-y} \theta e^{-\theta x} dx \right] dy = \int_0^1 \left(1 - e^{\theta(y-1-t)}\right) dy = 1 - e^{-\theta(1+t)} \cdot \frac{1}{\theta} e^{\theta y} \Big|_0^1 = 1 - e^{-\theta(1+t)} \left[\frac{e^\theta}{\theta} - \frac{1}{\theta}\right] =$$

$$\downarrow$$

$$-e^{-\theta x} \Big|_0^{1+t-y} = 1 - e^{\theta(y-1-t)}$$

$$= 1 - \frac{1}{\theta} e^{-\theta(1+t)} \left[e^\theta - 1\right] =$$

$$= 1 - \frac{1}{\theta} e^{-\theta t} + \frac{1}{\theta} e^{-\theta(1+t)}$$

$$t=0 \Rightarrow 1 - \frac{1}{\theta} + \frac{1}{\theta} e^{-\theta}$$

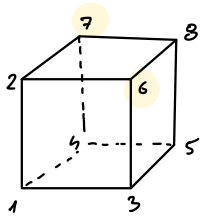
$$P(Z \le x \mid Z>0) = \frac{1 - e^{-x} - 1 + e^0}{}$$

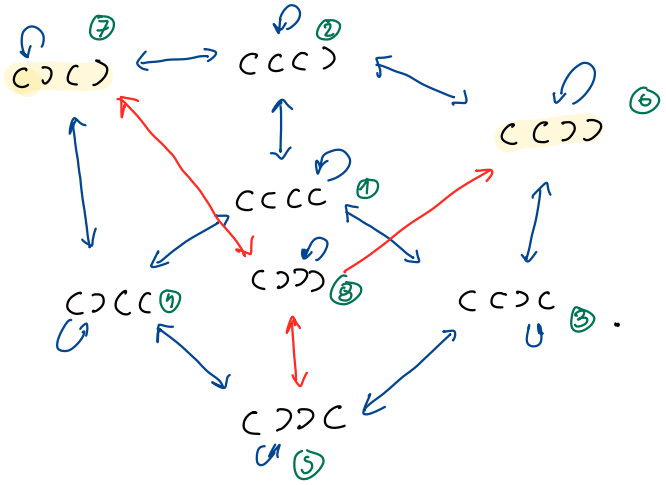
•) $P(X+Y=t) = \sum_y P(X+y=t) \cdot P(Y=y) =$

$$= \sum_y P(X=t-y) \cdot P(Y=y)$$

•) $f_{X+Y}(t) = \int_k f_x(k)\, f_z(t-k)\, dk$

EGZAMIN 2022 / ZAD 2





a)

$$t_1 = 1 + \frac{1}{4}t_1 + \frac{1}{4}t_2 + \frac{1}{4}t_3 + \frac{1}{4}t_4$$

$$t_2 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_7 + \frac{1}{4}t_1 + \frac{1}{4}t_2$$

$$t_3 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_5 + \frac{1}{4}t_1 + \frac{1}{4}t_3$$

$$t_4 = 1 + \frac{1}{4}t_5 + \frac{1}{4}t_7 + \frac{1}{4}t_1 + \frac{1}{4}t_4$$

$$t_5 = 1 + \frac{1}{4}t_3 + \frac{1}{4}t_4 + \frac{1}{4}t_8 + \frac{1}{4}t_5$$

$$t_8 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_7 + \frac{1}{4}t_5 + \frac{1}{4}t_8$$

$$\frac{3}{4}t_1 = 1 + \frac{1}{4}t_4 + \frac{1}{4}t_2 + \frac{1}{4}t_3$$

- $\frac{3}{4}t_2 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_7 + \frac{1}{4}t_1$

$$\frac{3}{4}t_3 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_5 + \frac{1}{4}t_1$$

$$\frac{3}{4}t_4 = 1 + \frac{1}{4}t_5 + \frac{1}{4}t_7 + \frac{1}{4}t_1$$

$$\frac{3}{4}t_5 = 1 + \frac{1}{4}t_3 + \frac{1}{4}t_4 + \frac{1}{4}t_8$$

- $\frac{3}{4}t_8 = 1 + \frac{1}{4}t_6 + \frac{1}{4}t_7 + \frac{1}{4}t_5$

$$\frac{1}{4}t_2 = \frac{1}{3} + \frac{1}{12}t_1$$

$$\frac{1}{4}t_8 = \frac{1}{3} + \frac{1}{12}t_5$$

$$\frac{3}{4}t_1 = 1 + \frac{1}{4}t_4 + \left(\frac{1}{3} + \frac{1}{12}t_1\right) + \frac{1}{4}t_3$$

$$\frac{3}{4}t_5 = 1 + \frac{1}{4}t_3 + \frac{1}{4}t_4 + \left(\frac{1}{3} + \frac{1}{12}t_5\right)$$

$$\frac{8}{12}t_1 = \frac{4}{3} + \frac{1}{4}t_4 + \frac{1}{4}t_3$$

$$\frac{8}{12}t_5 = \frac{4}{3} + \frac{1}{4}t_3 + \frac{1}{4}t_4$$

$$\Rightarrow \quad t_1 = t_5$$

$$t_2 = t_8$$

$$\frac{3}{4}t_1 = 1 + \frac{1}{4}t_1 + \frac{1}{4}t_2 + \frac{1}{4}t_3$$

$$\frac{3}{4}t_2 = 1 + \frac{1}{4}t_1$$

$$\frac{3}{4}t_3 = 1 + \frac{1}{4}t_1 + \frac{1}{4}t_1$$

$t_3 = t_4$

$$\frac{3}{4}t_4 = 1 + \frac{1}{4}t_1 + \frac{1}{4}t_1$$

$$\frac{3}{4}t_1 = 1 + \frac{1}{4}t_2 + \frac{1}{2}t_3$$

$$\frac{3}{4}t_2 = 1 + \frac{1}{4}t_1$$

$$\frac{3}{4}t_3 = 1 + \frac{1}{2}t_1 \qquad t_3 = \frac{4}{3} + \frac{2}{3}t_1$$

$$\frac{3}{4}t_1 = 1 + \frac{1}{3} + \frac{1}{12}t_1 + \frac{2}{3} + \frac{1}{3}t_1$$

$$\left(\frac{9}{12} - \frac{1}{12} - \frac{4}{12}\right)t_1 = 1 + \frac{1}{3} + \frac{2}{3} = 2$$

$$\frac{4}{12}t_1 = 2$$

$$t_1 = 6$$

6) $P(\text{pierwsze } (()))) = \frac{1}{2}$ przez symetrię

c) podobnie jak pierwsze tylko bez "+1" i z kilkoma zmianami znowu będzie można skorzystać z symetrii

:) Jeśli $X \sim N(\mu, \sigma^2)$ to $cX+a \sim (c\mu+a, c^2\sigma^2)$

$F_X(x) = P(X<x)$ $\quad f_X = (F_X)'$

$F_{cX+a}(x) = P(cX+a<x) = P(X < \frac{x-a}{c}) = F_X(\frac{x-a}{c})$

$f_{cx+a}(x) = (F_{cx+a}(x))' = (F_X(\frac{x-a}{c}))' = f_X(\frac{x-a}{c}) \cdot \frac{1}{c}$

$\frac{1}{c} f_X(\frac{x-a}{c}) = \frac{1}{c} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\frac{x-a}{c}-\mu)^2}{2\sigma^2}\right) =$

$= \frac{1}{\sqrt{2\pi}c\sigma} \exp\left(-\frac{(x-(a+c\mu))^2}{2\sigma^2c^2}\right) \sim N(c\mu+a, c^2\sigma^2)$

$X \sim N(\mu_1, \sigma_1^2)$ $\quad Y \sim N(\mu_2, \sigma_2^2)$ $\quad S \sim N(0,1)$

$X = \sigma_1 S + \mu_1$ $\quad Y \sim \sigma_2 S + \mu_2$

$X+Y = \sigma_1 S + \mu_1 + \sigma_2 S + \mu_2 = (\mu_1+\mu_2) + (\sigma_1 S + \sigma_2 S)$

przesunięcie ↓ suma ↓ dwóch prostszych rozkładów

$$f_{\sigma_1 S + \sigma_2 S}(z) = \frac{1}{2\pi\sigma_1\sigma_2} \int e^{-\left(\frac{x^2}{2\sigma_1^2} + \frac{(z-x)^2}{2\sigma_2^2}\right)} dx = *$$

$$\frac{x^2}{2\sigma_1^2} + \frac{x^2}{2\sigma_2^2} - \frac{2xz}{2\sigma_2^2} + \frac{z^2}{2\sigma_2^2} = \frac{1}{2}\left\{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)x^2 - \frac{1}{\sigma_2^2}2xz + \frac{z^2}{\sigma_2^2}\right\} =$$

$$\frac{1}{2}\left\{\left(\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right)^2 - \frac{z^2}{\sigma_2^4\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)} + \frac{z^2}{\sigma_2^2}\right\} =$$

$$\frac{1}{2}\left\{\left(\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right)^2 + \frac{z^2}{\sigma_2^2}\left[1 - \frac{1}{\frac{\sigma_2^2}{\sigma_1^2} + 1}\right]\right\} =$$

$$\frac{1}{2}\left\{\left(\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right)^2 + \frac{z^2}{\sigma_2^2}\left[\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right]\right\} =$$

$$\frac{1}{2}\left\{\left(\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right)^2 + \frac{z^2}{\sigma_1^2 + \sigma_2^2}\right\}$$

$$* = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right)^2} \cdot e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}} dx = \left\{\begin{array}{l} t = \sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,x - \frac{z}{\sigma_2^2\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \\ dt = \sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,dx \end{array}\right\}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt =$$

$\sqrt{\ }$ to plus to to jest 1 bo to jest prawdopodobieństwo całkowite dla $\mathcal{N}(0,1)$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}}$$

Zatem jest to f. gęstości rozkładu $\mathcal{N}(0, \sigma_1^2+\sigma_2^2)$