Lecture 5

Axiom 4 (How does the state evolve?)

What can be the most general transformation of a quantum state?

When describing quantum state we only care if it is normalized.

So we want the evolution to keep the vectors the same length.

General evolution $\mathcal{U}$ is a linear map which preserves lengths of all vectors

$$\boxed{\mathcal{U} \in \mathcal{L}(\mathcal{H}) : \forall_{|\psi\rangle \in \mathcal{H}} \ \| \hat{\mathcal{U}} |\psi\rangle \| = \| |\psi\rangle \| = 1} \qquad \text{Eq. 1}$$

such transformations are called unitary, and matrices associated are unitary matrices.

This is evolution in a Schrödinger picture, where it is state which evolves, not anything else.

Given that we have the most general transformation of states, we can write the most general evolution as:

$$\boxed{|\psi(t')\rangle = \hat{\mathcal{U}}(t', t) |\psi(t)\rangle} \qquad \text{Eq. 2}$$

We can also consider an infinitesimal evolution in time.

The Schrödinger equation has form

$$\boxed{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle} \qquad \text{Eq. 3}$$

where $\hat{H}(t)$ is self-adjoint operator, and $\hbar$ is a constant.*

hamiltonian

* $\hbar$ is conventionaly set to 1 but it main purpose is to make units of both sides of the equation the same

Is there any link between those two pictures? YES!

In special case when hamiltonian does not depend on time $[\hat{H}(t) = \hat{H}]$

we can solve Schr. eq.:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}\hat{H}|\psi(t)\rangle \Rightarrow |\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}\cdot t\right)|\psi(0)\rangle$$

NOTE: if $\hat{H}$ is self adjoint operator then $\exp(-i\hat{H})$ is an unitary operator (homework).

and additionally we can see that

$$\boxed{\hat{U}(t',t) = \exp\left(-\frac{i}{\hbar}\hat{H}(t'-t)\right)} \quad \text{Eq. 4.}$$

EXAMPLES:

NOTE: $\{I_2, \sigma_x, \sigma_y, \sigma_z\}$ is a basis for 2×2 hermitian matrices.

Unitary matrices 2x2

1) $\sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $\sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad$ $\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

BIT FLIP $\quad$ PHASE FLIP

$\sigma_x|0\rangle = |1\rangle \quad \sigma_x|1\rangle = |0\rangle$

$\sigma_z|0\rangle = |0\rangle \quad \sigma_z|1\rangle = -|1\rangle$

2) $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Hadamard gate

why need for "−"? UNITARY!

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

AXIOM 5 (COMPOSITE SYSTEMS)

So far we considered only one quantum system. What if we want to describe two systems at the same time (e.g. there is an interaction between them)?

We have two subsystems: A (with Hibert space $\mathcal{H}_A$) and B (with $\mathcal{H}_B$). Then the Hibert space of the system AB is tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

If system A is prepared in state $|\psi\rangle_A$ and system B in state $|\varphi\rangle_B$ then system AB has state $|\psi\rangle_A \otimes |\varphi\rangle_B$.

Properties of Tensor Product:

1) A has basis $\{|i\rangle_A\}_{i=0}^{m_A}$ and B - $\{|j\rangle_B\}_{j=0}^{m_B}$ then AB has basis $\{|i\rangle_A \otimes |j\rangle_B\}_{i=0,\, j=0}^{m_A \quad m_B}$ (dimension is multiplied) unlike carthesian product

* we will use notation $|i\rangle \otimes |j\rangle \equiv |ij\rangle$

2) inner product in AB is naturally defined as

$$\left({}_A\langle i| \otimes {}_B\langle j|\right)\left(|k\rangle_A \otimes |l\rangle_B\right) = {}_A\langle i|k\rangle_A \cdot {}_B\langle j|l\rangle_B = \delta_{ik}\,\delta_{jl}$$

vectors are orthogonal when at least one difference

3) tensor product of operators

$$(\hat{M} \otimes \hat{N})\left(|\psi\rangle_A \otimes |\varphi\rangle_B\right) = \hat{M}|\psi\rangle_A \otimes \hat{N}|\varphi\rangle_B$$

Examples: Two qubits

B \ A	$\|0\rangle$	$\|1\rangle$
$\|0\rangle$	$\|00\rangle$	$\|10\rangle$
$\|1\rangle$	$\|01\rangle$	$\|11\rangle$

four possible states.

the most general two qubit state is

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$ with normalization condition

Example: Extension

general n-qubit state lives in $\mathcal{H}_n = \mathbb{C}^2 \otimes \mathbb{C}^2 \ldots \mathbb{C}^2 = \mathbb{C}^{2\otimes n}$

which has dimension 2^n

EXAMPLE: TENSOR PRODUCT IN MATRIX NOTATION

Let $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{array}{cc} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \end{array}$$

Similarly with matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} \cdots \end{bmatrix}$

$$A \otimes B = \left.\underbrace{\begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}}_{2\times 2}\right\} 2\times 2$$

With the definition of composite systems we can now ask what states are possible to produce and which of them have some interesting properties. We will start with definitions:

Quantum state $|\psi\rangle_{AB}$ (in system AB) is called separable if it can be written as a tensor product of two states (in A and B).

$$\boxed{\exists\, |\theta\rangle_A, |\varphi\rangle_B : |\psi\rangle_{AB} = |\theta\rangle_A \otimes |\varphi\rangle_B}$$

If a quantum state is not separable it is entangled.

We can think about an entangled states as those which cannot be well described by looking at the parts of the system separately. "AB is more than A+B"

EXAMPLE: BELL STATE

Let's consider $|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$. It is perfectly correlated state where measuring 0 (1) in A will always yield 1 (0) in B. But this can be done also in classical world. What can't is the same behaviour after a base change.

Let us change $|\psi\rangle\big|_{|0\rangle, |1\rangle} \longmapsto |\psi\rangle\big|_{|+\rangle, |-\rangle}$ $\quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ $\Rightarrow$ $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$, $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

We will obtain that $|\psi\rangle\big|_{|+\rangle, |-\rangle} = \frac{|-+\rangle - |+-\rangle}{\sqrt{2}}$ DO THE CALCULATIONS

We got exactly the same form of the state. It means that it is maximaly entangled.

$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) =$
$ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$
$\quad 0 \quad\quad\quad \underbrace{\quad\quad\quad\quad\quad}_{0} \quad\quad\quad 0$

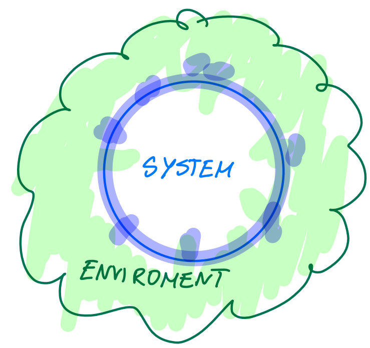
Key difference between classical and quantum correlation is that quantum are seen in more than one basis.

2.2. DENSITY OPERATORS

* where we do not have full knowledge of a system

HOW TO DESCRIBE OPEN* QUANTUM SYSTEMS?

So far we described systems which are isolated from the envivoment. But such systems does not really exist. For this we need to generalize / extend our theory for „open systems" i.e. where there is an interaction with an envivoment and where we do not have a full knowledge of a system.



Let's consider the following example:

we are given either state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

or $|\psi_2\rangle = \begin{cases} \xrightarrow{1/2} |0\rangle \\ \xrightarrow{1/2} |1\rangle \end{cases}$. What is a visible difference between them?

If we measure in a computational basis then we can't distinguish those states. BUT if we measure in basis $\{|+\rangle, |-\rangle\}$ then

$$|\langle + | \psi_1 \rangle|^2 = 1 \qquad |\langle + | 0 \rangle|^2 = \frac{1}{2} \qquad |\langle + | 1 \rangle|^2 = \frac{1}{2}$$

$$|\langle - | \psi_1 \rangle|^2 = 0 \qquad |\langle - | 0 \rangle|^2 = \frac{1}{2} \qquad |\langle - | 1 \rangle|^2 = \frac{1}{2}.$$

So in case of superposition we can change the base to always get answer "1" while in case of „mixture of states" we will always have this disambiguity.

We need something different to describe „mixtures of states".

We define density operator / density matrix as

$$\boxed{\rho \equiv \sum_i p_i \, |\psi_i\rangle\langle\psi_i|}$$

where $\{ \underset{\text{prob.}}{p_i}, \underset{\text{state}}{|\psi_i\rangle} \}$ is an ensemble of pure states.

How does one „measure" with density matrices?

Let's suppose we have a mixed state ρ with projective measurement operators (observables) $\{M_i\}$. Then obtaining the output 'm' given state 'k' is

$$p(m \mid i) = \langle\psi_i| M_m |\psi_i\rangle \overset{*}{=} \mathrm{Tr}\left(\langle\psi_i| M_m |\psi_i\rangle\right) =$$

* trace of number is a number

$$= \mathrm{Tr}\left(M_m |\psi_i\rangle\langle\psi_i|\right)$$ ← trace is cyclic

So total probability:

$$p(m) = \sum_i p(m|i) \cdot p_i = \sum_i p_i \, \mathrm{Tr}\left(M_m |\psi_i\rangle\langle\psi_i|\right) =$$

$$= \mathrm{Tr}\left(M_m \rho\right)$$

We have two "kinds" of density matrices

- for pure states $\rho = |\psi\rangle\langle\psi|$ (we have only one state)
- for mixed states $\rho \neq |\psi\rangle\langle\psi|$ (classical mixture of quantum states)

EXAMPLE: SUPERPOSITION VS. CORRELATIONS

Let's compare density matrices from initial example

1) $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \Rightarrow \rho_1 = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$

2) 50% $|0\rangle$ and 50% $|1\rangle \Rightarrow \rho_2 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

Let's measure it in $\{|+\rangle, |-\rangle\} \Rightarrow M_+ = |+\rangle\langle +| \quad M_- = |-\rangle\langle -|$

$$M_+ = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 0|\right) = \rho_1$$

We can think about it in matrix notation: $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

1) $\rho_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow M_+\rho_1 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$$p_+ = \mathrm{Tr}(M_+\rho_1) = 1 \qquad p_- = 1 - 1 = 0$$

2) $\rho_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow M_+\rho_2 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$

$$p_+ = \mathrm{Tr}(M_+\rho_2) = \frac{1}{2} \qquad p_- = 1 - \frac{1}{2} = \frac{1}{2}$$

We confirm our intuition and notation.

Properties of density operators:

* reveres also works: all conditions also give rise to a physical density matrix

(1) ~~SELF ADJOINT:~~ $\rho = \rho^{+}$

(2) TRACE CONDITION: $\mathrm{Tr}(\rho) = 1$ $\quad \sum_i \mathrm{Tr}(p_i |\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$

(3) POSITIVITY CONDITION: $\rho \geq 0$ → actually stronger than (1)

↳ $\langle\varphi|\rho|\varphi\rangle = \sum_i p_i \langle\varphi|\psi_i\rangle\langle\psi_i|\varphi\rangle = \sum_i p_i |\langle\varphi|\psi_i\rangle|^2 \geq 0 \quad \forall_{|\varphi\rangle}$

↳ those conditions guarantee orthonormal diagonalization, real and positive eigenvalues and sum of eigenvalues equal 1.

Can we easily distinguish pure states from mixed? Fortunetely YES!

If ρ is a density operator then

$$\rho \text{ - pure state} \iff \mathrm{Tr}(\rho^2) = 1$$

$$\rho \text{ - mixed state} \iff \mathrm{Tr}(\rho^2) < 1$$

Proof: orthonormal basis → we always can because it is positive

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \rho^2 = \sum_{i,j} p_i p_j |\psi_i\rangle\underbrace{\langle\psi_i|\psi_j\rangle}_{\delta_{ij}}\langle\psi_j|$$

$$= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|$$

$$\mathrm{Tr}(\rho^2) = \sum_i p_i^2 \leq \sum_i p_i$$

$$\sum_i p_i^2 \overset{?}{=} \sum_i p_i \Rightarrow \sum_i p_i(p_i - 1) = 0 \rightarrow p_i = 0 \vee p_i = 1$$

ale z $\sum p_i = 1$ musimy mieć jedno $p_i = 1$ reszta $p_i = 0$ czyli stan czysty

Where we can use density matrix formalism? „Perhaps deepest application is a descriptive tool for subsystems of composite quantum systems". Let's say we know that a quantum state lives in $\mathcal{H}_A \otimes \mathcal{H}_B$, but we can only see subsystem A.

Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB}. The reduced density operator for system A is defined by

$$\boxed{\rho^A \equiv \operatorname{Tr}_B(\rho^{AB})}$$

where $\operatorname{Tr}_B(\cdot)$ is partial trace over system B defined by

$$\operatorname{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2| \operatorname{Tr}(|b_1\rangle\langle b_2|)$$

where $|a_1\rangle, |a_2\rangle$ $(|b_1\rangle, |b_2\rangle)$ are any two vectors in the state space A (B).

We can shorten $\operatorname{Tr}(|b_1\rangle\langle b_2|)$ to $\langle b_2|b_1\rangle$

Partial trace tells us how much information we can access when we observe only a part of it

EXAMPLE: BELL STATE

Let $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$. Then

$$\rho^{AB} = |\psi\rangle\langle\psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)$$

$$|0\rangle\langle 0| \otimes |0\rangle\langle 0|$$

$$\rho^{A} = \mathrm{Tr}_B\left(\rho^{AB}\right) = \frac{1}{2}\left[\mathrm{Tr}_B\left(|00\rangle\langle 00|\right) + \mathrm{Tr}_B\left(|00\rangle\langle 11|\right) + \mathrm{Tr}_B\left(|11\rangle\langle 00|\right) + \mathrm{Tr}_B\left(|11\rangle\langle 11|\right)\right] =$$

$$= \frac{1}{2}\left[|0\rangle\langle 0| \cdot \langle 0|0\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |1\rangle\langle 1|\langle 1|1\rangle\right] =$$

$$= \frac{1}{2}\left[|0\rangle\langle 0| + |1\rangle\langle 1|\right] = \frac{1}{2} I$$

So the partial state is a <u>mixed state</u>. However the joint state is a <u>pure state</u>. Looking at only one part of it we loose all the knowledge (we get $|0\rangle$ and $|1\rangle$ with 50:50)

EXAMPLE: MEASUREMENT ON A PART OF THE SYSTEM

Let $\{|i\rangle\}_i$ is orthonormal basis in A and $\{|j\rangle\}_j$ in B.

Then $|\psi_\alpha\rangle_{AB} = \sum_{i,j} a^{\alpha}_{ij} |ij\rangle$ =>

$$\rho_{AB} = \sum p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_{\alpha ijkl} p_\alpha a^{\alpha}_{ij} (a^{\alpha}_{kl})^* |ij\rangle\langle kl|$$

$p(m_A) = \sum_{m_B} p(m_A \wedge m_B)$ $\{M^A_m\}_m$ $\{N^B_n\}_n$

$p(m_A \wedge m_B) = \mathrm{Tr}\left[(M^A_m \otimes N^B_n)\rho_{AB}\right]$

$\sum_n p(m_A \wedge m_B) = \mathrm{Tr}\left[(M^A_m \otimes \underbrace{[\sum_n N^B_n]}_{\mathbb{1}}\rho_{AB}\right]$

$p(m_A) = \mathrm{Tr}\left[(M_m \otimes \mathbb{1})\rho_{AB}\right]$ motivation

Let's consider measurement on only one subsystem: $M = M_A \otimes I_B$

$$\langle M\rangle_\rho = \mathrm{Tr}\left((M_A \otimes I_B)\rho_{AB}\right) = \sum_{\alpha ijkl} p_\alpha a^{\alpha}_{ij} (a^{\alpha}_{kl})^* \mathrm{Tr}\left(\langle kl| M_A \otimes I_B |ij\rangle\right) = \sum_{\alpha ijkl} (\cdot)\, \mathrm{Tr}\left(\langle k|M_A|i\rangle\langle l|j\rangle\right) =$$

$$\sum_{\alpha ijk} p_\alpha a^{\alpha}_{ij} (a^{\alpha}_{kj})^* \mathrm{Tr}\left(M_A |k\rangle\langle i|\right) = \mathrm{Tr}\left(M_A\left(\sum_{\alpha ijk} p_\alpha a^{\alpha}_{ij} (a^{\alpha}_{kj})^* |k\rangle\langle i|\right)\right) = \underline{\mathrm{Tr}\left(M_A \rho_A\right)}$$

Indeed

$$\rho_A := \mathrm{Tr}_B(\rho_{AB}) = \sum p_\alpha a_{ij}^\alpha (a_{kl}^\alpha)^* \mathrm{Tr}_B(|i\rangle\langle k| \otimes |j\rangle\langle l|) = \sum p_\alpha a_{ij}^\alpha (a_{kl}^\alpha)^* |i\rangle\langle k| \cdot \delta_{jl} =$$

$$= \sum p_\alpha a_{ij}^\alpha (a_{kj}^\alpha)^* |i\rangle\langle k|$$

so $\quad \mathrm{Tr}((M_A \otimes I_B)\rho_{AB}) = \mathrm{Tr}(M_A \rho_A)$ ← we „average out" the information in B

For pure states in $\mathbb{C}^2$ we can represent them as a sphere called Bloch sphere. Now we extend it for mixed state.

Any 2×2 hermitian matrix of trace 1 can be represented in "basis" $\{I_2, \sigma_x, \sigma_y, \sigma_z\}$

it is not a linear space

decomposition

so

$$\rho = \frac{1}{2} I_2 + x\sigma_x + y\sigma_y + z\sigma_z \qquad x, y, z \in \mathbb{R}$$

$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$

Now we also need to make sure that $\rho \geq 0$.

It is easy to calculate that $\det \rho = \frac{1}{4}(1 - x^2 - y^2 - z^2) = \frac{1}{4}(1 - \|\vec{r}\|^2)$.

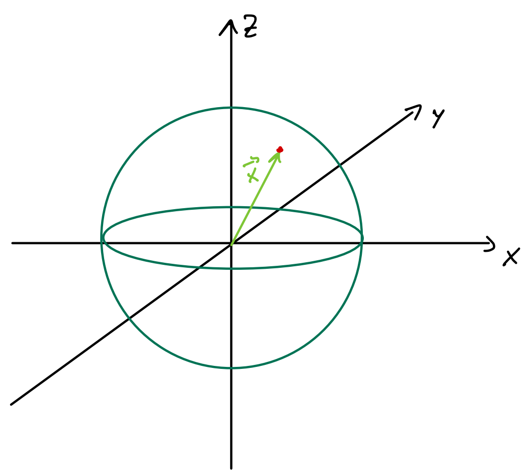
ρ is nonnegative when it's eigenvalues are non negative (here we use the fact that trace is 1)

2 numbers which sum and product is positive are also positive!

so $\quad \det \rho \geq 0 \Rightarrow \|\vec{r}\|^2 \leq 1$

So all 2×2 density matrices lay in Bloch ball

The boundary of Bloch ball are all states for which $\det \rho = 0$. This means that it has eigenvalues $\{0, 1\}$ so it is an ensemble of 100% one state. So it is Bloch sphere! containg all pure states.



How far away are two general quantum states?

Fidelity is a measure of how states are distinguishable. It is defined as

$$\boxed{F(\rho,\sigma)=\left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)}$$

Fidelity is

space orthogonal to the kernel

same as row space of a matrix

1) nonnegative
2) $=0$ when ρ,σ have support on mutually orthogonal spaces $\mathcal{N}(\rho)\perp\mathcal{N}(\sigma)$
3) $=1 \iff \rho=\sigma$

2.3 SCHMIDT DECOMPOSITION

USEFUL TOOL TO CHECK ENTANGLEMENT

<u>Theorem</u> (Schmidt decomposition) Suppose $|\psi\rangle$ is a pure state of composite system AB ($|\psi\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B$). Then there exists orthonormal states $\{|i_A\rangle\}\in\mathcal{H}_A$ and $\{|i_B\rangle\}\in\mathcal{H}_B$ such that

$$\boxed{|\psi\rangle=\sum_i\lambda_i\,|i_A\rangle\otimes|i_B\rangle}$$

where λ_i are non-negative real numbers satisfying $\sum_i\lambda_i^2=1$

<u>Proof</u>: Let $\{|i_A\rangle\}$ $(\{|\mu_B\rangle\})$ be an orthonormal basis for $\mathcal{H}_A$ $(\mathcal{H}_B)$

Then $|\psi\rangle=\sum_{i,\mu}a_{i\mu}\,|i_A\rangle\otimes|\mu_B\rangle=\sum_i|i_A\rangle\otimes\overbrace{\left[\sum_\mu a_{i\mu}|\mu_B\rangle\right]}^{|\tilde{i}_B\rangle}=\sum_i|i_A\rangle\otimes|\tilde{i}_B\rangle$

We need to check if $\{|\tilde{i}_B\rangle\}$ is orthogonal.

Now we will calculate ρ_A twice:

1) $\rho_A=\sum_i p_i\,|i\rangle\langle i|$ ← we write general state ρ in basis $|i\rangle$

2) $\rho_A=\mathrm{Tr}_B(|\psi\rangle\langle\psi|)=\mathrm{Tr}_B\left(\sum_{ij}|i_A\rangle\langle j_A|\otimes|\tilde{i}_B\rangle\langle\tilde{j}_B|\right)=\sum_{ij}|i_A\rangle\langle j_A|\cdot\langle\tilde{j}_B|\tilde{i}_B\rangle$

It means that $p_i \delta_{ij} = \langle \tilde{j}_B | \tilde{i}_B \rangle$

So indeed $\{ |\tilde{i}_B\rangle \}$ is orthogonal after all.
After scaling we obtain orthonormal vectors $|i_B\rangle \equiv \frac{1}{\sqrt{p_i}} |\tilde{i}_B\rangle$

So we obtain $|\psi\rangle = \sum_i \sqrt{p_i}\, |i_A\rangle \otimes |i_B\rangle$

NOTE: basis $|i_A\rangle$ and $|i_B\rangle$ depends on the state we want to decompose! in general we can't expand two vectors in the same basis!

NOTE: $\rho_B = \mathrm{Tr}_A (|\psi\rangle\langle\psi|) = \sum_i p_i |i_B\rangle\langle i_B|$

it means that ρ_A and ρ_B has the same eigenvalues!

Procedure of obtainig Schmidt decomposition:

given a density matrix ρ we need to diagonalize it using <u>SVD</u> It yields vectors as well as coefficients.

What for we can use Schmidt decomposition?

Entanglement (once again):

$\# p_i \neq 0$

-> $|\psi_{AB}\rangle$ is entangled if its <u>Schmidt number</u> > 1

-> $|\psi_{AB}\rangle$ is separable otherwise

From it follows

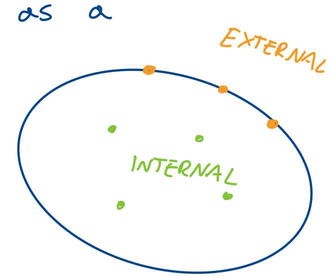
$$\boxed{|\psi\rangle_{AB} \text{ - is a product state} \iff \rho_A \text{ and } \rho_B \text{ are pure}}$$

2.4. ENSEMBLE INTEPRETATION

Natural question about density matrix is the structure they posses. What happens if we add two density matrices $\rho_1 + \rho_2 = ?$ We <u>do not</u> get another density matrix since $\mathrm{Tr}(\rho_1 + \rho_2) = 2 \neq 1$. What about convex combination $\lambda \rho_1 + (1-\lambda)\rho_2 = ?$ Teraz jest <u>OK</u> wszystko jest spełnione!

Density matrices are a <u>convex subset</u> of the real vector space of Hermitian operators.

External points are the points which cannot be written as a non-trivial combination of two other states.



Pure states are external points.

Mixed states are internal points.

Ok, so now a mixed state is a combination of pure states. But in how many ways this combination can be written?

Sets $\{|\tilde{\psi}_i\rangle\}$ and $\{|\tilde{\varphi}_j\rangle\}$ generate the same density matrix

i.e. $\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \sum_j |\tilde{\varphi}_j\rangle\langle\tilde{\varphi}_j|$ iff $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\varphi}_j\rangle$

where u is a unitary matrix of complex numbers.

Proof: Niels & Chuang (p. 104)

EXAMPLE: (TRIVIAL)

$\rho = \frac{1}{2} I$, now we can choose any orthonormal basis in $\mathbb{C}^2$ $\{|i\rangle\}$

and obtain $\rho = \frac{1}{2}\left[\sum_i |i\rangle\langle i|\right]$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

In particular $\rho = \frac{1}{2}\left[|0\rangle\langle 0| + |1\rangle\langle 1|\right] = \frac{1}{2}\left[|+\rangle\langle +| + |-\rangle\langle -|\right]$

PURIFICATION

Suppose we are given a mixed state ρ_A. It is possible (mathematicly) to introduce additional system R in which exist pure state $|AR\rangle$ such that

$$\rho_A = \mathrm{Tr}_R\left(|AR\rangle\langle AR|\right)$$

That is the pure state $|AR\rangle$ reduces to ρ_A when looking at system A alone.

Finding $|AR\rangle$ is called purification.

Proof: Let $\rho_A = \sum p_i |i_A\rangle\langle i_A|$ with $\{|i_A\rangle\}$ orthonormal basis

Let $\{|i_R\rangle\}$ orthonormal in R which is the same state space as A.

Then
$$|AR\rangle = \sum_i \sqrt{p_i}\, |i_A\rangle \otimes |i_R\rangle$$
exist because of Schmidt decomposition

Easy to calculate $\rho_A = Tr_R(|AR\rangle\langle AR|)$ ∎

EXAMPLES: DIFFERENT ENSEMBLES ONCE AGAIN

We are given state ρ_A which we want to purify. Let $\{|i_A\rangle\}$ be an eigenvectors of ρ_A (ρ_A is diagonal in this basis)

Let $\{|i_R\rangle\}$ be an orthonormal basis of R, the extended system.

Then we know there is purification of the form

$$|AR\rangle = \sum \sqrt{p_i}\, |i_A\rangle \otimes |i_R\rangle$$

But we had a freedom to choose basis in R. We can also choose $\{|j_R\rangle\}$. Those two basis are connected through an unitary transformation. So we can see that there exist infinitly many purifications of the system R. $|AR'\rangle = (I \otimes U)|AR\rangle$

$(I \otimes M)$

Furthermore: we also know that ρ_A is produced by a partial measurement on the subsystem R. So

2.5. Generalized Measurements

Let us first recall Axiom 3 (Measurement).

Set $\{M_m\}$ which $\sum_m M_m^\dagger M_m = \mathbb{1}$ and

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle \quad \text{and} \quad \frac{M_m |\psi\rangle}{\sqrt{\langle \psi | M_m^\dagger M_m | \psi \rangle}}$$

Sometimes we do not care about the state after the measurement (for instance photon is absorbed by a mirror and it does not make sense of talking abut state after the measurement). If it is a case then we can simplify this picture

POVM (Positive Operator-Valued Measure) Formalism

We are given measurements $\{M_m\}$. Let us define $E_m := M_m^\dagger M_m$

Because $\sum_m M_m^\dagger M = \mathbb{1} \Rightarrow \sum_m E_m = \mathbb{1}$. Now, if we are not interested in a post measurement state we can define POVM to be set of $\{E_m\}$ such that $E_m \geq 0$ and $\sum_m E_m = \mathbb{1}$.

There is an equivalence between those pictures.

POVM $\xrightarrow{M_m = \sqrt{E_m}}$ **AXIOM 3**

POVM:

$\{E_m\}$

$E_m \geq 0$ & $\sum_m E_m = \mathbb{1}$

$p(m) = \langle \psi | E_m | \psi \rangle$

AXIOM 3:

$\{M_m\}$

$\sum_m M_m^\dagger M_m = \mathbb{1}$

$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$

AXIOM 3 $\xrightarrow{E_m = M_m^\dagger M_m}$ **POVM**

NOTE: POVM are perfectly equivalent to projective measurements augmented by an unitary evolution. We will come to it soon.

$$\text{POVM} = \text{PVM} + \mathcal{U}$$

why to even introduce it? " Turns out that there are important problems in quantum computation and q. information, the answer to which involves a general measurements, rather than a projective measurement."

EXAMPLE:

We would like to distinguish between two quantum states

$$|\psi_1\rangle = |0\rangle \qquad \text{and} \qquad |\psi_2\rangle = (|0\rangle + |1\rangle)/\sqrt{2} = |+\rangle$$

The problem is that those states are not orthogonal and so they are imposible to distinguish with 100%. However it is possible to perform a measurements which distinguishes perfectly but only sometimes, and never mis-identify. Let's consider

A_1, A_2, A_3 - normalization factors

$$E_1 = A_1 |1\rangle\langle 1|$$
$$E_2 = A_2 |-\rangle\langle -| \quad \rightarrow$$
$$E_3 = \mathbb{1} - E_1 - E_2$$

those operators are positive and satisfy completness relation.

Now we have two scenarios:

1) we recieve $|\psi_1\rangle = |0\rangle$

$$p(1) = A_1 \langle 0|1\rangle\langle 1|0\rangle = 0$$
$$p(2) = A_2 \langle 0|-\rangle\langle -|0\rangle = \frac{A_2}{2}$$
$$p(3) = 1 - \frac{A_2}{2}$$

2) we recive $|\psi_2\rangle = |+\rangle$

$$p(1) = A_1 \langle +|1\rangle\langle 1|+\rangle = \frac{A_1}{2}$$
$$p(2) = A_2 \langle +|-\rangle\langle -|+\rangle = 0$$
$$p(3) = 1 - \frac{A_1}{2}$$

So if we measure 2 we are sure it was $|\psi_1\rangle$ and if we measure 1 we are sure it was $|\psi_2\rangle$. We got it at the cost of sometimes not knowing anything abut the state (measurement 3).

How can we do a generalized measurement (POVM or AXIOM3) with projective measurements only?

Let Q be a system and M auxiliary system. Let define U
additiomy

$$U(\underset{Q}{|\psi\rangle} \otimes \underset{M}{|0\rangle}) \equiv \sum_m M_m |\psi\rangle \otimes \underbrace{|m\rangle}_{\text{orthonormal basis}}$$

U is unitary

$$(\langle\varphi| \otimes \langle 0|) U^+ U (|\psi\rangle \otimes |0\rangle) = \sum_{mm'} \langle\varphi|M_m^+ M_{m'}|\psi\rangle \cdot \langle m|m'\rangle =$$

$$= \sum_m \langle\varphi| M_m^+ M_m |\psi\rangle = \langle\varphi|\psi\rangle$$

We checked it only on states $|\psi\rangle \otimes |0\rangle$ but it is possible to extend U to be unitary in whole space $Q \otimes M$. (homework)

Now we perform projective measurement: $P_m \equiv I_Q \otimes |m\rangle\langle m|$

$$p(m) = (\langle\psi| \otimes \langle 0|) U^+ P_m U(|\psi\rangle \otimes |0\rangle) =$$

$$= \sum_{m',m''} (\langle\psi|M_{m'}^+ \otimes \langle m'|)(I_Q \otimes |m\rangle\langle m|)(M_{m''}|\psi\rangle \otimes |m''\rangle) =$$

$$= \sum_{m'm''} \langle\psi|M_{m'}^+ M_{m''}|\psi\rangle \cdot \langle m'|m\rangle\langle m|m''\rangle =$$

$$= \langle\psi|M_m^+ M_m|\psi\rangle$$ → like in AXIOM 3

IMPORTANT NOTE: CREATION OF ENTANGLEMENT BETWEEN A QUANTUM SYSTEM AND A DEVICE THAT MEASURE IT IS THE ESSENCE OF QUANTUM MEASUREMENTS.

$$\begin{tikzcd} |\psi\rangle_{AB} \arrow[r, "U_\mathcal{E}"] & U_\mathcal{E}|\psi\rangle_{AB} \arrow[d, "Tr_B(\cdot)"] \\ \rho_A \arrow[u, "Purification"] \arrow[r, "\mathcal{E}"] & \mathcal{E}(\rho_A) \end{tikzcd}$$

2.6. QUANTUM CHANNELS

Motivation: we know from the axioms that pure states evolution is unitary. Furthermore any mixed state can be seen as a part of a greater pure state. So we can purify a mixed state, apply unitary evolution and then "trace out" the auxilliary system. But we can also introduce new mechanism to evolve mixed states directly.

Definition: Quantum channel $\mathcal{E}$ is a map from the set of density operators of the input space Q_1 to the set of density operators for the output space Q_2, which satisfy:
$$\mathcal{E}: \underset{Q_1}{\underset{\cap}{\rho}} \mapsto \underset{Q_2}{\underset{\cap}{\mathcal{E}(\rho)}}$$

1) $Tr(\mathcal{E}(\rho)) = 1$ → output of a quantum channel needs to be a proper density matrix

2) $\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i)$, $\sum_i p_i = 1$ → convex - linear map

3) $\mathcal{E}$ → completely positive ⇒ $\forall_{m} \ (\mathbb{1}_m \otimes \mathcal{E})(\rho_{RQ_1}) \geq 0$ for $\rho_{RQ_1} \geq 0$

NOTE: 3) implies normal positivity for $m = 0$

Theorem: The map $\mathcal{E}$ is quantum channel if and only if
$$\boxed{\mathcal{E}(\rho) = \sum_i E_i \rho E_i^+}$$
for some operators $\{E_i\}$ with $\sum_i E_i^+ E_i = \mathbb{1}$. → set $\{E_i\}$ is called Krauss operators

Proof: Niels & Chuang, p. 368.

NOTE: set $\{E_i\}$ is not unique!

EXAMPLE:

$E_1 = \frac{\mathbb{1}_2}{\sqrt{2}}$ $E_2 = \frac{Z}{\sqrt{2}}$

$F_1 = |0\rangle\langle 0|$ $F_2 = |1\rangle\langle 1|$

they give rise to the same quantum channel

Are channels revertible like unitary transformation? The answer is:

only special cases are revertible.

We are given a channel $\mathcal{E}$. Let us find a channel $\mathcal{E}'$ such that $(\mathcal{E}' \circ \mathcal{E})(\rho) = \rho$. But since $\mathcal{E}$ is convex linear we can limit ourselves to pure states only:

$$(\mathcal{E}' \circ \mathcal{E})(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|$$

$$(\mathcal{E}' \circ \mathcal{E})(|\psi\rangle\langle\psi|) = \mathcal{E}'\left(\mathcal{E}(|\psi\rangle\langle\psi|)\right) = \mathcal{E}'\left(\sum_a M_a |\psi\rangle\langle\psi| M_a^+\right) = \sum_a \mathcal{E}'\left(M_a |\psi\rangle\langle\psi| M_a^+\right) =$$

$$\sum_{ab} N_b M_a |\psi\rangle\langle\psi| M_a^+ N_b^+ = |\psi\rangle\langle\psi| \quad \forall_{|\psi\rangle}$$

We can deduce that $N_b M_a = \lambda_{ba} \mathbb{1}$ ← proportional to identity

We can also see

$$M_b^+ M_a = M_b^+ \left(\sum_\mu N_\mu^+ N_\mu\right) M_a = \sum_\mu \lambda_{\mu b}^* \lambda_{\mu a} \mathbb{1} \equiv \underset{\text{real, positiv}}{\beta_{ba}} \mathbb{1}$$

We can write $M_a = U_a \sqrt{M_a^+ M_a}$ ← polar decomposition still satisfy $\sum_a M_a^+ M_a = \mathbb{1}$

$$M_a = \underset{\text{unitary}}{U_a} \sqrt{M_a^+ M_a} = \sqrt{\beta_{aa}}\, U_a$$

So

$$M_b^+ M_a = \sqrt{\beta_{aa}\beta_{bb}}\, U_b^+ U_a = \beta_{ba} \mathbb{1} \quad \Rightarrow \quad \boxed{U_a = \frac{\beta_{ba}}{\sqrt{\beta_{aa}\beta_{bb}}} U_b}$$

It means that all Kraus operators are proportional to the same unitary matrix, so

$$U_i := \frac{\beta_{1i}}{\sqrt{\beta_{ii}\beta_{11}}} U_1$$

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i M_i |\psi\rangle\langle\psi| M_i^+ = \sum_i \beta_{ii} U_i |\psi\rangle\langle\psi| U_i^+ =$$

$$= \sum_i \beta_{ii} \frac{\beta_{1i}^2}{\beta_{ii}\beta_{11}} U_1 |\psi\rangle\langle\psi| U_1^+ = \frac{\sum_i \beta_{1i}^2}{\beta_{11}} U_1 |\psi\rangle\langle\psi| U_1^+ =$$

$$= \tilde{U} |\psi\rangle\langle\psi| \tilde{U} \quad \text{where} \quad \tilde{U} = \sqrt{\frac{\sum_i \beta_{1i}^2}{\beta_{11}}}\, U_1$$

unitary transformation!

So the channel can be reverted by another channel if it is unitary.

Decoherence is <u>irreversible</u>. Once system A is entangled to B we lose information not having access to system B.