# Computational Complexity — tutorial 4

NP-completeness, PSpace-completeness

**Some reminder about complexity spaces (not necessary now):**

$$\mathsf{LogSpace} \subseteq \mathsf{NLogSpace} \subseteq \mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{PSpace} = \mathsf{NPSpace} \subseteq \mathsf{ExpTime} \subseteq \mathsf{NExpTime} \subseteq \dots$$

*Space hierarchy theorem:* for any space-constructible function $f$, $\mathsf{DSpace}(o(f(n)) \subsetneq \mathsf{DSpace}(f(n))$; analogously for $\mathsf{NSpace}$. For example, $\mathsf{LogSpace} \subsetneq \mathsf{PSpace}$.

*Savitch's theorem:* for any function $f \in \Omega(\log(n))$, $\mathsf{NSpace}(f(n)) \subseteq \mathsf{DSpace}(f(n)^2)$. As a simple corollary, $\mathsf{PSpace} = \mathsf{NPSpace}$. But unknown if $\mathsf{LogSpace} \stackrel{?}{=} \mathsf{NLogSpace}$.

*Immerman–Szelepcsényi theorem:* for any function $f \in \Omega(\log(n))$, $\mathsf{NSpace}(s(n)) = \mathsf{co\text{-}NSpace}(s(n))$. As a simple corollary, $\mathsf{NLogSpace} = \mathsf{co\text{-}NLogSpace}$.

*Some known inequalities:* $\mathsf{NLogSpace} \subsetneq \mathsf{PSpace}$ (by space hierarchy and Savitch), $\mathsf{P} \subsetneq \mathsf{ExpTime}$ (halting problem-like argument).

**NP-completeness.** Some well-known NP-complete problems:

- NP Halting Problem. Given a non-deterministic Turing Machine $\mathcal{M}$, input $w$ and an integer $k$ (written in unary), is there an accepting run of $\mathcal{M}(w)$ which terminates in $k$ steps?
- Boolean Satisfiability (SAT). Given a boolean formula $\varphi$ over boolean variables $x_1, x_2, \dots, x_n$, is it satisfiable for any values of $x_1, \dots, x_n$? (That is, is $\exists_{x_1} \exists_{x_2} \dots \exists_{x_n} \varphi(x_1, \dots, x_n)$ true?)
- 3-CNF-SAT. Similar to above, but $\varphi$ is of the form $\varphi = \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_m$, and each $\psi_i$ is a disjunction (OR) of 3 literals ($x_i$ or $\neg x_i$). E.g., $\varphi = (x_1 \vee x_3 \vee \neg x_2) \wedge (x_1 \vee \neg x_3 \vee \neg x_4)$.

**PSPACE-completeness.** Some well known PSpace-complete problems:

- PSpace Halting Problem. Given a non-deterministic Turing Machine $\mathcal{M}$, input $w$ and an integer $k$ (written in unary), is there an accepting run of $\mathcal{M}(w)$ whose memory usage is bounded from above by $k$?
- Quantified Boolean Formula. Given a boolean formula $\varphi$ (with quantifiers) without any free variables (e.g. $\exists x_1 \forall x_2 \exists x_3 (x_1 \wedge (\neg x_2 \vee x_3))$), is it true? Note that we can assume that all quantifiers occur at the beginning of the formula.

For your enjoyment after the tutorial: G. Aloupis, E. D. Demaine, A. Guo, G. Viglietta. *„Classic Nintendo Games are (Computationally) Hard."* Available at `https://arxiv.org/abs/1203.1895`.

**1.** Prove that the following computational problems are NP-complete. For simplicity, assume that graphs are given as $n \times n$ adjacency matrices, and all integers on input are given in binary.

(a) **[solved]** Given three boolean formulas $\varphi_1, \varphi_2, \varphi_3$, decide if at least two of them are satisfiable at the same time.

(b) **[solved]** Clique problem: given an undirected graph $G$ and an integer $k$, does $G$ contain a clique (a complete graph) on $k$ vertices?

(c) Independent Set problem: given an undirected graph $G$ and an integer $k$, does $G$ contain an independent set of size $k$ (that is, $k$ vertices without any edges between them)?

(d) Vertex Cover problem: given an undirected graph $G$ and an integer $k$, is there a subset $A$ of $k$ vertices of $G$ such that each edge is incident to some vertex of $A$?

(e) $\{0, 1\}$-Linear Programming problem (in a decision form): given an integer matrix $A \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$, is there an **integer** vector $x \in \{0, 1\}^n$ such that $Ax \leqslant b$?

($Ax \leqslant b$ is a set of linear inequalities of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leqslant b_i$.)

If $x$ can be arbitrary $\mathbb{Z}^n$, then the problem is NP-complete, but this is non-trivial. Why?

*Note*: If $x$ can be arbitrary $\mathbb{R}^n$, then the problem is in P, but the algorithms are hard.

(f) Subset Sum problem: given a multiset of $n$ non-negative integers $\{a_1, \ldots, a_n\}$, and an integer $A$ (target), is there a subset of the multiset whose sum is $A$?

*What if $A$ was written in unary instead of binary?*

**2.** Generalized Geography is a two-player impartial game played on a directed graph $G$. A token is placed in some vertex $v_0 \in V(G)$. The players move alternately, taking a token from some $v \in V(G)$ and moving it along an edge to another vertex $w \in V(G)$ (so $vw \in E(G)$ must hold). However, the token **cannot** appear multiple times at any vertex. Whoever cannot make a legal move, loses.

Prove that determining which player wins Generalized Geography, assuming both players play optimally, is PSpace-complete.

**3.** (homework 2017) Let *pattern* be a word consisting of letters and question marks. A pattern $p$ matches a word $w$ if $p$ can be obtained by taking an infix of $w$ and replacing some of its characters by question marks. For instance, `ab?d?` matches `abcde`, `ababadddd`, but not `abcd` or `abaaaadx`. Consider the following problem: given an alphabet $\Sigma$ and two sets $P$ and $N$ of patterns, decide whether there exists a word $w \in \Sigma^*$ matching all patterns in $P$ and no pattern in $N$. Prove that this problem is PSpace-complete.