Chapter 4:
Uniform quasi-wideness

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1 Introduction

In the first chapter we have defined nowhere dense classes by forbidding large cliques as depth-r minors. In this chapter we study a dual characterization, expressed in terms of *distance-r independence*. Recall that a vertex subset $A \subseteq V(G)$ in a graph G is distance-r independent if any two different vertices $a, b \in A$ are at distance larger than r.

The intuition is that in a large sparse graph, one should be able to find many vertices that are pairwise far from each other, i.e., form a large distance-r independent set. This intuition is not entirely correct. Consider for example a star S, i.e., a tree of depth 1: no two vertices of S are at distance greater than 1. However, if we were allowed to delete a bounded number of vertices, we could delete the center of the star and in the resulting graph we are left with many distance-r independent vertices, whatever value for r we choose.

We will formalize this concept of deleting a few elements to find large distance-r independent sets by introducing the notion of *uniform quasi-wideness*. We will then show that the new concept is equivalent to nowhere denseness. Later, we present several combinatorial and algorithmic applications of uniform quasi-wideness.

2 Uniform wideness

Let us first consider the simpler concept of *uniform wideness*, which will help to understand uniform quasi-wideness.

Definition 2.1. A class of graph $\mathcal{C}$ is called *uniformly wide* if for every $r \in \mathbb{N}$ there is a function $N_r \colon \mathbb{N} \to \mathbb{N}$ such that for every $m \in N$, $G \in \mathcal{C}$ and $A \subseteq V(G)$ with $|A| \geq N_r(m)$, there exists $B \subseteq A$ with $|B| \geq m$ such that B is distance-r independent in G

In other words, a class of graphs is uniformly wide if for every value of r, in every *huge* set A we still find a *large* distance-r independent set. The appropriate definition of *huge* and *large* depends on the value of r we care for. It is not difficult to see that uniformly wide classes are very simple classes, as the next theorem shows.

Theorem 2.2. *A class $\mathcal{C}$ of graphs is uniformly wide if and only if $\mathcal{C}$ is a class of bounded degree, i.e., there is a number d such that the maximum degree $\Delta(G)$ of every $G \in \mathcal{C}$ is bounded by d.*

Proof. Assume first that $\mathcal{C}$ is uniformly wide. By definition, for every $r \in \mathbb{N}$ there is a function $N_r \colon \mathbb{N} \to \mathbb{N}$ such that for all $m \in N$, $G \in \mathcal{C}$, and $A \subseteq V(G)$ with $|A| \geq N_r(m)$ there exists $B \subseteq A$ with $|B| \geq m$ such that B is distance-r independent in G. We claim that for each $G \in \mathcal{C}$, the maximum degree of G is smaller than $N_2(2)$. Take any vertex v of G and let $A = N(v)$. Then all vertices of A are pairwise at distance at most 2, hence there is no set $B \subseteq A$ of size 2 that would be distance-2 independent. We infer that $|N(v)| = |A| < N(2,2)$, and this must hold for every vertex v of G.

Conversely, assume that $\mathcal{C}$ has bounded degree, and let d be an integer such that the maximum degree $\Delta(G)$ of every $G \in \mathcal{C}$ is bounded by d. Define $N_r(m) = m \cdot (d+1)^r$. We claim that $\mathcal{C}$ is uniformly wide with function N_r. To see this, let $G \in \mathcal{C}$ and $A \subseteq V(G)$ with $|A| \geq N_r(m)$ for some integers $r, m \in \mathbb{N}$. We can now greedily pick elements from A to the set B as follows. Choose an arbitrary vertex $v \in A$ and put it into the set B, then remove all elements at distance at most r from v from the set A. As every vertex has degree at most d, we remove at most $(d+1)^r$ vertices from A in each step, which after m steps gives us the desired set B. □

3 Uniform quasi-wideness

We now slightly change the definition of wideness to allow the deletion of a small number of vertices.

Definition 3.1. A class of graph $\mathcal{C}$ is called *uniformly quasi-wide* if for every $r \in \mathbb{N}$ there exists a function $N_r \colon \mathbb{N} \to \mathbb{N}$ and a constant $s_r \in \mathbb{N}$ such that for all $m \in N$, $G \in \mathcal{C}$, and $A \subseteq V(G)$ with $|A| \geq N_r(m)$, there exists $S \subseteq V(G)$ with $|S| \leq s_r$ and $B \subseteq A - S$ with $|B| \geq m$ such that B is distance-r independent in $G - S$.

In other words, a class of graphs is uniformly quasi-wide if for every value of r and for every *huge* set A, we can delete a *very small* number of vertices such that we find a *large* distance-r independent subset of A in $G - S$. Again the appropriate definitions of *huge*, *large* and *very small* depend on the value of r we care for. This is governed by the functions $N_r(\cdot)$ and constants s_r. Constants s_r are sometimes called the *margins*, while $N_r(\cdot)$ are the *wideness functions*.

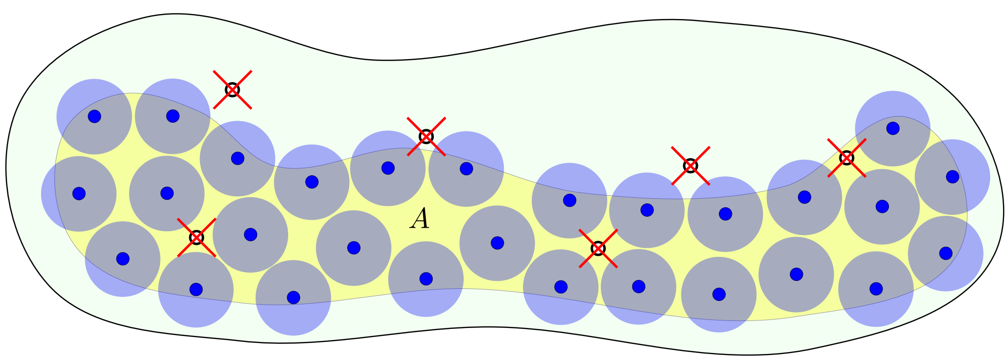

Figure 1: Definition of uniform quasi-wideness. In a huge set A (yellow) we may find a large subset B (blue) that is distance-r independent after removing a small subset of vertices S (crossed out), whose size depends only on the radius r. In case r is even, distance-r independence of B in $G - S$ is equivalent to saying that balls of radius $r/2$ around vertices of B in $G - S$ are disjoint, as depicted in the figure.

The rest of this section is devoted to proving the somewhat surprising fact that uniform quasi-wide classes are exactly the nowhere dense classes.

Theorem 3.2. *A class $\mathcal{C}$ of graphs is uniformly quasi-wide if and only if it is nowhere dense.*

We split the proof into two lemmas. The direction from left to right is easy to prove.

Lemma 3.3. *If a class $\mathcal{C}$ is uniformly quasi-wide, then $\mathcal{C}$ is nowhere dense.*

Proof. Assume $\mathcal{C}$ is uniformly quasi-wide and this is witnessed by constants s_r and functions $N_r(\cdot)$. Suppose K_t is a depth-r topological minor in some $G \in \mathcal{C}$. We prove that then $t < N_{2r+1}(2s+2)$, where $s \coloneqq s_{2r+1}$; this will imply that $\mathcal{C}$ is nowhere dense.

Suppose otherwise, that $t \geq N_{2r+1}(2s+2)$. Fix a depth-r topological minor model ϕ of a clique K on t vertices in G. Let $A = \phi(V(K))$. Consider any vertex subset $S \subseteq V(G)$ of size at most s, and let $A' \subseteq A$ be constructed from A as follows: whenever $\phi(u) \in S$ for some $u \in V(K)$, remove $\phi(u)$ from A', and whenever an internal vertex of $\phi(uv)$ belongs to S for some $uv \in E(K)$, remove both u and v from A'. Thus, when constructing A' from A we remove at most $2s$ vertices, implying $|A - A'| \leq 2s$. Now observe that every two vertices of A' are at distance at most $2r+1$ in $G - S$, because the path between them in the model ϕ was left untouched by the removal of S. Hence, any subset $B \subseteq A - S$ that is $(2r+1)$-independent in $G - S$ contains at most $2s+1$ vertices: at most $2s$ vertices of $A - A'$ and at most 1 vertex of A'. Since $|A| = t \geq N_{2r+1}(2s+2)$ and S was chosen arbitrarily, this is a contradiction. Hence $t < N_{2r+1}(2s+2)$ and $\mathcal{C}$ is nowhere dense. □

The other direction is much harder to prove.

Lemma 3.4. *If a class $\mathcal{C}$ is nowhere dense, then $\mathcal{C}$ is uniformly quasi-wide.*

We split the proof into several steps. In the following, fix the function

$$t(r) = \omega_r(\mathcal{C}) + 1$$

and a graph $G \in \mathcal{C}$. Thus, for all $r \in \mathbb{N}$ we have that $K_{t(r)} \not\preccurlyeq_r G$. Also fix a large $A \subseteq V(G)$. Our strategy is to inductively construct sequences

$$G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_r \qquad \text{and} \qquad S_1, S_2, \ldots, S_r,$$

where G_i are graphs and S_i are vertex sets such that for all $i \in \{1, \ldots, r\}$, we have

1. $G_i = G_{i-1} - S_i$ and
2. $S_i \subseteq V(G_{i-1})$.

Moreover, we will find a sequences

$$A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_r \qquad \text{and} \qquad m_0 \geq m_1 \geq \ldots \geq m_r = m,$$

where A_i are vertex sets and m_i are integers, such that for all $i \in \{1, \ldots, r\}$,

1. $A_i \subseteq A$ is distance-i independent in $G_{i-1} - S_i$, $A_i \cap S_i = \emptyset$,
2. $|A_i| \geq m_i$, and
3. $S_i = \emptyset$ if i is odd and $|S_i| < t(i/2)$ if i is even.

We will then return the set A_r of size m which is distance-r independent in $G_r = G - S$, where $S = \bigcup_{1 \le i \le r} S_r$. The construction will be applicable provided initially the invariant $|A_0| \ge m_0$ holds, hence we will set $N_r(m)$ simply as the obtained m_0. More precisely, for each $i \in \{1, \ldots, r\}$ we will provide a lower bound on m_{i-1} such that in A_{i-1} of size at least m_{i-1} we will be able to find a suitable A_i of size at least m_i. Then we will be able to trace these lower bounds from $m_r = m$ back to the final m_0.

The following two lemmas will imply that it suffices to consider only the cases $i = 1$ and $i = 2$. Their proofs are immediate.

Lemma 3.5. *Let A be a distance-$2j$ independent set in G. Let $H \preccurlyeq_j G$ be the depth-j minor of G obtained by contracting the disjoint distant-j neighborhoods $N_j^G[v]$ for $v \in A$ to single vertices. The vertex of H resulting from contracting $N_j^G[v]$ will be identified with the original vertex v of G, thus via this identification A is both a subset of vertices of G and a subset of vertices of H. Then any subset of A is distance-$(2j+1)$ independent in G if and only if it is distance-1 independent in H.*

Lemma 3.6. *Let A be a distance-$(2j+1)$ independent set in G. Let $H \preccurlyeq_j G$ be the depth-j minor of G obtained by contracting the disjoint distance-j neighborhoods $N_j^G[v]$ for $v \in A$ to single vertices. The vertex of H resulting from contracting $N_j^G[v]$ will be identified with the original vertex v of G, thus via this identification A is both a subset of vertices of G and a subset of vertices of H. Then, for any $S \subseteq V(H) - A = V(G) - \bigcup_{v \in A} N_j(v)$, it holds that a subset of A is distance-$(2j+2)$ independent in $G - S$ if and only if it is distance-2 independent in $H - S$.*

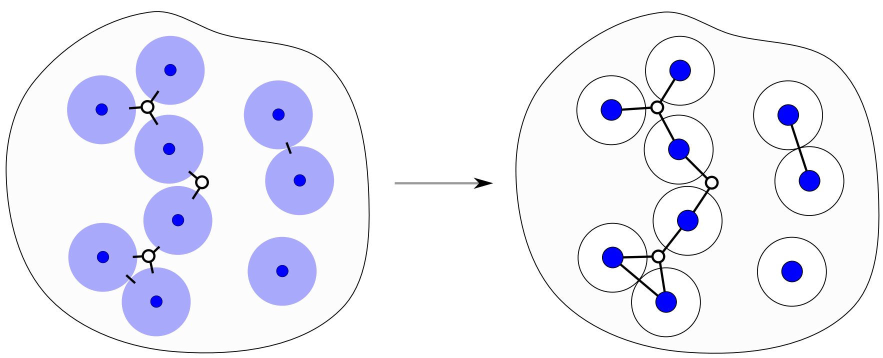

Figure 2: Reduction to cases $i = 1$ and $i = 2$ in Lemmas 3.5 and 3.6: contracting balls of radius j around vertices of a distance-$2j$ independent set turns distance-$(2j+1)$ independence into distance-1 independence, and distance-$(2j+2)$ independence into distance-2 independence.

The construction will be done iteratively for $i = 1, 2, \ldots, r$, where in step i we wish to construct A_i and S_i by examining the graph G_{i-1} and the distance-$(i-1)$ independent set A_{i-1} in it. The whole procedure will work as follows, assuming that for the cases $i = 1$ and $i = 2$ we have already given the construction. Starting with the set $A = A_0$, using the case $i = 1$ we will find a large independent subset $A_1 \subseteq A$ without deleting any vertices, hence $S_1 = \emptyset$, as claimed. Now that A_1 is distance-1 independent, using the case $i = 2$ we will find a small set S_2 and a large subset A_2 of A_1 which is distance-2 independent in the graph $G_2 = G_1 - S_2$ (hence, the set is distance-2 independent after deleting S_2, just as required in the definition of uniform quasi-wideness). As A_2 is distance-2 independent in G_2, we can contract the disjoint 1-neighborhoods of elements of A_2 (identifying contracted vertices with elements of A_2, as in Lemmas 3.5 and 3.6), thus obtaining a depth-1 minor $H \preccurlyeq_1 G_2$. Using the case $i = 1$ again, we find a large subset $A_3 \subseteq B_2$ that is distance-1 independent in H and apply Lemma 3.5 to conclude that it is, in fact, distance-3 independent in G_3. We continue with this set in the graph G_3. Again, we contract the disjoint 1-neighborhoods of elements of A_3 (identifying contracted vertices with elements of A_3), thus obtaining a depth-1 minor $H \preccurlyeq_1 G_3$. Using case $i = 2$ in H we find a small set $S_4 \subseteq V(H) - A_3 = V(G_3) - \bigcup_{a \in A_3} N_1^{G_3}[a]$ and a large set $A_4 \subseteq A_3$ which is distance-2 independent in $H - S_4$. We apply Lemma 3.6 to conclude that A_4 is distance-4 independent in $G_4 = G_3 - S_4$; the lemma is applicable since $S_4 \subseteq V(H) - A_4$. We continue this argumentation for r steps to construct the graphs G_i and sets A_1 and S_i with the desired properties.

It remains to show cases $i = 1$ and $i = 2$: how to construct A_1 out of A_0, and how to construct A_2, S_2 out of A_1. One of the main ingredients for this is Ramsey's Theorem.

Theorem 3.7. *Let $a, b \in \mathbb{N}$. Then there exists a number $R(a, b)$ such that for every coloring of the edges of a complete graph on $R(a, b)$ vertices with colors red and blue we will either find a clique on a vertices whose edges are all blue or a clique on b vertices whose edges are all red.*

Proof. We prove by induction on $a + b$ that it suffices to take $R(a, b) = R(a - 1, b) + R(a, b - 1)$. Clearly, for all $n \in \mathbb{N}$ we may take $R(n, 1) = R(1, n) = 1$; then, it is easy to see by induction that the above recurrence will yield

$$R(a, b) \le \binom{a + b - 2}{a - 1}.$$

Assume that we have established the bounds for $R(a-1, b)$ and $R(a, b-1)$ and consider a complete graph K on $R(a - 1, b) + R(a, b - 1)$ vertices whose edges are colored red and blue. Pick a vertex v and partition the remaining vertices into two sets A and B, such that for every vertex $w \in A$ the edge vw is blue and for every vertex $w \in B$ if the edge vw is red. We have $|A| + |B| + 1 = R(a - 1, b) + R(a, b - 1)$, and hence either $|A| \ge R(a - 1, b)$ or $|B| \ge R(a, b - 1)$. In the first case, by induction we know that A contains either a clique on $a - 1$ vertices with all edges blue, or a clique on b vertices with all edges red. In the latter subcase we are immediately done, and in the former case we may add v to this clique to obtain a clique on a vertices will all edges blue. The second case is analogous. This finishes the proof of the theorem. □

We may now give the construction for $i = 1$. Recall that $t(0)$ is such that $K_{t(0)} \not\preccurlyeq_0 G$, that is, $K_{t(0)}$ is not a subgraph of G. Suppose we are given a set A of size $|A| \ge m_0 := \binom{m_1 + t(0) - 2}{t(0) - 1}$, where m_1 is the target size of a distance-1 independent set we are interested in. By Ramsey's theorem, in $G[A]$ we may either find a clique of size $t(0)$ or an independent set of size m_1. The former case, however, cannot happen since $K_{t(0)}$ is not a subgraph of G. So we obtain an independent set of size m_1, as promised. To lift this to the case of $i = 2j + 1$ using Lemma 3.6, as explained before, we will apply this argument to a graph that is a j-shallow minor of the original graph, hence we will need to set $m_{i-1} := \binom{m_i + t(j) - 2}{t(j) - 1}$. Note again that $S_i = \emptyset$ in this case.

The case $i = 2$, which will lift to the induction step for even i, is much harder. In this case we assume that in our graph G we have already found a huge distance-1 independent set A_1, and we want to find a large distance-2 independent set in it, possibly after removing some small set of vertices S_1 that is disjoint from A_1; we will have that $|S_1| < t := t(1)$. Denote by D the set of all neighbors of vertices of A_1 and consider the graph G' defined as the subgraph of G on the vertex set $A_1 \cup D$ where we preserve only edges with one endpoint in A_1 and second in D. Clearly G' is bipartite, with bipartition $A_1 \uplus D$. Since A_1 is independent in G, it is easy to see that any subset of A_1 is distance-2 independent in G if and only if it is distance-2 independent in G'; so we may focus on G'. We will use the following extension of Ramsey's Theorem for a finite number of colors.

Theorem 3.8. *Let $n_1, \ldots, n_k \in \mathbb{N}$. There exists a number $R(n_1, \ldots, n_k)$ such that for every coloring of the edges of a complete graph on $R(n_1, \ldots, n_k)$ vertices with k different colors $c_1, \ldots, c_k$ we will find for some $1 \le i \le k$ a clique on n_i vertices all of whose edges are colored with color c_i.*

This variant of Ramsey's Theorem can easily be proved by induction on the number of colors, using the two-color case. We will apply it to prove the following lemma.

Lemma 3.9. *Let G be a bipartite graph with sides A and B. Let $m, t, d \in \mathbb{N}$. If $|A| \ge R(t, \ldots, t, m)$, where t is repeated $\binom{d-1}{2}$ times, then at least one of the following assertions holds.*

(a) A contains a set $A' \subseteq A$ of size m such that no two vertices of A' have a common neighbor;

(b) in G there is a 1-subdivision of K_t with all principal vertices contained in A; or

(c) B contains a vertex of degree at least d.

Let us motivate the statement of the lemma by examining its application to the graph G' we discussed before. The lemma says that provided A_1 is sufficiently large, we will either find a distance-2 independent set, which is exactly what we are looking for, or a 1-subdivision of K_t, which should not happen due to $K_t \not\preccurlyeq_1 G$, or we may find a vertex v in D that has degree at least d. We will add this vertex to the set S_1 of vertices to delete and inductively continue with the set $A_1' = N_{G'}(v) \cap A_1$. We will apply the lemma again to the bipartite graph induced by A_1' and its neighborhood and with v deleted (with a smaller value d'), again, giving us either a large distance-2 independent subset (in which case we are done), a 1-subdivision of K_t (which is again not possible by assumption), or another vertex of high degree. We apply the lemma again and again, always on the subset of A_1 induced by the neighborhood of the high degree vertex, which eventually, if the initial value of d was chosen large enough, gives us a complete bipartite graph $K_{t,t}$. This however, is not possible, as $K_{t,t}$ contains K_t itself as a depth-1 minor. We conclude that before we could apply the lemma t times, we must have found a large distance-2 independent set. Precise argument will follow, but now we give a proof of Lemma 3.9.

Proof of Lemma 3.9. Assume that B does not contain a vertex of degree at least d (otherwise we conclude that the third assertion holds). Enumerate the vertices of B as $b_1, \ldots, b_n$. Let K be the complete graph with vertex set A whose edges we will color with $\binom{d-1}{2} + 1$ colors. We initially consider all edges as colorless. Now we consider the vertices $b_1, \ldots, b_n$ in increasing order. In each step i, $1 \le i \le n$, we consider the set $X = N(b_i) \cap A$ and color each edge uv for $u, v \in X$ which has not previously received a color with an integer between 1 and $\binom{d-1}{2}$ such that no two edges that are colored in this step get the same color. This is possible as the degree of b_i is assumed to be at most $d-1$. Finally, we color all edges which do not have received a color in the steps $1, \ldots, n$ with color $\binom{d-1}{2} + 1$.

As $|A| \ge R(t, \ldots, t, m)$, in the resulting colored graph we can either find a clique of size t whose edges are all colored with one of the colors $1, \ldots, \binom{d-1}{2}$, or a clique of size m whose edges are all colored with color $\binom{d-1}{2}+1$. In the latter case, the vertices of the clique define a subset $A' \subseteq A$ such that no two vertices in A' have a common neighbor, as stated in the first assertion of the lemma.

In the former case, all edges of the monochromatic clique have been added at different steps in the construction, as all edges receive different colors in each individual step. Hence the edges have been colored in $\binom{t}{2}$ different steps, and each edge can be associated with a different vertex b_i that caused coloring of exactly this edge. Hence we find a 1-subdivision of K_t with all principal vertices in A. This is the second assertion of the lemma. $\square$

As we discussed, the idea is to apply Lemma 3.9 to the graph G' defined earlier not once, but t times. For convenience, we write $R_d(t,m)$ for $R(t,\ldots,t,m)$ where the first argument is repeated $\binom{d-1}{2}$ times. Then let

$$R^\star(s,t,m) \coloneqq \begin{cases} t & \text{if } s = 0; \\ R_k(t,m) & \text{if } s \geq 1, \text{ where } k = R^\star(s-1,t,m). \end{cases}$$

The next lemma explains the iterative application of Lemma 3.9.

Lemma 3.10. *Let G be a bipartite graph with partitions A and B. If $|A| \geq R^\star(t,t,m)$, then at least one of the following assertions holds.*

(a) A contains a set $A' \subseteq A$ of size m and B contains a set S of size less than t such that no two vertices of A' have a common neighbor outside of S;

(b) in G there is a 1-subdivision of K_t with all principal vertices contained in A;

(c) in G there is a complete bipartite subgraph $K_{t,t}$.

Proof. We will iteratively find vertices $s_1, s_2, \ldots$ and subsets $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \ldots$ with $|A_i| \geq R^\star(t-i,t,m)$ and $A_i \subseteq N(s_1) \cap N(s_2) \cap \ldots \cap N(s_i)$. Suppose $s_1,\ldots,s_i$ and A_i are already defined for some $i < t$. Then apply Lemma 3.9 to the graph $G[A_i \cup B_i]$ with $d = R^\star(t-i-1,t,m)$, where $B_i = B - \{s_1,\ldots,s_i\}$. This application yields either objects witnessing the satisfaction of the first or the second assertion (for $S = \{s_1,\ldots,s_i\}$), or gives a vertex s_{i+1} that has at least $R^\star(t-i-1,t,m)$ neighbors in A_i. In the former two cases we may stop the iteration, and in the latter case we may define $A_{i+1} \coloneqq A_i \cap N(s_{i+1})$ and proceed. Finally, observe that if $s_1,\ldots,s_t$ and A_t (with $|A_t| \geq R^\star(0,t,m) = t$) have been constructed, then $\{s_1,\ldots,s_t\} \cup A_t$ is a complete bipartite graph in G, so the third assertion holds. $\square$

Using Lemma 3.10 we may solve directly the case $i = 2$ of the main construction. Assuming that $|A_1| \geq m_1 \coloneqq R^\star(t,t,m_2)$, where m_2 is the requested size of a distance-2 independent set after this step, apply Lemma 3.10 to the bipartite graph G' we defined. This application cannot yield either a 1-subdivision of K_t or a $K_{t,t}$ subgraph of G', since both these graphs contain K_t as a depth-1 minor, which is excluded since we assumed $K_t \not\preccurlyeq_1 G$. The last conclusion — a set $S_1 \subseteq D$ with $|S_1| < t$ together with a subset $A_2 \subseteq A_1$ with $|A_2| \geq m_2$ that is distance-2 independent in $G' - S$ — is exactly what we were looking for.

As we discussed earlier, the case $i = 2$ presented above lifts to all even i by applying Lemma 3.6. More precisely, we apply case $i = 2$ to the graph H obtained by contracting distance-$(i/2)$ neighborhoods of vertices A_i, and the independent set A_i in H. Observe that in this setting, to exclude assertions (b) and (c) in Lemma 3.10 it suffices to take $t \coloneqq t(i/2)$. Indeed, if in the contracted graph H we find either a 1-subdivision of K_t with all principal vertices in A_i, or a $K_{t,t}$ with one side contained in A_i and second outside of A_i, then in the graph before contractions, both of these would yield a depth-$(i/2)$ minor model of K_t, a contradiction.

To summarize, we put

$$m_{i-1} \coloneqq \begin{cases} \binom{m_i + t(i/2) - 2}{t(i/2) - 1} & \text{if } i \text{ is odd;} \\ R^\star(t(i/2), t(i/2), m_i) & \text{if } i \text{ is even.} \end{cases}$$

By requesting $m_r \coloneqq m$, this gives a value of $N_r(m) \coloneqq m_0$ for which the whole construction can be performed. This concludes the proof of Lemma 3.4, which was the missing part of the proof of Theorem 3.2.

We give two remarks about the proof. First, it should be clear that it is algorithmic: a straightforward implementation of the proof gives a polynomial-time procedure that given $r, m \in \mathbb{N}$ together with G and A of appropriate size, outputs suitable S and B. Note that the procedure needs to have constants $t(i)$ for $i \leq r$ hard-coded. Second, the provide proof yields tower-Ramsey bounds on the function $N_r(m)$, due to stacking applications of Ramsey's Theorem in the proof of Lemma 3.10. There is a smarter (though, more involved) way of executing the proof of this lemma, which actually yields polynomial bounds. As a consequence, the following statement is true: if $\mathcal{C}$ is nowhere dense, then $\mathcal{C}$ is uniformly quasi-wide with margins s_r and wideness functions $N_r(\cdot)$, where for every fixed $r \in \mathbb{N}$, the function $N_r(m)$ is a polynomial of m. Note that the degree of this polynomial (highly) depends on the class $\mathcal{C}$ and integer r.

4 The splitter game

We now provide an intuitive game characterization of nowhere denseness, which is close in spirit to uniform quasi-wideness. Historically, it was introduced as a tool in the design of an almost linear-time FPT algorithm for FO model checking on nowhere dense graph classes.

Definition 4.1. Let G be a graph and let $\ell, m, r \in \mathbb{N}$. The (ℓ, m, r)-*Splitter Game* on G is played by two players, Connector and Splitter, as follows. We let $G_0 := G$. In round i of the game, Connector chooses a vertex $v_i \in V(G_{i-1})$. Then Splitter picks a subset $W_i \subseteq N_r^{G_{i-1}}(v_i)$ of size at most m. We let $G_i := G_{i-1}[N_r^{G_{i-1}}(v_i)] - W_i$. Splitter wins if $G_i = \emptyset$. Otherwise the game continues on G_i. If Splitter has not won after ℓ rounds, then Connector wins.

In the Splitter Game, a *strategy* for Splitter is, well, what one expects it to be. Formally, it is a function σ that maps every partial play $(v_1, W_1, \ldots, v_i, W_i, v_{i+1})$ to a new move $W_{i+1} \subseteq N_r^{G_i}[v_{i+1}]$ of Splitter. Similarly for Connector. A strategy σ is a *winning strategy* for Splitter in the (ℓ, m, r)-splitter game on G if Splitter wins every play in which he follows the strategy σ. If Splitter has a winning strategy on G, we say that he *wins* the (ℓ, m, r)-splitter game on G.

We first show that nowhere denseness guarantees that Splitter wins the Splitter Game within a bounded number rounds.

Lemma 4.2. *Let $\mathcal{C}$ be a nowhere dense class of graphs. Then for every $r \in \mathbb{N}$ there are $\ell \in \mathbb{N}$ and $m \in \mathbb{N}$ such that for every $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-Splitter Game on G.*

Proof. As $\mathcal{C}$ is nowhere dense, it is also uniformly quasi-wide, say with margins $(s_r)_{r \in \mathbb{N}}$ and wideness functions $(N_r(\cdot))_{r \in \mathbb{N}}$. Fix $r \in \mathbb{N}$ and let $\ell \coloneqq N_r(2s_r + 1)$ and $m \coloneqq \ell \cdot (r + 1)$. Note that both ℓ and m only depend on $\mathcal{C}$ and r. We claim that for any $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-Splitter Hame on G; for this, we present a suitable winning strategy.

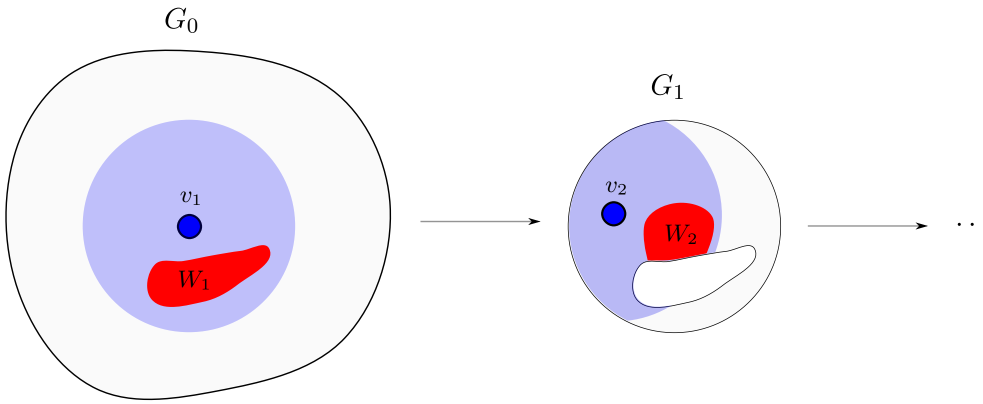

Figure 3: First two rounds of the Splitter Game. In the ith round, Connector first picks a vertex v_i and the arena gets restricted to the distance-r neighborhood of v_i. Then the Splitter removes a set W_i of at most m vertices from the arena. The goal of Splitter is to obtain an empty graph within ℓ rounds, the goal of Connector is to prevent this.

Let $G \in \mathcal{C}$ be a graph. In the (ℓ, m, r)-Splitter Hame on G, Splitter uses the following strategy. In the first round, if Connector chooses $v_1 \in V(G_0)$, where $G_0 \coloneqq G$, then Splitter chooses $W_1 \coloneqq \{v_1\}$. Now let $i \geq 1$ and suppose that $v_1, \ldots, v_{i-1}, G_1, \ldots, G_{i-1}, W_1, \ldots, W_{i-1}$ have already been defined. Suppose Connector chooses $v_i \in V(G_{i-1})$. We define W_i as follows. For each $j < i$, choose a path $P_{j,i}$ in G_{j-1} of length at most r connecting v_j and v_i. Such a path must exist as $v_i \in V(G_i) \subseteq V(G_j) \subseteq N_r^{G_{j-1}}[v_j]$. We let $W_i \coloneqq \bigcup_{1 \leq j < i} V(P_{j,i}) \cap N_r^{G_{i-1}}[v_i]$; in other words, the move W_i of Splitter consists of all vertices of all the paths $P_{j,i}$ that are still in the arena. Note that $|W_i| \leq (i-1) \cdot (r+1) \leq m$, as the paths have length at most r and hence consist of at most $r+1$ vertices. It remains to be shown is that the length of any such play is bounded by ℓ.

Assume toward a contradiction that Connector may survive for ℓ rounds. Let $(v_1, \ldots, v_\ell, G_1, \ldots, G_\ell, W_1, \ldots, W_\ell)$ be a play witnessing this, where the moves of the Splitter are according to the presented strategy. As $\ell = N_r(2s_r + 2)$, for $A \coloneqq \{v_1, \ldots, v_\ell\}$ there is a set $S \subseteq V(G)$ with $|S| \leq s_r$ such that $A - S$ contains subset B of size $2s_r + 2$ that is distance-r independent in $G - S$. Suppose $B = \{v_{i_1}, \ldots, v_{i_{2s_r+2}}\}$ with $i_1 < \ldots < i_{2s_r+2}$; for brevity we write $w_j \coloneqq v_{i_j}$.

We now consider the pairs (w_{2j-1}, w_{2j}) for $1 \leq j \leq s_r + 1$. By construction, $Q_j \coloneqq P_{i_{2j-1}, i_{2j}}$ is a path of length at most r from w_{2j-1} to w_{2j} in $G_{i_{2j-1}-1}$. We now observe that paths Q_j, for $j \in \{1, \ldots, s_r + 1\}$, are pairwise disjoint. Indeed, if $1 \leq j < j' \leq s_r + 1$, then the whole path Q_j was removed by the Splitter in round i_{2j} (formally, $V(Q_j) \cap V(G_{i_{2j-1}}) \subseteq W_{i_{2j}}$), hence it is entirely disjoint with the vertex set of the graph $G_{i_{2j'-1}-1}$ due to $j' > j$. On the other hand, since $Q_{j'}$ is entirely contained in the graph $G_{i_{2j'-1}-1}$ by definition, indeed Q_j and $Q_{j'}$ are disjoint. Now, since S contains at most s_r vertices, some path Q_j has to be entirely disjoint with S. However, this means that w_{2j-1} and w_{2j} do not belong to S and are at distance at most r in $G - S$. This contradicts the assumption that B is distance-r indendent in $G - S$ and finished the proof. $\square$

We now observe that also the converse of Lemma 4.2 holds.

Lemma 4.3. *Let $\mathcal{C}$ be a class of graphs. If for every $r \in \mathbb{N}$ there are $\ell, m \in \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-splitter game, then $\mathcal{C}$ is nowhere dense.*

Proof. We prove the contrapositive. Suppose $\mathcal{C}$ is somewhere dense, hence $\mathcal{C}$ admits all complete graphs as depth-r minors, for some fixed depth $r \in \mathbb{N}$. Then we claim that for every choice of $\ell, m \in \mathbb{N}$, there is a graph $G \in \mathcal{C}$ such that Connector wins the $(\ell, m, 3r+1)$-splitter game on G.

Fix $\ell, m \in \mathbb{N}$. We choose $G \in \mathcal{C}$ such that G contains the complete graph $K := K_{\ell m+1}$ as a depth-$(3r+1)$ minor. Let ϕ be a minor model of K in G and for every vertex u of K, let $\gamma(u)$ be the center of the branch set $\phi(u)$, that is, a vertex of $\phi(u)$ that is at distance at most r from all the vertices of $\phi(u)$. Connector uses the following strategy to win the $(\ell, m, 3r+1)$-Splitter Game on G. First, Connector chooses $\gamma(u)$ for any vertex u of K. The distance-$(3r+1)$ ball around $\gamma(u)$ contains the whole branch sets of all vertices of K. Splitter removes any m vertices. We actually allow him to remove the complete branch sets (under ϕ) containing all m vertices he chose. In round 2 we may thus assume that we still find the complete graph $K_{(\ell-1)m+1}$ as a depth-r minor of the current arena. By continuing to play in this way until, after round ℓ the arena still contains some vertices and the Connector wins. ☐

Lemmas 4.2 and 4.3 together imply the following characterization of nowhere denseness in terms of the Splitter Game.

Theorem 4.4. *For a class of graphs $\mathcal{C}$, the following conditions are equivalent:*

- *$\mathcal{C}$ is nowhere dense;*
- *for every $r \in \mathbb{N}$ there exists $\ell, m \in \mathbb{N}$ such that for every $G \in \mathcal{C}$, Splitter wins the (ℓ, m, r)-Splitter Game on G.*

5 Algorithmic applications: parameterized Distance-r Dominating Set

As an algorithmic application of uniform quasi-wideness we now design an efficient algorithm for the (parameterized) Distance-r Dominating Set problem on nowhere dense classes. Recall that in a graph G, a subset of vertices D is *distance-r dominates* a subset of vertices A if every vertex of A is at distance at most r from a vertex of D. Further, D is an *distance-r dominating set* of G if D distance-r dominates the whole vertex set. By $\mathrm{dom}_r(G, A)$ we denote the smallest size of a set that distance-r dominates A in G, and $\mathrm{dom}_r(G)$ — the distance-r domination number of G — is the smallest size of a distance-r dominating set in G.

We will consider the Distance-r Dominating Set problem defined in the decision variant as follows: given a graph G and parameter k, is it true that $\mathrm{dom}_r(G) \leq k$? In general, our goal will be two-fold: (a) to design an efficient algorithm assuming r and k are small, and (b) to reduce the size of the input instance to a function of r and k only. In the terminology of parameterized complexity, point (a) is to design an efficient *fixed-parameter* algorithm for the problem, and point (b) is to design a *kernelization* procedure.

The main idea is that in the considered problem, every vertex of the graph can serves two roles. First, it is a potential *dominator*: a vertex that we may pick to the dominating set so that it dominates other vertices. Second, it is a *dominatee*: a vertex that imposes a constraint that it needs to be dominated. In the beginning, all the vertices start with both roles. We separate the roles and reduce the number of essential dominatees and of essential dominators separately.

We start with reducing the number of essential dominatees. To formalize the notion of being essential for getting dominated, we introduce the following definition.

Definition 5.1. Let G be a graph and $k \in \mathbb{N}$. A set $Z \subseteq V(G)$ is a *distance-r domination core* for G and k if every set $D \subseteq V(G)$ of size at most k which distance-r dominates Z, also distance-r dominates $V(G)$.

Thus, intuitively a distance-r domination core is a subset of vertices whose domination forces the domination of the whole graph. Obviously, $Z = V(G)$ is always a distance-r domination core, but we will be looking for small domination cores.

Fix a nowhere dense class $\mathcal{C}$ of graphs. Then $\mathcal{C}$ is uniformly quasi-wide, say with margins $(s_r)_{r\in\mathbb{N}}$ and wideness functions $(N_r(\cdot))_{r\in\mathbb{N}}$. Fix $r, k \in \mathbb{N}$ and let $s := s_{2r}$. The next, slightly surprising lemma shows that in a distance-r domination core that is too large one can always find an *irrelevant dominatee* that can be safely removed.

Lemma 5.2. *Suppose $G \in \mathcal{C}$ and let $Z \subseteq V(G)$ be a vertex subset satisfying*

$$|Z| \geq N_{2r}\left((k+1)(r+1)^s + 1\right).$$

Then we can compute in polynomial time a vertex $w \in Z$ such that for any set $D \subseteq V(G)$ satisfying $|D| \leq k$, the following equivalence holds:

$$D \text{ distance-}r \text{ dominates } Z \text{ if and only if } D \text{ distance-}r \text{ dominates } Z - \{w\}.$$

Proof. By Theorem 3.2 and the remark after its proof, we can find in polynomial time sets $S \subseteq V(G)$ and $B \subseteq Z - S$ such that $|S| \leq s$, $|B| \geq (k+1)(r+1)^s + 1 > (k+1)(r+1)^s$ and B is distance-$2r$ independent in $G - S$. For each $v \in B$, compute $\mathrm{profile}_r[v, S]$, the distance-r profile of v on S. Recall here that $\mathrm{profile}_r[v, S]$ is a function from S to $\{0, 1, \ldots, r, \infty\}$ such that for $a \in S$ we put $\mathrm{profile}_r[v, S](a) = \mathrm{dist}(v, a)$ if this distance is at most r, and $\mathrm{profile}_r[v, S](a) = \infty$ otherwise. Clearly, we can compute these distance profiles in polynomial time. Note that there are at most $(r+1)^s$ different distance-r profiles on S. Since $|B| > (k+1)(r+1)^s$, there are $k+2$ elements $b_1, \ldots, b_{k+2} \in B$ which have the same distance profile. Now we choose $w := b_1$ and show that for any set $D \subseteq V(G)$ with $|D| \leq k$, D distance-r dominates Z if and only if D distance-r dominates $Z - \{b_1\}$.

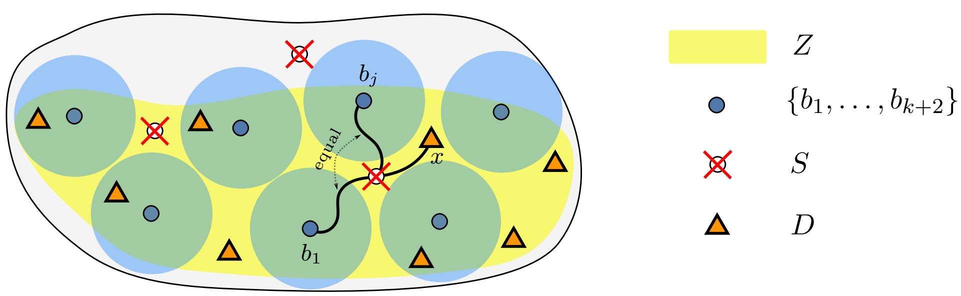

Figure 4: Situation in the proof of Lemma 5.2.

The direction from left to right is obvious. Now, suppose D distance-r dominates $Z - \{b_1\}$. Consider the sets $W_i := N_r^{G-S}[b_i]$ for $i \in \{2, \ldots, k+2\}$. Since B is distance-$2r$ independent in $G - S$, the sets W_i are pairwise disjoint. Since there are $k+1$ of these sets, at least one of them, say W_j, does not contain any element of D. However, since $b_j \in Z - \{b_1\}$ and D distance-r dominates $Z - \{b_1\}$, there is a path of length at most r from some element $x \in D$ to b_j. This path must go through an element of S. Since b_1 and b_j have the same distance-r distance profiles on S, we conclude that there is also a path of length at most r from x to b_1 and therefore D distance-r dominates Z. $\square$

An immediate corollary of Lemma 5.2 is that we can always find a small domination core. Simply start with $Z = V(G)$, which is trivially a distance-r domination core, and apply the above procedure to remove an irrelevant dominatee from the distance-r domination core Z until $|Z| < N_{2r}((k+1)(r+1)^s + 1)$.

Corollary 5.3. *There is an algorithm running in polynomial time that given a graph $G \in \mathcal{C}$ computes a distance-r domination core of G for parameter k of size smaller than $N_{2r}((k+1)(r+1)^s + 1)$.*

Having a small domination core is already sufficient to design an efficient parameterized algorithm for the problem.

Lemma 5.4. *Given a graph G with n vertices and m edges, vertex subset Z, and numbers $k, r \in \mathbb{N}$, one can decide whether $\mathrm{dom}_r(G, Z) \leq k$ in time $2^{k|Z|} \cdot |Z|^{\mathcal{O}(1)} \cdot (n + m)$.*

Proof. First, we compute the set system

$$\mathcal{F} = \{N_r^G[u] \cap Z \colon u \in V(G)\}.$$

This can be done in time $\mathcal{O}(|Z|(n+m))$ by running a BFS from every vertex of Z, recording for every vertex $u \in V(G)$ the profile of its distances to vertices of Z, and collecting sets $N_r^G[u] \cap Z$ in a trie. Next, observe that $\mathrm{dom}_r(G, Z) \leq k$ if and only if Z can be covered by at most k sets from $\mathcal{F}$. As $|\mathcal{F}| \leq 2^{|Z|}$, this can be done in time $2^{k|Z|} \cdot |Z|^{\mathcal{O}(1)}$ by investigating all subsets of $\mathcal{F}$ of size at most k. $\square$

Corollary 5.5. *For any nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$, the* Distance-r Dominating Set *problem on an n-vertex graph from $\mathcal{C}$ can be solved in time $f(k) \cdot n^c$ for some function f and a universal constant c, indepndent of $\mathcal{C}$.*

Proof. Using Corollary 5.3, in polynomial time we compute a distance-r domination core Z for G and k of size at most $g(k)$, for some function g depending only on $\mathcal{C}$ and r. By the definition of a distance-r domination core we have that $\mathrm{dom}_r(G) \leq k$ if and only if $\mathrm{dom}_r(G, Z) \leq k$. Then we use Lemma 5.4 to verify whether $\mathrm{dom}_r(G, Z) \leq$ in time $2^{k \cdot g(k)} \cdot g(k)^{\mathcal{O}(1)} \cdot n^2$ and check whether it is not larger than k. $\square$

In the above proof we did not try to optimize the function $f(k)$, but let us take a closer look on how we can estimate its magnitude. First, the size of the core Z is at most $g(k) = N_{2r}((k+1)(r+1)+1)$, which, by the remark after the proof of Theorem 3.2, is actually polynomial in k. Next, the exponential factor $2^{k|Z|}$ in the statement of Lemma 5.4 comes from a brute-force bound of $2^{|Z|}$ on the size of the family $\mathcal{F}$. However, we know that if G is drawn from a fixed nowhere dense class $\mathcal{C}$ and r is fixed — which is the setting considered in Corollary 5.5 — then the size of $\mathcal{F}$ is bounded by $\mathcal{O}_\varepsilon(|Z|^{1+\varepsilon})$ for any $\varepsilon > 0$; this is just neighborhood complexity in nowhere dense classes. Therefore, the algorithm of Lemma 5.4 in this situation works in time $g(k)^{k(1+\varepsilon)+\mathcal{O}(1)} \cdot (n+m)$. As $g(k) = k^{\mathcal{O}(1)}$, we conclude that the running time of the whole algorithm is $2^{\mathcal{O}(k \log k)} \cdot n^c$; this is not bad at all.

We now move to the second algorithmic corollary, namely reducing the instance size to a function of k. For this, we additionally need to reduce the number of essential *dominators*, that is, the number of vertices that may be used to dominate other vertices. Obviously, only vertices at distance at most r from a vertex belonging to a distance-r domination core Z are relevant. Furthermore, if there are two vertices $v, v' \in V(G)$ with $N_r[v] \cap Z = N_r[v'] \cap Z$, it suffices to keep one of v and v' as a representative, as they serve the same role when we consider dominating vertices in Z.

Theorem 5.6. *Suppose $\mathcal{C}$ is a nowhere dense class of graphs and $r \in \mathbb{N}$ is fixed. Then there exists a function $p\colon \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm that on input $G \in \mathcal{C}$ and $k \in \mathbb{N}$ computes an induced subgraph $H \subseteq G$ on at most $p(k)$ vertices and a vertex subset $Z \subseteq V(H)$ such that $\mathrm{dom}_r(G) \leq k$ if and only if $\mathrm{dom}_r(H, Z) \leq k$*

Proof. Using the algorithm of Corollary 5.3, we first compute a distance-r domination core Z in G of size smaller than $N_{2r}\left((k+1)(r+1)^s + 1\right)$.

Now, for every vertex $v \in V(G)$ we compute the set

$$L_v \coloneqq N_r^G[v] \cap Z$$

Consider two vertices $v, v' \in V(G)$ *equivalent* if $L_v = L_{v'}$. Clearly, the number of equivalence classes of this relation is at most $2^{|Z|}$, hence let A be any set of at most $2^{|Z|}$ vertices containing one element from each equivalence class. Construct a set W as follows: start with putting $A \cup Z$ into W and then, for every pair of vertices $v \in A$ and $z \in Z$, if $\mathrm{dist}_G(v, z) \leq r$ then add the vertices of any path of length at most r between v and z to W. Clearly the size of W computed in this manner is bounded by a function of k only, and we are left with verifying that $H \coloneqq G[W]$ and Z satisfy the asserted property: $\mathrm{dom}_r(G) \leq k$ if and only if $\mathrm{dom}_r(H, Z) \leq k$. Since Z is a distance-r domination core in G, we have $\mathrm{dom}_r(G) \leq k$ iff $\mathrm{dom}_r(G, Z) \leq k$. Hence, it suffices to prove that $\mathrm{dom}_r(G, Z) = \mathrm{dom}_r(H, Z)$.

In one direction, if $D \subseteq V(H)$ distance-r dominates Z in H, then D also distance-r dominates Z in G, because H is an induced subgraph of G. This proves that $\mathrm{dom}_r(G, Z) \leq \mathrm{dom}_r(H, Z)$.

In the other direction, take any $D \subseteq V(G)$ that distance-r dominates Z in G. For each $x \in D$, some vertex x' that is equivalent to x has been included in A. Let $D' \coloneqq \{x' \colon x \in D\}$; clearly $|D'| \leq |D|$ and $D' \subseteq A \subseteq W$. It is now straightforward to see that D' distance-r dominates Z in H, since for each $x \in D$ the corresponding vertex $x' \in D'$ distance-r dominates exactly the same vertices of Z in H as x distance-r dominated in G. This is because we explicitly added to H a path of length at most r between x' and every vertex of Z that was distance-r dominated by x' in G. This proves that $\mathrm{dom}_r(G, Z) \geq \mathrm{dom}_r(H, Z)$, so $\mathrm{dom}_r(G, Z) = \mathrm{dom}_r(H, Z)$ and we are done. □

Again, let us discuss the obtained bounds on the function $p(k)$. First, as before, the computed core Z has size at most $g(k) = N_{2r}((k+1)(r+1)+1)$, which is actually polynomial in k. Next, the size of the set A is bounded by the number of possible distance-r neighborhoods in Z, which due to G being drawn from a fixed nowhere dense class $\mathcal{C}$, is actually bounded by $\mathcal{O}_\varepsilon(|Z|^{1+\varepsilon})$ for any $\varepsilon > 0$, instead of the trivial $2^{|Z|}$ upper bound that we used. Hence, the total number of vertices in H is bounded by a $\mathcal{O}_\varepsilon(g(k)^{2+\varepsilon})$, which is polynomial in k. This means that we have obtained a *polynomial kernel* for DISTANCE-r DOMINATING SET on any nowhere dense class $\mathcal{C}$.

The degree of the polynomial upper bound on the size of the above kernel, while being dependent only on $\mathcal{C}$ and r, is generally large and highly depends on them. However, with other ideas and more technical work, one can improve the result to an (almost) linear kernel, as explained in the following theorems.

Theorem 5.7. *Let $\mathcal{C}$ be a class of graphs of bounded expansion and $r \in \mathbb{N}$ be fixed. Then there is a polynomial-time algorithm that on input $G \in \mathcal{C}$ and $k \in \mathbb{N}$ computes an induced subgraph $H \subseteq G$ on $\mathcal{O}(k)$ vertices and a vertex subset $Z \subseteq V(H)$ such that* $\mathrm{dom}_r(G) \leq k$ *if and only if* $\mathrm{dom}_r(H, Z) \leq k$

Theorem 5.8. *Let $\mathcal{C}$ be a nowhere dense class of graphs and $r \in \mathbb{N}$ be fixed. Then for every $\varepsilon > 0$ there is a polynomial-time algorithm that on input $G \in \mathcal{C}$ and $k \in \mathbb{N}$ computes an induced subgraph $H \subseteq G$ on $\mathcal{O}(k^{1+\varepsilon})$ vertices and a vertex subset $Z \subseteq V(H)$ such that* $\mathrm{dom}_r(G) \leq k$ *if and only if* $\mathrm{dom}_r(H, Z) \leq k$

We actually already know all the tools needed for the proofs of the above theorems. However, they are a bit technical and consist of several steps, so we choose to omit them in this edition of the course. An interested reader is referred to the lecture notes from the previous edition of the course for a proof of Theorem 5.7.