Sparsity — tutorial 4

Generalized coloring numbers

Problem 1. For every $r \in \mathbb{N}$, compute $\mathrm{adm}_r(\text{Forests})$, $\mathrm{scol}_r(\text{Forests})$, and $\mathrm{wcol}_r(\text{Forests})$.

Problem 2. Give a two-line proof using generalized coloring numbers that if $\mathcal{C}$ has bounded expansion, then $\mathcal{C} \bullet K_c$ has bounded expansion for every constant $c \in \mathbb{N}$.

Definition 0.1. For $r \in \mathbb{N}$, a graph G, vertex subset $S \subseteq V(G)$, and vertex $v \in S$, define $b_r(v, S)$ to be the largest cardinality of a family $\mathcal{P}$ of paths in G with the following properties:

- each path $P \in \mathcal{P}$ has length at most r, leads from v to some other vertex of S, and all its internal vertices do not belong to S; and
- for all distinct $P, P' \in \mathcal{P}$, we have $V(P) \cap V(P') = \{v\}$.

Problem 3. Fix $r \in \mathbb{N}$ and let G be a graph. Consider the following procedure of computing a vertex ordering $\sigma(v_1, \ldots, v_n)$ of G. Assuming $v_{i+1}, \ldots, v_n$ are already defined, let $S_i \coloneqq V(G) - \{v_{i+1}, \ldots, v_n\}$, and choose v_i to be any vertex $v \in S_i$ that minimizes $b_r(v, S_i)$. Prove that if σ is any vertex ordering obtained using this procedure, then σ has the optimum r-admissibility, i.e., $\mathrm{adm}_r(G, \sigma) = \mathrm{adm}_r(G)$.

Problem 4. Give an algorithm with runtime $\mathcal{O}(nm)$ that given $r \in \mathbb{N}$, a graph G, vertex subset $S \subseteq V(G)$, and $v \in S$, computes a family of paths $\mathcal{P}_v$ as in the definition of $b_r(v, S)$, but of size at least $\frac{b_r(v,S)}{r}$.

Problem 5. Give an algorithm with runtime $\mathcal{O}(n^3 m)$ that given $r \in \mathbb{N}$ and a graph G, computes a vertex ordering σ of G with $\mathrm{adm}_r(G, \sigma) \leq r \cdot \mathrm{adm}_r(G)$.

Problem 6. Let G be a graph and let $r \in \mathbb{N}$. Prove that if $u, v \in V(G)$ are two distinct vertices, then every path of length at most r between u and v is hit by the set

$$\mathrm{WReach}_r[G, \sigma, u] \cap \mathrm{WReach}_r[G, \sigma, v].$$

Problem 7. Let G be a graph, let $r \in \mathbb{N}$, and let σ be a vertex ordering of G. For every vertex $u \in V(G)$, define the *cluster* of u as follows:

$$C_u \coloneqq \{v \in V(G) :\ u \in \mathrm{WReach}_{2r}[G, \sigma, v]\}.$$

Prove that the following conditions hold:

- each cluster has radius at most $2r$;
- each vertex of $V(G)$ appears in at most $\mathrm{wcol}_{2r}(G, \sigma)$ clusters; and
- for each vertex $u \in V(G)$, the r-neighborhood $N_r^G[u]$ of u in G is entirely contained in some cluster.

Such family of clusters is called an *r-neighborhood cover* of G with radius $2r$ and overlap $\mathrm{wcol}_{2r}(G, \sigma)$.

Problem 8. Let $\mathcal{I}_k$ be the class of intersection graphs of families of closed intervals on a line with ply at most k. In other words, a graph G belongs to $\mathcal{I}_k$ if and only if we can associate a closed interval $I_u \subseteq \mathbb{R}$ with every vertex $u \in V(G)$ such that $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$, and no $x \in \mathbb{R}$ belongs to more than k intervals from $\{I_u\}_{u \in V(G)}$.

Prove that $\mathrm{wcol}_r(\mathcal{I}_k) \leq \binom{r+k-1}{r}$ for all $r \in \mathbb{N}$.

Problem 9. Prove that for every $r \in \mathbb{N}$ and graph G, we have

$$\mathrm{wcol}_r(G) \leq 2^{r+1} \cdot (\mathrm{adm}_r(G) - 1)^r.$$