ALGOMANET Sparsity

# Tutorial 1: Measuring sparsity

January 20th, 2020

**Problem 1.** Prove that a graph is 1-degenerate if and only if it is a forest.

**Problem 2.** Prove that a $k$-degenerate graph on $n$ vertices is $k$-colorable and has at most $1 + 2^k \cdot n$ different cliques.

**Problem 3.** The *arboricity* of a graph $G$, denoted $\mathrm{arb}(G)$, is the minimum number $k$ such that the edge set of $G$ can be partitioned into $k$ sets, each inducing a forest. Prove that

$$\mathrm{arb}(G) \leqslant \mathrm{degeneracy}(G) \leqslant 2 \cdot \mathrm{arb}(G) - 1.$$

**Problem 4.** Prove that a $d$-degenerate $n$-vertex graph contains less than $n/2$ vertices of degree at least $4d$.

**Problem 5.** Give a linear-time algorithm that given a graph, computes its degeneracy together with a suitable vertex ordering witnessing it.

**Problem 6.** Consider two classes of graphs $\mathcal{G}_1, \mathcal{G}_2$ and an operator $\mathcal{G}_1 \oplus \mathcal{G}_2$ defined as follows. A graph $G$ belongs to $\mathcal{G}_1 \oplus \mathcal{G}_2$ if there exist graphs $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ with $|V(G)| = |V(G_1)| = |V(G_2)|$ and bijections $\pi_1 : V(G) \to V(G_1)$ and $\pi_2 : V(G) \to V(G_2)$ such that $uv \in E(G)$ if and only if $\pi_1(u)\pi_1(v) \in E(G_1)$ or $\pi_2(u)\pi_2(v) \in E(G_2)$. Prove or disprove the following statements:

1. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are of bounded degerenacy, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is of bounded degeneracy.
2. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are of bounded expansion, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is of bounded expansion.
3. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are nowhere dense, then $\mathcal{G}_1 \oplus \mathcal{G}_2$ is nowhere dense.

**Problem 7.** Prove that if $J \preceq_a H$ and $H \preceq_b G$, then $J \preceq_{2ab+a+b} G$.

**Problem 8.** For a class $\mathcal{C}$ and $d \in \mathbb{N}$, we denote

$$\mathcal{C} \nabla d \coloneqq \{\, H \;:\; H \preceq_d G \text{ for some } G \in \mathcal{C} \,\}.$$

Prove that if $\mathcal{C}$ has bounded expansion, then for every $d \in \mathbb{N}$ the class $\mathcal{C} \nabla d$ also has bounded expansion. Similarly, prove that if $\mathcal{C}$ is nowhere dense, then for every $d \in \mathbb{N}$ the class $\mathcal{C} \nabla d$ is also nowhere dense.

**Definition 1.** A graph $H$ is a *depth-$d$ topological minor* of $G$ if there exists an *embedding* $\psi$ of $H$ into $G$ such every vertex $u$ of $H$ is embedded into a different vertex $\psi(u)$ of $G$, and every edge $uv$ of $H$ is embedded as path $\psi(uv)$ of length at most $2d+1$ and connecting $\psi(u)$ with $\psi(v)$ in $G$, so that paths $\{\psi(e) \colon e \in E(H)\}$ are pairwise internally vertex-disjoint.

**Problem 9.** Prove that if $G$ contains $K_s$ as a depth-$d$ minor, where $s = 2 + t^{2(d+1)}$, then $G$ contains $K_t$ as a depth-$(3d+1)$ topological minor.

**Problem 10.** We define *topologically nowhere dense* classes in the same way as nowhere dense classes, but using the notion of a bounded-depth topological minor. Prove that a class of graphs is nowhere dense if and only if it is topologically nowhere dense.

**Problem 11.** Prove that for a graph class $\mathcal{C}$, the following conditions are equivalent:

- The class $\mathcal{C}$ is somewhere dense.
- There is $d \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, the $d$-subdivision of $K_n$ is a subgraph of some graph from $\mathcal{C}$.

**Problem 12.** Suppose $G$ is a graph and $A \subseteq V(G)$ some subset of its vertices. Define the following equivalence relation $\sim_A$ on the vertices of $V(G) - A$:

$$u \sim_A v \qquad \text{if and only} \qquad N[u] \cap A = N[v] \cap A.$$

Prove that

- in $V(G) - A$, the number of vertices with at least $2\nabla_0(G)$ neighbors in $A$ is at most $|A|$; and
- $\sim_A$ has at most $\left(4^{\nabla_1(G)} + \nabla_1(G)\right) \cdot |A| + 1$ equivalence classes.