# Sparsity — tutorial 3

Measuring sparsity

**Problem 2.** Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant $c_r$ such that the following holds. For every graph $G \in \mathcal{C}$ and every its vertex subset $A \subseteq V(G)$, there exists a vertex subset $B \supseteq A$ with the following properties:

- $|B| \leqslant c_r|A|$, and
- for every pair of vertices $u, v \in A$, if $\mathrm{dist}_G(u,v) \leqslant r$ then $\mathrm{dist}_{G[B]}(u,v) = \mathrm{dist}_G(u,v)$.

**Solution.** We will use the following two facts; the first one was proved in the first tutorial, the second one was given in the second.

**Fact 1** (Closure lemma)**.** Suppose $\mathcal{C}$ is a class of bounded expansion. Then for every $r \in \mathbb{N}$ there exists a constant $d_r$ such that the following holds. For every graph $G \in \mathcal{C}$ and every subset $A$ of its vertices, there exists a vertex subset $B \supseteq A$ such that $|B| \leqslant d_r|A|$ and for every vertex $u \in V(G) - B$, at most $d_r$ vertices of $B$ can be reached from $u$ by a path of length at most $r$ whose internal vertices do not belong to $B$.

**Fact 2** (Stability under lexproduct)**.** If $G$ is a graph and $r, c \in \mathbb{N}$, then

$$\nabla_r(G \bullet K_c) \leqslant 2c^2(r+1)^2\nabla_r(G) + c.$$

We now present a solution to Problem 1. First, we apply Lemma 1 to the set $A$, yielding a set $A'$ with the asserted properties: $|A'| \leqslant d_r|A|$ and for every vertex $u \in V(G) - A'$, at most $d_r$ vertices of $A'$ can be reached from $u$ by a path of length at most $r$ whose internal vertices do not belong to $A'$. Next, for each pair of distinct vertices $u, v \in A'$, select an arbitrary path $P_{u,v}$ that connects $u$ and $v$, and whose internal vertices do not belong to $A'$, and which is the shortest among the paths satisfying these properties; in case there is no such path, put $P_{u,v} = \emptyset$. Then define $B$ to be $A'$ plus the vertex sets of all paths $P_{u,v}$ that have length at most $r$.

We first claim that $B$ has indeed the required property of preserving distances up to $r$. More precisely, take any distinct $u, v \in A$ with $\mathrm{dist}_G(u,v) \leqslant r$. Let $R$ be a shortest path between $u$ and $v$ in $G$, and let $a_1, a_2, \ldots, a_q$ be consecutive vertices of $A'$ visited on $R$, where $u = a_1$ and $v = a_q$. For each $i = 1, 2, \ldots, q-1$, let $R_i$ be the segment of $R$ between $a_i$ and $a_{i+1}$. Then the existence of $R_i$ certifies that some path of length at most $|R_i|$ between $a_i$ and $a_{i+1}$ was added when constructing $B$ from $A'$, and hence $\mathrm{dist}_{G[B]}(a_i, a_{i+1}) \leqslant |R_i|$. Consequently, by the triangle inequality we infer that

$$\mathrm{dist}_{G[B]}(u,v) \leqslant \sum_{i=1}^{q-1} \mathrm{dist}_{G[B]}(a_i, a_{i+1}) \leqslant \sum_{i=1}^{q-1} |R_i| = |R| = \mathrm{dist}_G(u,v).$$

However, the opposite inequality $\mathrm{dist}_{G[B]}(u,v) \geqslant \mathrm{dist}_G(u,v)$ follows directly from the fact that $G[B]$ is an induced subgraph of $G$. Hence indeed $\mathrm{dist}_G(u,v) = \mathrm{dist}_{G[B]}(u,v)$.

We are left with showing that $|B| \leqslant c_r|A|$ for some constant $c_r$. First, we have $|A'| \leqslant d_r|A|$, so we only need to upper bound the ratio $\frac{|B|}{|A'|}$. Let $H$ be a graph on vertex set $A'$, where $uv \in E(A')$ if and only if $P_{u,v}$ exists and has length at most $r$, and hence its vertex set was added in the costruction of $B$. Clearly $|B| \leqslant |A'| + (r-1) \cdot |E(H)|$, so it suffices to prove an upper bound on $|E(H)|$.

Take any $w \in B - A'$, and consider for how many pairs $\{u, v\}$ it can hold that $w \in P_{u,v}$. If $\{u, v\}$ is such a pair, then in particular both $u$ and $v$ can be reached from $w$ by a path of length at most $r$ that internally avoids $A'$. However, we know that the number of such vertices is at most $d_r$, so the number of such pairs $\{u, v\}$ is at most $\tau = \binom{d_r}{2}$. Consequently, we observe that graph $H$ is an $(r-1)$-shallow minor (actually even an $\lceil (r-1)/2 \rceil$-shallow topological minor) of $G \bullet K_\tau$: when each vertex $w \in B - A'$ is replaced with $\tau$ copies, then we can realize all the paths $P_{u,v}$ in $G \bullet K_\tau$ so that they are internally vertex-disjoint. Now we know by Fact 1 that $\nabla_{r-1}(G \bullet K_\tau)$ is bounded by a function of $\nabla_{r-1}(G)$ and $\tau$. Both $\nabla_{r-1}(G)$ and $\tau$ are bounded by constants, namely by $\nabla_{r-1}(\mathcal{C})$ and $\binom{d_r}{2}$ respectively, and hence so does $\nabla_{r-1}(G \bullet K_\tau)$. As $|E(H)| \leqslant \nabla_{r-1}(G \bullet K_\tau) \cdot |A'|$, $|B| \leqslant |A'| + (r-1)|E(H)|$, and $|A'| \leqslant c_r|A|$, we are done. $\square$