Sparsity — tutorial 12

VC dimension and hitting sets

Problem 1. Let $\mathcal{C}$ be a nowhere dense class of graphs and let $G \in \mathcal{C}$. For $r \in \mathbb{N}$ and a vertex ordering σ of G, let

$$\mathcal{W}_{r,\sigma} = \{\mathrm{WReach}_r[G, \sigma, v] \;:\; v \in V(G)\}.$$

Prove that there exists a constant k, depending only on $\mathcal{C}$ and r (and not on G and σ), such that $\mathcal{W}_{r,\sigma}$ has VC-dimension at most k.

Problem 2. Let G be a graph. Consider the following linear program for fractional dominating sets.

Dominating Set LP

- **Variables:** x_v for all $v \in V(G)$
- **Objective:** minimize $\sum_{v \in V(G)} x_v$
- **Constraints:**
  - $\sum_{w \in N[v]} x_w \geq 1$ for all $v \in V(G)$; and
  - $x_v \geq 0$ for all $v \in V(G)$.

Assume G is d-degenerate. Let $D_1 = \{v \in V(G) \;:\; x_v \geq 1/(3d)\}$ and $D_2 = V(G) \setminus N[D_1]$, that is, D_1 consists of all vertices which get weight at least $1/(3d)$ and D_2 are the vertices that are not dominated by D_1. Clearly, $D_1 \cup D_2$ is a dominating set of G.

Prove that $|D_1 \cup D_2| \leq 3dk$, where k is the size of a minimum dominating set of G.

Problem 3. Let $\mathcal{F}$ be a set system over a finite ground set A. We define the *dual* set system $\mathcal{F}^\star$ as follows: the ground set of $\mathcal{F}^\star$ is $\mathcal{F}$, and for each $e \in A$ we add to $\mathcal{F}^\star$ the set $\{X \in \mathcal{F} \colon e \in X\}$. Prove that if the VC dimension of $\mathcal{F}$ is k, then the VC dimension of $\mathcal{F}^\star$ is at most $2^{k+1} - 1$.

Problem 4. Let $\mathcal{F}$ be a set family on ground set A. A *set cover* for A with sets from $\mathcal{F}$ is a subset $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup_{F \in \mathcal{G}} F = A$. Prove that there exists a randomized polynomial-time algorithm which given a set system $\mathcal{F}$ of VC dimension $d \geq 2$, computes a set cover for $\mathcal{F}$ that has size at most $\mathcal{O}(d \cdot \tau(\mathcal{F}) \ln \tau(\mathcal{F}))$ with probability at least $\frac{1}{2}$.