Sparsity — homework 1

Measuring sparsity, deadline: October 23rd, 2017, 14:15 CET

Problem 1. The *arboricity* of an undirected graph G, denoted $\mathrm{arb}(G)$, is the smallest integer k such that the edge set of G can be partitioned into k subsets, each inducing a forest. Prove that for every graph G it holds that

$$\mathrm{arb}(G) \leqslant \deg(G) \leqslant 2 \cdot \mathrm{arb}(G) - 1,$$

where $\deg(G)$ is the degeneracy of G.

Problem 2. Let $\mathcal{C}$ be a somewhere dense graph class that is closed under taking subgraphs. Prove that there exists $r \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists a graph $G \in \mathcal{C}$ and a vertex subset $A \subseteq V(G)$ of size n with the following property: for each subset $B \subseteq A$ there exists some vertex $u \in V(G)$ such that $B = N_G^r[u] \cap A$.

Problem 3. Suppose $\mathcal{F}$ is a family of closed Euclidean balls in $\mathbb{R}^d$, not necessarily of equal radii and not necessarily disjoint. The *ply* of $\mathcal{F}$ is the maximum number of balls that intersect at one point; that is, $\mathcal{F}$ has ply at most ρ iff every point in $\mathbb{R}^d$ is in at most ρ balls of $\mathcal{F}$. For $\rho, d \in \mathbb{N}$, let $\mathcal{B}_{\rho,d}$ be the class of intersection graphs of families of balls of ply at most ρ in $\mathbb{R}^d$; that is, $G \in \mathcal{B}_{\rho,d}$ if with every vertex of G we can associate a closed ball in $\mathbb{R}^d$ so that the balls form a family of ply at most ρ and two vertices are adjacent in G if and only if the corresponding balls intersect.

Prove that for all fixed $\rho, d \in \mathbb{N}$, there is a polynomial $p(\cdot)$ of degree d such that $\nabla_r(\mathcal{B}_{\rho,d}) \leqslant p(r)$ for all $r \in \mathbb{N}$.