Parameterized algorithms — tutorial 6

Treewidth 2

Problem 1. Prove that if $a_1, \ldots, a_n$ are nonnegative numbers not greater than $1/2$ and $\sum_{i=1}^{n} a_i \leqslant 1$, then these numbers can be partitioned into two groups such that the sum of numbers within each group is at most $2/3$.

Problem 2. Prove that H is a minor of G if and only if H can be obtained from G by means of the following three operations: vertex removal, edge removal, and edge contraction. Conclude that if H is a minor of G, then $\mathsf{tw}(H) \leqslant \mathsf{tw}(G)$.

Problem 3. Prove that the following conditions are equivalent for a graph H:

- H is planar;
- the class of H-minor-free graphs has bounded treewidth (i.e. there exists a universal constant c_H upper bounding the treewidth of every H-minor-free graph).

Problem 4. Use the fact that the function in the Grid Minor Theorem for planar graphs is linear in the treewidth to prove that the treewidth of a connected planar graph is linear in its radius.

Problem 5. Prove that there planar graphs of radius 1 that have unbounded pathwidth.

Problem 6. Determine whether bidimensionality can be used to give parameterized algorithms with running time $2^{\mathcal{O}(\sqrt{k})} \cdot n$ or $2^{\mathcal{O}(\sqrt{k}\log k)} \cdot n$ on planar graphs for the following problems:

- Distance-r Independent Set parameterized by solution size k, where r is a constant;
- Weighted Independent Set: find an independent set of size k and maximum possible weight;
- Dominating Set parameterized by solution size k;
- Feedback Vertex Set parameterized by solution size k;
- k-Path and Directed k-Path;
- Steiner Tree parameterized by the target tree size, and parameterized by the number of terminals.

Theorem 1. Given a planar graph G and integer k one can in randomized time $2^{\mathcal{O}(\sqrt{k}\log^2 k)} \cdot n^{\mathcal{O}(1)}$ enumerate a family $\mathcal{F}$ of subsets of vertices of G with the following properties:

- $|\mathcal{F}| \leqslant 2^{\mathcal{O}(\sqrt{k}\log^2 k)} \cdot n^{\mathcal{O}(1)}$;
- for each $A \in \mathcal{F}$, the graph $G[A]$ has treewidth $\mathcal{O}(\sqrt{k}\log k)$; and
- for each $X \subseteq V(G)$ such that $|X| \leqslant k$ and $G[X]$ is connected, there exists $A \in \mathcal{F}$ such that $X \subseteq A$.

Problem 7. Give an algorithm for Subgraph Isomorphism on graphs of treewidth at most t with running time $t^{\mathcal{O}(p)} \cdot n$, where p is the size of the pattern graph H.

Problem 8. Design EPTASes for Distance-r Independent Set and Dominating Set on planar graphs: given a planar graph G, the algorithm should compute a solution of size at least $(1-\varepsilon)\mathsf{OPT}$, respectively at most $(1+\varepsilon)\mathsf{OPT}$, in time $2^{\mathcal{O}(1/\varepsilon)} \cdot n$.

Problem 9. Let G be a plane graph. A *face-vertex curve* in G is a non-self-crossing curve in the plane that intersects the embedding of G only at vertices. The *face-vertex distance* between two vertices u, v is the smallest number of faces visited by a face-vertex curve connecting u and v. Prove that every planar graph of face-vertex radius r has treewidth at most $6r$.