ANOTHER NOTE ON DUALITY BETWEEN MEASURE AND CATEGORY

MARCIN KYSIAK

SUMMARY. We show that no Erdös–Sierpiński mapping from $\mathbb{R}$ to $\mathbb{R}$ can be additive.

1. Introduction

Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideals of Lebesgue measure zero and meager subsets of the real line, respectively.

Definition 1.1. A bijection $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called an Erdös–Sierpiński mapping (on $\mathbb{R}$) if

$$\forall X \subseteq \mathbb{R} \ \ (X \in \mathcal{M} \Leftrightarrow f[X] \in \mathcal{N}) \ \ \& \ \ (X \in \mathcal{N} \Leftrightarrow f[X] \in \mathcal{M}).$$

It is well known that the existence of an Erdös–Sierpiński mapping is consistent with ZFC:

Theorem 1.2 (Erdös – Sierpiński, see [1])**.** *Assume CH. Then there exists an Erdös – Sierpiński mapping.*

The existence of such a function is independent from ZFC. Recall that for an ideal $\mathcal{I}$, we define $\mathrm{non}(\mathcal{I}) = \min\{|X| : X \notin \mathcal{I}\}$. The following fact shows that it is consistent that an Erdös – Sierpiński mapping does not exist:

Theorem 1.3 (see [1])**.** *It is consistent that* $\mathrm{non}(\mathcal{M}) \neq \mathrm{non}(\mathcal{N})$.

An Erdös–Sierpiński mapping is an isomorphism of the structures $\langle \mathbb{R}, \mathcal{N} \rangle$ and $\langle \mathbb{R}, \mathcal{M} \rangle$ and thus preserves basic set-theoretical properties of the ideals of measure and category. It need not, however, preserve additive properties of these ideals, i.e. properties which concern translations of sets from an ideal (see [1] for examples of such properties). This motivates the following question, attributed to Cz. Ryll-Nardzewski:

Is it consistent that there exists an Erdös–Sierpiński mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ such, that

$$\forall x, y \in \mathbb{R} \ \ f(x + y) = f(x) + f(y)?$$

In [2] T. Bartoszyński showed that the answer is negative if we replace $\mathbb{R}$ by the space 2^ω with addition modulo 2 (equipped with the canonical natural product measure). In this paper we modify his proof to obtain the same answer for the original question about $\mathbb{R}$.

2000 *Mathematics Subject Classification.* 03E15 (28A99).

Key words and phrases. Erdös–Sierpiński mapping, duality of measure and category.

The main result of this paper was included earlier in the author's Master's Thesis written under supervision of P. Zakrzewski, University of Warsaw, 2000.

It is usually much more convenient to investigate properties of measure and category in 2^ω than in $\mathbb{R}$ as technical parts of proofs are easier to handle there. On the other hand, the space $\mathbb{R}$ is a much more natural object for a mathematician than the space 2^ω. Although these spaces reveal many similarities, it was (in the author's opinion) not obvious how to make Bartoszyński's argument work for $\mathbb{R}$. Roughly speaking, the problem is caused by the different relations of the group structure with measure and topology in these spaces.

The basic idea of our proof is based on the original proof for 2^ω; we introduce, however, a couple of tricks which make these arguments work for $\mathbb{R}$.

In our opinion, there is a need for developing a universal technique which could "translate" additive properties of the ideals of measure and category between $\mathbb{R}$ and 2^ω. The purpose of writing this paper, beside solving the problem for $\mathbb{R}$, was to present of the methods we used for this particular problem, in hope that they might become a part of a more universal technique.

2. Notation

We use a standard set-theoretical notation. In particular, by ω we denote the set of natural numbers. We identify a natural number n with the set of its predecessors: $n = \{0, 1, ..., n-1\}$. In particular $2 = \{0, 1\}$. When $X \subseteq \omega$, by 2^X we denote the set of binary sequences with domain X.

We use an additive notation for group operations. If G is a group, $A \subseteq G$ and $t \in G$ then by $t + A$ we denote a translation of A by t, i.e. the set $\{t + a : a \in A\}$. Similarly, when $A, B \subseteq G$, then by $A + B$ we denote the set $\{a + b : a \in A, b \in B\}$.

We will say that a family $\mathcal{A}$ of subsets of a group G is translation invariant, if for every $A \subseteq G$ and for every $t \in G$ we have $A \in \mathcal{A} \Leftrightarrow A + t \in \mathcal{A}$.

A family $\mathcal{I} \subseteq P(X)$ is called an ideal on X if it is closed under taking finite unions and subsets.

A family $\mathcal{I} \subseteq P(X)$ is called a filter on X if it is closed under taking finite intersections and supersets. When $\mathcal{I}$ is an ideal on X, then by $\mathcal{I}^*$ we denote it's dual filter, i.e. the family $\{X \setminus A : A \in \mathcal{I}\}$.

When A is any set and κ is a cardinal number, by $[A]^\kappa$ we denote the family of subsets of A of cardinality κ.

Further notation will be introduced throughout the paper.

3. Translatability of ideals and filters

Let us begin with the notion of κ-translatability introduced by Carlson in [3]:

Definition 3.1. Let $\mathcal{I}$ be a translation invariant ideal on a commutative group G. We say that $\mathcal{I}$ is κ-translatable, if

$$\forall A \in \mathcal{I} \ \exists B \in \mathcal{I} \ \forall T \in [G]^\kappa \ \exists t \in G$$
$$T + A \subseteq t + B.$$

The following characterization was also observed in [3]:

Proposition 3.2. *Let $\mathcal{I}$ be a translation invariant ideal on a commutative group G and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then $\mathcal{I}$ is κ-translatable if and only if*

$$\forall A \in \mathcal{F} \ \exists B \in \mathcal{F} \ \forall T \in [G]^\kappa \ \exists t \in G$$
$$T + B \subseteq t + A.$$

One of the most important results concerning the notion of κ-translatability in [3] was the following theorem:

Theorem 3.3. *The ideal $\mathcal{M}$ is ω-translatable.*

It is clear, that if an additive Erdös – Sierpiński mapping existed, than the ideal $\mathcal{N}$ would also be ω-translatable. Hence, to show that an additive Erdös–Sierpiński mapping does not exist, it is sufficient to show that the ideal $\mathcal{N}$ is not 2-translatable. We will prove this theorem in the next section.

4. Main result

4.1. **More specific notation and terminology.** For technical reasons, we will work rather in the set $(0,1]$ with addition modulo 1 than in $\mathbb{R}$. We can also think of this group either as $\mathbb{R}/\mathbb{Z}$ or the unit circle on the complex plane. Let $\mathcal{N}((0,1])$ denote the ideal of null (with respect to the Lebesgue measure on $(0,1]$) subsets of this space.

If not otherwise stated, by a binary expansion of a number $x \in (0,1]$ we mean the non-terminating expansion, i.e. the one that is not eventually equal to 0. Note, that every $x \in (0,1]$ has such an expansion.

We will identify a number $x \in (0,1]$ with such a binary expansion, treated as a binary sequence defined on the set $\omega\setminus\{0\}$. For instance, if $x = 0,10101010...$ then $x(1) = 1$ and $x(2) = 0$ and so on.

If s is a finite binary sequence, than by $[s]$ we denote the set of those numbers in $(0,1]$, which have non-terminating binary expansion extending s. Also if $I \subseteq \omega$ and $J \subseteq 2^I$, then by $[J]$ we denote the set $\{x \in (0,1] : x \restriction I \in J\}$.

By μ we denote the Lebesgue measure on $(0,1]$. By μ_2 we denote the Lebesgue measure on $(0,1]^2$.

4.2. **Some basic facts.**

Lemma 4.1. *Let $K = (a,b) \subseteq \mathbb{R}$ and let $X \subseteq K$ be a measurable set of positive measure. Then there exists $x \in X$ such, that:*

$$\mathrm{dist}(x, \mathbb{R}\setminus K) \geq \frac{\mu(X)}{3}.$$

Proof. Otherwise we would have $X \subseteq [a, a+\frac{\mu(X)}{3}) \cup (b-\frac{\mu(X)}{3}, b]$, and thus $\mu(X) \leq \frac{2}{3}\mu(X)$ - a contradiction. □

Definition 4.2. Let S be a real number such that $0 < S \leq 1$. We say that $X \subseteq (0,1]$ is S-periodic, if

$$\forall x \in (0,1] \;\; x \in X \Leftrightarrow x + S \in X.$$

We leave the proof of the following fact as an easy exercise:

Proposition 4.3. *For every $S \in (0,1]$ the family of S-periodic sets forms a translation-invariant algebra of subsets of $(0,1]$.* □

Proposition 4.4. *Let I be a finite subset of ω and $J \subseteq 2^I$. Then $[J]$ is $\frac{1}{2^{minI-1}}$ periodic.*

Proof. Note, that the number $\frac{1}{2^{minI-1}}$ has a terminating binary expansion which is equal to 0 on the set $[\min I, \infty)$. Thus adding a number $\frac{1}{2^{minI-1}}$ to an arbitrary number $x \in (0,1]$ does not change $x \restriction I$. □

The last proposition can be partially reversed:

Proposition 4.5. *If X is $\frac{1}{2^m}$ periodic, then the question whether x belongs to X does not depend on the first m places of the binary expansion of x.*

Proof. Adding the number

$$\frac{1}{2^m} = 0,\underbrace{00\ldots0}_{m-1\,\text{zeros}}1000...$$

to x, we change $x \restriction \{1,...,m\}$ into the lexicographically next element of $2^{\{1,\ldots,m\}}$ while the following places of the expansion remain unchanged. Thus adding $\frac{1}{2^m}$ to x several times we can obtain any initial segment of binary expansion of length m without affecting the fact that x does (or does not) belong to X. □

4.3. **Main theorem.**

Theorem 4.6. *The ideal $\mathcal{N}((0,1])$ of null subsets of $(0,1]$ is not 2-translatable.*

Proof. We begin with the following lemma (compare also with lemma 7 from [2]):

Lemma 4.7. *If J' is a Borel subset of $(0,1]$, $\delta = 1-\mu(J')$ and $\varepsilon > \delta^2$ then the set $S = \{t \in (0,1] : \mu(J' \cup (t+J')) > 1-\varepsilon\}$ is measurable and*

$$\mu(S) \geq 1 - \frac{\delta^2}{\varepsilon}.$$

Proof. Let $Z = \{\langle z,t\rangle \in (0,1]^2 : z \in J' \cup (t+J')\}$. Obviously Z is Borel. Observe that $S = \{t \in (0,1] : \mu(Z_t) > 1-\varepsilon\}$, so S is measurable.

For every $z \in (0,1]$ we have

$$Z_z = \{t \in (0,1] : z \in J' \cup (t+J')\} = \{t \in (0,1] : z \in J' \vee t \in z - J'\}.$$

Thus

$$(0,1] \setminus Z_z = ((0,1] \setminus J') \times ((0,1] \setminus (z-J')),$$

so

$$\mu(Z_z) = 1-\delta^2.$$

From the Fubini theorem we obtain:

$$1-\delta^2 = \mu_2(Z) = \int_S \mu((Z)_t)d\mu + \int_{(0,1]\setminus S} \mu((Z)_t)d\mu.$$

We have

$$\int_S \mu((Z)_t)d\mu \leq \mu(S)$$

and

$$\int_{(0,1]\setminus S} \mu((Z)_t)d\mu \leq (1-\varepsilon)(1-\mu(S)).$$

Finally

$$1-\delta^2 \leq \mu(S) + (1-\varepsilon)(1-\mu(S))$$

and after easy transformations:

$$\mu(S) \geq 1 - \frac{\delta^2}{\varepsilon}.$$

□

Now we prove our main theorem. Following the idea of the proof for 2^ω, we define a set A, which will witness non-translatability of $\mathcal{N}((0,1])$ in the following way:

Fix a partition of the set $\omega\backslash\{0\}$ into consecutive finite disjoint intervals $\langle I_n : n > 0\rangle$ such that $|I_n| = 2^{n+3}$. One can easily check that the following inequalities hold:

- $\frac{1}{2^{|I_n|}} < \frac{1}{n^5}$
- $\frac{1}{2^{|I_n|}} \leq \frac{1}{12}(\frac{1}{n^2} - \frac{1}{n^5})$ for $n > 1$.

For every $n > 0$ choose $J_n \subseteq 2^{I_n}$ such that:

$$1 - \frac{1}{n^2} + \frac{1}{n^5} > \frac{|J_n|}{2^{|I_n|}} > 1 - \frac{1}{n^2}.$$

Moreover, let us demand that J_n consists of the first $|J_n|$ consecutive (in the sense of the lexicographical ordering of 2^{I_n}) elements of 2^{I_n}.

We will define the set A as follows: we take $F = \bigcap_m \bigcup_{n>m}((0,1]\backslash[J_n])$ and put $A = (0,1]\backslash F$. Clearly $\mu(F) = 0$, so $A \in \mathcal{N}^*$.

We will show that for every $B \in \mathcal{N}^*$ there are numbers x_1, x_2 such that:

$$\forall x \in (0,1]\ \ (x_1 + B) \cup (x_2 + B) \nsubseteq (x + A).$$

In fact, we will put $x_1 = 0$ and construct $y \in (0,1]$ such that

$$\forall x \in (0,1]\ \ B \cup (y + B) \nsubseteq (x + A).$$

For this purpose let us take any closed set C of positive measure such that $C \subseteq B\backslash\mathbb{Q}$. We will construct $y \in (0,1]$ such that:

$$\forall x \in (0,1]\ \ C \cup (y + C) \nsubseteq (x + A).$$

Without loss of generality we may assume that for every open set $U \subseteq (0,1]$ we have $\mu(U \cap C) > 0 \vee U \cap C = \emptyset$ (if not, consider $C' = C \setminus \bigcup\{(p,q) : p,q \in \mathbb{Q} \wedge \mu((p,q) \cap C) = 0\}$ instead). Notice that in this situation for every $s \in 2^{<\omega}$ we also have $\mu([s] \cap C) > 0 \vee [s] \cap C = \emptyset$. Although the set $[s]$ is not open in $(0,1]$, it is of the form $(p,q]$, where $p,q \in \mathbb{Q}$. So if $(p,q] \cap C \neq \emptyset$ then $(p,q) \cap C \neq \emptyset$, because $C \cap \mathbb{Q} = \emptyset$.

Let $\lambda_n = 1 - \mu([J_n])$ and $\varepsilon_n = \frac{1}{4}\lambda_n$. We inductively define:

(1) A strictly increasing sequence of natural numbers $\langle n_k : k \in \omega\rangle$,
(2) Sets $J'_{n_k} \subseteq 2^{I_{n_k}}$ such that for every $c \in C$ and $s \in J'_{n_{k+1}}$ exists $c' \in C$ such that $c\restriction I_1 \cup ... \cup I_{n_k} = c'\restriction I_1 \cup ... \cup I_{n_k}\ \ \wedge\ \ c'\restriction I_{n_{k+1}} = s$,
(3) Sets $S_k \subseteq (0,1]$ such that $\mu(S_k) > 1 - \frac{1}{4^{k+1}}$ and

$$\forall t \in S_k\ \ \mu([J'_{n_k}] \cup (t + [J'_{n_k}])) > 1 - \varepsilon_{n_k}.$$

Remark 4.8. Condition 2. guarantees that for every choice of $u_k \in J'_{n_k}$ exists $c \in C$ such that

$$\forall k \in \omega\ \ c\restriction I_{n_k} = u_k.$$

Indeed: using 2. we may construct $c \in C$ inductively. Having defined $c\restriction I_1, \cup ... \cup I_{n_k} \in C\restriction I_1, \cup ... \cup I_{n_k}$ for some $k \in \omega$ we look at $I_{n_{k+1}}$. We know that $c\restriction I_1, \cup ... \cup I_{n_k}$ is a restriction of an element of C to the set $I_1, \cup ... \cup I_{n_k}$, hence, by 2., the sequence $c\restriction I_1, \cup ... \cup I_{n_k}$ can be extended onto $I_1, \cup ... \cup I_{n_{k+1}}$ such that $c \restriction I_{n_{k+1}} = u_{k+1}$ and $c\restriction I_1, \cup ... \cup I_{n_{k+1}} = d\restriction I_1, \cup ... \cup I_{n_{k+1}}$ for some $d \in C$. As C is closed, we see that $c \in C$.

Now we are ready to give the construction. Suppose that we have already defined n_i, J'_{n_i}, S_i, for $i \leq k$. We look at $C\restriction I_1 \cup ... \cup I_{n_k}$ - a (finite!) set of restrictions of elements of C to the interval $I_1 \cup ... \cup I_{n_k}$. Let t be the number of elements of this set; we may write $C\restriction I_1 \cup ... \cup I_{n_k}$ as $\{s_1, ...s_t\}$. By our assumptions about C we know that for every $s \in C\restriction I_1 \cup ... \cup I_{n_k}$ the set $[s] \cap C$ has positive measure.

Using the Lebesgue density theorem we can find $l > n_k$ and $r_s \supset s$ for every $s \in C\restriction I_1 \cup ... \cup I_{n_k}$ such that $\mathrm{dom}(r_s) = I_1 \cup ... \cup I_l$ and the set

$$P = 2^{I_1 \cup ... \cup I_l} \times \bigcap_{i=1}^{t} \{x\restriction (|r_{s_i}|, \omega) : r_{s_i} \subseteq x \in C\} \subseteq (0,1]$$

has positive measure. To see this, observe that for every $s \in C\restriction I_1 \cup ... \cup I_{n_k}$ we can find its extension r_s such that $\mu([r_s] \cap C) > \frac{t-1}{t}\mu([r_s])$. Then

$$\bigcap_{i=1}^{t} \{x\restriction (|r_{s_i}|, \omega) : r_{s_i} \subseteq x \in C\}$$

has positive measure in $2^{[\max I_l + 1, \infty)}$.

For $m > l$ we look at the sets

$$J'_m = \{x\restriction I_m : x \in P\}$$

We have

$$P \subseteq \{x \in (0,1] : \forall m > l \;\; x\restriction I_m \in J'_m\}$$

so the set on the right side of the above inclusion has positive measure, hence:

$$\prod_m \mu([J'_m]) > 0.$$

Let $\delta_m = 1 - \mu([J'_m])$, then we have:

$$\prod_m (1 - \delta_m) > 0.$$

One can easily check that

$$\prod_m (1 - \frac{1}{2^{k+3}} \sqrt{\frac{1}{m^2} - \frac{1}{m^5}}) = 0,$$

so for infinitely many $m \in \omega$

$$1 - \delta_m > 1 - \frac{1}{2^{k+3}} \sqrt{\frac{1}{m^2} - \frac{1}{m^5}}.$$

Let n_{k+1} be the first such $m > l$. One can easily verify that the set $J'_{n_{k+1}}$ satisfies condition 2.

Notice that the inequality

$$1 - \delta_{n_{k+1}} > 1 - \frac{1}{2^{k+3}} \sqrt{\frac{1}{(n_{k+1})^2} - \frac{1}{(n_{k+1})^5}}$$

holds, so

$$\delta^2_{n_{k+1}} < \frac{1}{4^{k+3}} (\frac{1}{(n_{k+1})^2} - \frac{1}{(n_{k+1})^5}).$$

On the other hand, the set $J_{n_{k+1}}$ has been chosen in such a way that

$$\frac{1}{(n_{k+1})^2} - \frac{1}{(n_{k+1})^5} < \lambda_{n_{k+1}}$$

(recall that we have put $\lambda_{n_{k+1}} = 1 - \mu([J_{n_{k+1}}]).$) If we multiply both sides by $\frac{1}{4^{k+3}}$ we get:

$$\frac{1}{4^{k+3}}(\frac{1}{(n_{k+1})^2} - \frac{1}{(n_{k+1})^5}) < \frac{\lambda_{n_{k+1}}}{4^{k+3}}.$$

Finally:

$$\delta^2_{n_{k+1}} < \frac{1}{4^{k+3}}(\frac{1}{{n_{k+1}}^2} - \frac{1}{{n_{k+1}}^5}) < \frac{\lambda_{n_{k+1}}}{4^{k+3}},$$

so

$$\frac{\delta^2_{n_{k+1}}}{\lambda_{n_{k+1}}} < \frac{1}{4^{k+3}}.$$

In other words

$$\frac{\delta^2_{n_{k+1}}}{\frac{1}{4}\lambda_{n_{k+1}}} < \frac{1}{4^{k+2}}.$$

We have put $\varepsilon_{n_{k+1}} = \frac{\lambda_{n_{k+1}}}{4}$, hence we have:

$$\frac{\delta^2_{n_{k+1}}}{\frac{1}{4}\lambda_{n_{k+1}}} = \frac{\delta^2_{n_{k+1}}}{\varepsilon_{n_{k+1}}} < \frac{1}{4^{k+2}}.$$

In particular $\delta^2_{n_{k+1}} < \frac{\varepsilon_{n_{k+1}}}{4}$. From lemma 4.7 we immediately get the set $S_{k+1} = \{t \in (0,1] : \mu([J'_{n_{k+1}}] \cup (t + [J'_{n_{k+1}}])) > 1 - \varepsilon_{n_{k+1}}\} \subseteq (0,1]^2$ as needed. This finishes the construction.

Notice that as $\mu(S_k) > 1 - \frac{1}{4^{k+1}}$, we see that $\bigcap_k S_k \neq \emptyset$. Let us take any y such that $y \in \bigcap_k S_k$. We will check that

$$\forall x \in (0,1] \;\; (C) \cup (y + C) \nsubseteq (x + A).$$

For this purpose take any $x \in (0,1]$. From the condition 3. and from the fact that $y \in S_k$ and $\mu([J_{n_k}])) = 1 - \lambda_{n_k} < 1 - \varepsilon_{n_k}$ we see that for every $k \in \omega$

$$[J'_{n_k}] \cup (y + [J'_{n_k}]) \nsubseteq x + [J_{n_k}].$$

Moreover, we know that $\mu([J'_{n_k}] \cup (y + [J'_{n_k}])) \geq 1 - \varepsilon_{n_k}$; on the other hand $\mu(x + [J_{n_k}]) = 1 - \lambda_{n_k}$. Hence

$$\mu([J'_{n_k}] \cup (y + [J'_{n_k}]) \setminus (x + [J_{n_k}])) \geq (1 - \varepsilon_{n_k}) - (1 - \lambda_{n_k}) = \lambda_{n_k} - \varepsilon_{n_k},$$

so for every $k \in \omega$ either

$$\mu([J'_{n_k}] \setminus (x + [J_{n_k}])) \geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2}$$

or

$$\mu((y + [J'_{n_k}]) \setminus (x + [J_{n_k}])) \geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2}.$$

We will assume that the second case holds for infinitely many k. As our proof will not refer to any specific properties of y, if this is true for the first case only, the same arguments work if we put $y = 0$.

Let us denote by $U \subseteq \omega$ the set of such k that

$$\mu((y + [J'_{n_k}]) \setminus (x + [J_{n_k}])) \geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2}.$$

We will construct $z \in C$ such that $y + z \notin x + A$. To do this, for every $k \in U$ we will choose $v_k \in (0,1]$ to have $y + v_k \in (y + [J'_{n_k}]) \setminus (x + [J_{n_k}])$.

Notice, that as the sets $(x - y) + [J_{n_k}]$ and $[J'_{n_k}]$ are $\frac{1}{2^{|I_1 \cup \ldots \cup I_{n_k - 1}|}}$ periodic, this property of v_k does not depend on the first $\min I_{n_k} - 1$ terms of it's binary expansion.

On the other hand, if the distance of v_k from the set $(x-y)+[J_{n_k}]$ is positive, this property does not depend on sufficiently far terms of the binary expansion of v_k. We will show, that it is possible to choose v_k such that the numbers of the terms which are "essential" for this property belong to I_{n_k}. To get this, it is sufficient to choose v_k in such way that $\mathrm{dist}(v_k,(x-y)+[J_{n_k}]) > \frac{1}{2^{|I_1\cup\ldots\cup I_{n_k}|}}$. We will show, that it is possible.

Let us look closer at the set $[J_{n_k}]$. As we have noticed, it is $\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}$ periodic. As the elements if J_{n_k} were lexicographically consecutive, the interval $(0,1]$ can be divided into $2^{|I_1\cup\ldots\cup I_{n_k-1}|}$ subintervals $[s]=(p_s,q_s]$ for $s\in 2^{I_1\cup\ldots\cup I_{n_k-1}}$, such that there are $r_s\in(p_s,q_s)$ such that $[J_{n_k}]\cap(p_s,q_s]=(p_s,r_s]$. In other words:

$$[J_{n_k}]=\bigcup_s (p_s,r_s]; \qquad (0,1]\setminus[J_{n_k}]=\bigcup_s (r_s,q_s].$$

We know that

$$\mu((y+[J'_{n_k}]))\setminus(x+[J_{n_k}]))\geq\frac{\lambda_{n_k}-\varepsilon_{n_k}}{2},$$

so

$$\mu(((y-x)+[J'_{n_k}]))\setminus[J_{n_k}])\geq\frac{\lambda_{n_k}-\varepsilon_{n_k}}{2}.$$

The set $(y-x)+[J'_{n_k}]$ is also $\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}$ - periodic. Moreover, it is easy to see that if $s,t\in 2^{I_1\cup\ldots\cup I_{n_k-1}}$ then

$$(p_s,q_s]=(p_t,q_t]+\frac{i}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}$$

and

$$(r_s,q_s]=(r_t,q_t]+\frac{i}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}$$

for some $i<2^{|I_1\cup\ldots\cup I_{n_k-1}|}$. It follows that for every $s,t\in 2^{I_1\cup\ldots\cup I_{n_k-1}}$ the following holds:

$$\mu(((y-x)+[J'_{n_k}]\cap[r_s,q_s))=\mu(((y-x)+[J'_{n_k}])\cap[r_t,q_t)).$$

Hence, for every $s\in 2^{I_1\cup\ldots\cup I_{n_k-1}}$

$$\mu(((y-x)+[J'_{n_k}])\cap[r_s,q_s))\geq\frac{\lambda_{n_k}-\varepsilon_{n_k}}{2}\,\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}.$$

Fix any $s\in 2^{I_1\cup\ldots\cup I_{n_k-1}}$. By lemma 4.1 there is $v_k\in[J'_{n_k}]$ such that $(y-x)+v_k\in(r_s,q_s)$ and

$$\mathrm{dist}((y-x)+v_k,(0,1]\setminus(r_s,q_s))=\mathrm{dist}((y-x)+v_k,[J_{n_k}])\geq\frac{\lambda_{n_k}-\varepsilon_{n_k}}{6}\,\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}.$$

From v_k we expect that

$$\mathrm{dist}((y-x)+v_k,[J_{n_k}])=\mathrm{dist}(v_k,(x-y)+[J_{n_k}])>\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k}|}}.$$

Thus it is sufficient to show that

$$\frac{\lambda_{n_k}-\varepsilon_{n_k}}{6}\,\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}>\frac{1}{2^{|I_1\cup\ldots\cup I_{n_k}|}};$$

equivalently:

$$\frac{\lambda_{n_k}-\varepsilon_{n_k}}{6}>\frac{2^{|I_1\cup\ldots\cup I_{n_k-1}|}}{2^{|I_1\cup\ldots\cup I_{n_k}|}}=\frac{1}{2^{|I_{n_k}|}}.$$

Let us remind that we have put $\varepsilon_{n_k} = \frac{1}{4}\lambda_{n_k}$, so in particular $\lambda_{n_k} - \varepsilon_{n_k} > \frac{\lambda_{n_k}}{2}$. From this, we get:

$$\frac{\lambda_{n_k} - \varepsilon_{n_k}}{6} > \frac{\lambda_{n_k}}{12}.$$

On the other hand, we know that $\lambda_{n_k} \geq \frac{1}{{n_k}^2} - \frac{1}{{n_k}^5}$.

Finally:

$$\frac{\lambda_{n_k} - \varepsilon_{n_k}}{6} > \frac{\lambda_{n_k}}{12} \geq \frac{1}{12}(\frac{1}{{n_k}^2} - \frac{1}{{n_k}^5}) \geq \frac{1}{2^{|I_{n_k}|}}$$

(the last inequality follows from the choice of I_n).

We have constructed $v_k \in [J'_{n_k}]$ such that

$$z\restriction I_{n_k} = v_k \restriction I_{n_k} \Longrightarrow y + z \notin x + [J_{n_k}]$$

for any $z \in (0,1]$. But $v_k \restriction I_{n_k} \in J'_{n_k}$, so by remark 4.8, there exists $z \in C$ such that

$$\forall k \in U \;\; z\restriction I_{n_k} = v_k \restriction I_{n_k}.$$

We see that $y + z \in y + C$ but for infinitely many $k \in \omega$ we have $y + z \notin x + [J_{n_k}]$. Hence:

$$y + z \in (y + C) \cap (x + F) = (y + C)\backslash(x + A),$$

so

$$(y + C) \nsubseteq (x + A).$$

□

Theorem 4.9. *The ideal $\mathcal{N}$ is not 2-translatable.*

Proof. Notice that the group $(0,1]$ with addition modulo 1 is isomorphic with $\mathbb{R}/\mathbb{Z}$. Let $\pi : \mathbb{R} \longrightarrow (0,1]$ be the natural epimorphism. Note, that π preserves the measure ideal, i.e.:

$$X \in \mathcal{N} \Leftrightarrow \pi[X] \in \mathcal{N}((0,1]).$$

Now suppose that the ideal $\mathcal{N}$ is 2-translatable. Take any $A \in \mathcal{N}((0,1])$, and consider $\pi^{-1}[A] \in \mathcal{N}$. Let $B \subseteq (0,1]$ be such that for every $t_1, t_2 \in \mathbb{R}$ exists $t \in \mathbb{R}$ that

$$(t_1 + \pi^{-1}[A]) \cup (t_2 + \pi^{-1}[A]) \subseteq t + B.$$

Then for every $s_1, s_2 \in (0,1]$ we would have:

$$(s_1 + A) \cup (s_2 + A) \subseteq s + \pi[B]$$

for some $s \in (0,1]$. To see this, consider t_i such that $\pi(t_i) = s_i$ for $i \in \{1,2\}$ and take $s = \pi(t)$. But that means that $\mathcal{N}((0,1])$ is 2-translatable - a contradiction. □

5. Final remarks and acknowledgments

In [2] T. Bartoszyński showed that there is no additive Erdös–Sierpiński mapping from 2^ω to 2^ω in the same way, i.e. showing that the ideal $\mathcal{N}(2^\omega)$ is not 2-translatable. It was known earlier (see [3]) that the ideal $\mathcal{M}(2^\omega)$ of meager subsets of 2^ω is ω-translatable.

Let us point out that our proof gives us a little stronger property than the negation of 2-translatability. In fact, by careful analysis of our proof, we can obtain

Corollary 5.1. *There exists a set $A \in \mathcal{N}$ such that for every $B \in \mathcal{N}$ the set of those $y \in \mathbb{R}$ such that*

$$\forall x \in \mathbb{R} \;\; A \cup (y + A) \not\subseteq x + B$$

has full measure.

Proof. Observe that to choose y we could take any element of the set $\bigcap_k S_k$, which appeared in our construction. This set was constructed to have measure not smaller than $\frac{2}{3}$. By refining the appropriate estimations we could obtain instead a set of measure greater than $1 - \varepsilon$ for any given $\varepsilon > 0$. The corollary follows easily. □

Using the same techniques one can improve T. Bartoszyński's proof to obtain an analogous corollary for 2^ω.

The result for $\mathbb{R}$ which is presented here was included earlier in the author's Master's Thesis written under supervision of prof. P. Zakrzewski. I would like to thank him for a great supervision and a lot of help with preparing this paper.

References

[1] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, Ltd, 1995.

[2] T. Bartoszyński, *A note on duality between measure and category*, Proceedings of the American Mathematical Society **128** (2000), no. 9, 2745–2748.

[3] T. J. Carlson, *Strong measure zero and strongly meager sets*, Proceedings of the American Mathematical Society **118** (1993), no. 2, 577–586.

Institute of Mathematics, Polish Academy of Sciences
(Instytut Matematyczny Polskiej Akademii Nauk);
ul. Śniadeckich 8; 00-950 Warszawa; Poland
E-mail address: mkysiak@impan.gov.pl