Probability on graphs
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Problem set 9

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Problem 1. Consider a reversible Markov chain with state space S, transition matrix P and stationary distribution π. For any function $f : S \to \mathbb{R}$ let $\mathrm{Var}_\pi(P^t f)$ denote the variance of the random variable $(P^t f)(X_0)$, where $X_0 \sim \pi$. Likewise, for $f, g : S \to \mathbb{R}$ let $\mathrm{Cov}_\pi(f, g)$ denote the covariance of $f(X_0)$ and $g(X_0)$ with $X_0 \sim \pi$.

Recall that γ_* denotes the absolute spectral gap of P.

(a) Prove for any f the inequality $\mathrm{Var}_\pi(P^t f) \leq (1-\gamma_*)^{2t}\mathrm{Var}_\pi(f)$.

(b) Deduce that for any f, g we have $\mathrm{Cov}_\pi(P^t f, g) \leq (1-\gamma_*)^t \sqrt{\mathrm{Var}_\pi(f), \mathrm{Var}_\pi(g)}$.

(c) Let $G = (V, E)$ be a d-regular graph with n vertices. Let A be its adjacency matrix, i.e., the matrix with entries $A_{ij} = 1$ if $\{i, j\} \in E$ and 0 otherwise. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues and let $\beta = \max\{|\lambda_2|, |\lambda_n|\}$.

For $A, B \subseteq V$ define

$$e(A, B) = \{(x, y) \in A \times B : \{x, y\} \in E\}.$$

Prove the following inequality:

$$\left| |e(A, B)| - \frac{d|A||B|}{n} \right| \leq \beta\sqrt{|A||B|}.$$

Problem 2. Consider a reversible Markov chain with state space S, transition matrix P and stationary distribution π. Recall that for $p \in [1, \infty]$ we defined

$$d^{(p)}(t) = \sup_{x \in S} \left\| \frac{P^t(x, \cdot)}{\pi(\cdot)} - 1 \right\|_p$$

and

$$t_{mix}^{(p)}(\varepsilon) = \inf\{t \geq 0 : d^{(p)}(t) \leq \varepsilon\}.$$

(a) Prove the equality $d^{(\infty)}(2t) = \left[d^{(2)}(t)\right]^2$.

(b) Consider the lazy random walk on the complete graph with n vertices. Show that the separation distance satisfies $s(2) \leq \frac{1}{4}$, but $t_{mix}^{(\infty)}(1/4)$ is of the order of $\log n$. Is the answer different for the simple (non-lazy) random walk?