# Probability on graphs

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# Problem set 7

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**Problem 1.** Let $m_+ = m_+(n)$, $m_- = m_-(n)$ be such that $m_+ = o(\sqrt{n})$ and $\sqrt{n} = o(m_-)$. Let $\mu_n$ be the $\mathrm{Bin}(n, 1/2)$ distribution, $\nu_n$ the $\mathrm{Bin}(n - m_+, 1/2)$ distribution and $\eta_n$ – the $\mathrm{Bin}(n - m_-, 1/2)$ distribution. Prove that

$$d_{\mathrm{TV}}(\mu_n, \nu_n) \to 0$$

and

$$d_{\mathrm{TV}}(\mu_n, \eta_n) \to 1$$

as $n \to \infty$.

**Problem 2.** Let $G = (V, E)$ be a $d$-regular graph, $d \geq 3$. Consider the simple random walk on $G$ and assume that it is irreducible and aperiodic. Prove the following lower bound on the mixing time

$$t_{\mathrm{mix}}(\varepsilon) \geq \frac{\log\left(|V|(1-\varepsilon)/3\right)}{\log(d-1)}.$$

**Problem 3.** Consider the coupon collector problem with $n$ coupons – at each step we draw one coupon out of $n$ possible types, with each type of coupon equally likely. Let $\tau$ be the first time at which at least one coupon of each type has been collected.

(a) Prove that $\mathbb{E}\tau = n \sum\limits_{k=1}^{n} \frac{1}{k}$ and for any $c > 0$ we have

$$\mathbb{P}(\tau > \lceil n \log n + cn \rceil) \leq e^{-c}.$$

(b) Let $I_j(t)$ be the indicator of the event that the $j$-th type of coupon has not been collected by time $t$. Let $R(t) = \sum\limits_{j=1}^{n} I_j(t)$. Prove that the random variables $I_j(t)$ are negatively correlated and setting $p_t = \left(1 - \frac{1}{n}\right)^t$ we have

$$\mathbb{E}R_t = np_t,$$
$$\mathrm{Var}R_t \leq np_t(1 - p_t) \leq \frac{n}{4}.$$