Probability on graphs

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Problem set 4

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Problem 1. Let $X_1, \ldots, X_n$ be independent random variables with X_i having Bernoulli distribution with parameter p_i. Let $\lambda = \sum_{i=1}^{n} p_i$. Prove that if $X = X_1 + \ldots + X_n$ and $Y \sim \text{Poiss}(\lambda)$, then there exists a coupling (X, Y) such that

$$\mathbb{P}(X \neq Y) \leq \sum_{i=1}^{n} p_i^2.$$

Deduce that if $X \sim \text{Bin}(n, \frac{\lambda}{n})$ and $Y \sim \text{Poiss}(\lambda)$, then there exists a coupling (X, Y) for which $\mathbb{P}(X \neq Y) \leq \frac{\lambda^2}{n}$.

Problem 2. Prove that if T is the total progeny of a branching process with offspring distribution X satisfying $\mu = \mathbb{E}X < 1$, then

$$\mathbb{E}T = \frac{1}{1-\mu}.$$

Use this to deduce that for $\lambda < 1$ the mean size of the connected component of a given vertex in a $G(n, \frac{\lambda}{n})$ graph satisfies $\mathbb{E}|\mathcal{C}(v)| \leq \frac{1}{1-\lambda}$.

Problem 3. As in the lecture, let S_t be the number of active vertices in the exploration process in the $G(n, p)$ graph at time t.

(a) For any $k \leq n$ prove the following equality in distribution:

$$S_t + t - k \sim \text{Bin}(n - k, 1 - (1-p)^t).$$

(b) Show that for $l \geq t$ we have, conditionally on S_t,

$$S_l + (l - t) - S_t \sim \text{Bin}(n - t - S_t, 1 - (1-p)^{l-t}).$$

Problem 4. Given a graph G, an ε-*cut* is a partition of $V(G)$ into two disjoint sets A and B such that $|A| = \varepsilon n$, $B = (1-\varepsilon)n$ and there are no edges between A and B.

(a) Prove that if $|\mathcal{C}_{max}|$ is the size of the largest connected component of G, then there exists an ε-cut in G with $\varepsilon \geq 0$ satisfying

$$\left|\varepsilon - \frac{1}{2}\right| \leq \frac{1}{2}\frac{|\mathcal{C}_{max}|}{n}.$$

(b) Prove that if $\lambda > 4\log 2$, then with high probability the largest component in $G(n, \frac{\lambda}{n})$ has size at least $\delta(\lambda)n$ for some constant $\delta(\lambda) > 0$ depending on λ. *Hint:* look at the probability that there exists an ε-cut with ε close to $\frac{1}{2}$.