Probability on graphs
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Problem set 10

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Problem 1. Consider the lazy random walk on $\mathbb{Z}_n$ and let $f : \mathbb{Z}_n \to \mathbb{R}$ be the function $f(i) = \mathbb{1}_{\{n/4 \leq i \leq 3n/4\}}$.

(a) What bound on the relaxation time (and hence mixing time) do you obtain if you plug f into the variational characterization of the spectral gap in terms of the Dirichlet energy $\mathcal{E}(f, f)$?

(b) Provide an estimate of how large $\mathrm{Var}_\pi(P^t f)$ is, depending on t, and use part (a) of Problem 1 from the previous problem set to obtain a lower bound on the relaxation time. How does this compare to the bound from (a)?

Problem 2. Consider a lazy random walk on a finite binary tree of depth k. Let $n = 2^{k+1} - 1$ be the number of vertices of the tree. Prove that the mixing time of this chain satisfies $t_{mix} = \Theta(n)$.

Problem 3. Show that for a Markov chain on the state space S for any set $A \subseteq S$ we have $Q(A, A^c) = Q(A^c, A)$ (even if the chain is not reversible). Deduce that any chain supported on a tree is reversible.