Probability on graphs

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Problem set 1

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Problem 1. Let $G \sim G(n,p)$ and let $\mathcal{A}$ be the property that G contains at least one cycle. Find the threshold function $p^* = p^*(n)$ for $\mathcal{A}$.

Problem 2. Let $G \sim G(n, 1/2)$. Find the largest possible $k = k_n$ such that G contains a clique of size at least k_n a.a.s.

Problem 3. As in the lecture, for any graph H let

$$m(H) = \max\left\{\frac{e(H')}{v(H')} \,:\, H' \subseteq H, e(H') \geq 1\right\},$$

where $e(H')$ is the number of edges in H', $v(H')$ is the number of vertices in H' and the maximum is over all subgraphs H' of H with at least one edge.

Let $G \sim G(n,p)$. Prove that for a fixed graph H the threshold for the property "G contains at least one copy of H" is $p^*(n) = n^{-1/m(H)}$.

Problem 4. Consider the set $[n] = \{1, \ldots, n\}$. A *k-term arithmetic progression* is a subset of $[n]$ of the form $\{m, m+a, \ldots, m+(k-1)a\}$ for some $m, a \in [n]$. Let S_p be a randomly chosen subset of $[n]$ obtained by including each $i \in [n]$ independently with probability p. For fixed $k \geq 3$ find the threshold $p^* = p^*(n)$ for the property "S_p contains at least one k-term arithmetic progression".