# Probability on graphs

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# Problem set 3

Michał Kotowski

**Problem 1.** Suppose $G \sim G(n,p)$ where $p = \frac{c}{n}$ for some constant $c > 0$. Let $X_k$ denote the number of cycles of length $k$ in $G$. Find the limiting joint distribution of $(X_3, X_4)$ as $n \to \infty$.

**Problem 2.** Suppose $G \sim G(n,p)$ where $p = \frac{\log n + t}{n}$ for some constant $t \in \mathbb{R}$. Prove that the number of isolated vertices in $G$ converges in distribution to a Poisson random variable with parameter $e^{-t}$.

**Problem 3.** Let $G = (V, E)$ be a graph with a vertex set $V$ and edge set $E$. For any subset of vertices $S \subseteq V$ let $\partial S$ denote the set of edges in $E$ which have one endpoint in $S$ and the other one in $V \setminus S$. Let

$$\mathrm{ex}(S) = \frac{|\partial S|}{\min\{|S|, |V \setminus S|\}}.$$

Suppose that $G \sim G(n,p)$ for $p = \frac{100 \log n}{n}$. Prove that there exist constants $\alpha > 0$, $\beta < 1$ such that

$$\mathbb{P}\left(\min_{S \subseteq V} \mathrm{ex}(S) < \alpha\right) < \beta.$$

**Problem 4** (Bonus)**.** Draw a picture/comic/painting/graphic illustrating the existence of a threshold for a chosen graph property. The style of your work should somehow match the type of the threshold (coarse, as for triangle containment, or sharp, as for connectivity), but otherwise is up to you. In this problem the artistic value will be more important than mathematical content.