Probability on graphs

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Problem set 1

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Problem 1. Let $G \sim G(n,p)$ and let $\mathcal{A}$ be the property that G contains at least one cycle. Find the threshold function $p^* = p^*(n)$ for $\mathcal{A}$.

Problem 2. Let $\mathcal{A}$ be an increasing graph property. Let $N = \binom{n}{2}$ and consider p and m such that $p = \frac{m}{N}$. Assume furthermore that $\sqrt{pN} \to \infty$ and $\sqrt{pN}\frac{1-p}{p} \to \infty$ as $n \to \infty$. Prove that for n large enough we have

$$\mathbb{P}\left(G(n,m) \in \mathcal{A}\right) \leq 3\mathbb{P}\left(G(n,p) \in \mathcal{A}\right).$$

Problem 3. Let $f, g : \{0,1\}^n \to \mathbb{R}$ be increasing functions (i.e, changing any input bit from 0 to 1 does not decrease the value of the function). Let $\mathbb{E}$ denote the expectation with respect to a product measure on $\{0,1\}^n$. Prove that

$$\mathbb{E}\left(fg\right) \geq \left(\mathbb{E}f\right)\left(\mathbb{E}g\right).$$

Hint: induction on n and taking conditional expectations.

Problem 4. Let $G \sim G(n,p)$ and let $0 < p < 1$ be a constant independent of n. Prove that a.a.s. the graph G has diameter 2 (i.e., the maximal graph distance between any two vertices is 2). Can you prove the same result for some $p = o(1)$?