On non-stability of one-dimensional non-periodic ground states

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Abstract

We address the problem of stability of one-dimensional non-periodic ground-state configurations in classical lattice-gas models with respect to finite-range perturbations of interactions. We show that a relevant property of ground-state configurations in this context is their homogeneity. The so-called strict boundary condition says that the number of finite patterns of a configuration has bounded fluctuations on any finite subset of the lattice $\mathbf{Z}$. We show that if the strict boundary condition is not satisfied and interactions between particles decay faster than $1/r^{\alpha}$ with $\alpha > 2$, then ground-state configurations are not stable. In the Thue-Morse ground state, number of finite patterns may fluctuate as much as the logarithm of the lenght of a lattice subset. We show that the Thue-Morse ground state is unstable for any $\alpha > 1$ with respect to arbitrarily small four-body interactions favoring the presence of molecules consisting of two up or down spins. We also investigate Sturmian systems defined by irrational rotations on the circle. They satisfy the strict boundary condition but nevertheless they are unstable for $\alpha > 3$.

1 Introduction

Since the discovery of quasicrystals [1], one of the problems in statistical mechanics is to construct microscopic models of interacting atoms or molecules in which all configurations minimizing energy, the so-called ground-state configurations, are non-periodic. Here we will discuss models in which although all ground-state configurations are non-periodic, they all look the same, they cannot be distinguished locally. More precisely, they support the unique translation-invariant measure called the non-periodic ground state of the system.

There were constructed many classical lattice-gas models without periodic ground-state configurations [2, 3, 4, 5]. Some of these models are based on non-periodic tilings of the plane with Wang square-like tiles [6], in particular Robinson's tilings [7]. In such tilings, tiles can cover the plane but only in a non-periodic way. Let us point out that centers of tiles form a regular two-dimensional lattice Z^2 but assignments of tiles to vertices are non-periodic. Now, types of tiles can be identified with types of particles, then interactions between particles correspond to matching rules. Namely, the interaction energy between two particles which match as tiles is 0, if they don't match, the energy is positive, say 1. It is easy to see that ground-state configurations correspond to tilings, therefore there are no periodic ones. Interactions in such models are obviously non-frustrated - all interactions attain their minima (equal to 0) in ground-state configurations.

A desired property of non-periodic ground-state configurations is their stability against small perturbations of interactions between particles. For two-dimensional systems with finite-range non-frustrated interactions and with unique non-periodic ground-state measure, the relevant property is the so-called strict boundary condition - the requirement that the number of finite patterns can fluctuate at most proportional to the boundary of a lattice subset (a precise definition is given in Section 2) [5]. It was shown in [5] that the strict boundary condition is equivalent to the stability of ground-state configurations. More precisely, non-periodic ground states are stable against small perturbations of the range d if and only if the strict boundary condition is satisfied for all finite patterns of sizes smaller or equal to d in all local tilings. So far we do not have an example of a two-dimensional finite-range classical lattice-gas model without periodic ground-state configurations which satisfies the strict boundary condition. For example, it was proved that the non-periodic ground state based on Robinson's tilings does not satisfy the strict boundary condition and therefore is unstable with respect to arbitrarily small chemical potentials [8].

One of the goals of our paper is to show that the strict boundary condition is relevant also for one-dimensional models. Situation is quite different here. It is known that one-dimensional systems without periodic ground-state configurations require infinite-range interactions [9, 10, 11]. It follows that one-dimensional non-periodic ground states cannot be stable with respect to small perturbations in any reasonable space of infinite-range interactions. In the space l_1 of summable interactions we can cut the tail of an arbitrary small l_1-norm and obtain a finite-range Hamiltonian which have at least one periodic ground-state configuration.

One-dimensional two-body interactions producing only non-periodic ground-state configurations were presented in [12, 13, 14, 15, 16]. Hamiltonians in these papers consist of two-body repelling interactions between particles and a chemical potential favoring them. Such interactions are obviously frustrated. Ground states of these models form Cantor sets called devil's staircases. Non-periodicity is present only for certain values (of measure zero) of chemical potentials - an arbitrary small change of a chemical potential destroys a non-periodic ground state.

In [17], a non-frustrated infinite-range, exponentially decaying four-body Hamiltonian was constructed, with the unique ground-state measure supported by Thue-Morse sequences [18]. Recently in [19] there were constructed non-frustrated two-body (augmented by some finite-range interactions) Hamiltonians producing exactly the same ground states as in the frustrated model of [12, 13, 14].

Here we investigate the stability of non-periodic ground-state configurations with respect to finite-range perturbations. We show that the strict boundary condition plays here an important role. Our general result is that if the strict boundary condition is not satisfied and interactions between particles decay faster than $1/r^{\alpha}$ with $\alpha > 2$, then ground-state configurations are not stable, see Theorem 2.2.

In the Thue-Morse ground state, the number of finite patterns may fluctuate as much as the logarithm of the lenght of a lattice subset. We show that such a ground state is unstable with respect to arbitrarily small two-body interactions favoring the presence of molecules consisting of two up or down spins, see Theorem 4.1.

We also investigate Sturmian systems defined by irrational rotations on the circle, which satisfy the strict boundary condition [19]. Hamiltonians having Sturm sequences as ground-state configurations we recently constructed in [19]. We show that if $\alpha > 3$, then Sturmian systems are not stable, see Theorem 5.6.

2 Strict boundary condition and non-stability of non-periodic ground states

A frequency of a finite pattern in an infinite configuration is defined as the limit of the number of occurrences of this pattern in a segment of lenght L divided by L as $L \to \infty$. All sequences in any given Sturmian system have the same frequency for each pattern. We are interested now whether the fluctuations of the numbers of occurrences are bounded. If that is the case, configurations are said to satisfy the **strict boundary condition** [5] or rapid convergence of frequencies to their equilibrium values [20, 21].

Definition 2.1. *Given a sequence $X = (x_n) \in \{0,1\}^{\mathbb{Z}}$ and a finite word w, define the frequency of w as*

$$\xi_w = \lim_{N\to\infty} \frac{\#\{|n| \leq N \mid x_n \ldots x_{n+|w|-1} = w\}}{2N+1}.$$

Furthermore, for a segment $A \subset \mathbb{Z}$, denote by $X(A)$ the sub-word $(x_n)_{n\in A}$. We say that a sequence X satisfies the **strict boundary condition** *(quick convergence of frequencies) if for any word w and a segment $A \subset \mathbb{Z}$, the number of appearances of w in $X(A)$, $n_w(X(A))$, satisfies the following inequality:*

$$|n_w(X(A)) - \xi_w|A|| < C_w,$$

where $C_w > 0$ is a constant which depends only on the word w.

We consider classical lattice-gas models with unique ground-state measures supported by non-periodic ground-state configurations. The configuration space of a system is denoted by $\Omega = \{0,1\}^{\mathbb{Z}}$, where 0 means the absence of a particle at a given lattice site, and 1 its presence. Let $f(r) = 1/r^\alpha$ be the energy of one-dimensional interaction between particles at a distance r.

Theorem 2.2. *If non-periodic ground-state configurations do not satisfy the strict boundary condition and the interaction energy decays as $1/r^{\alpha}$ with $\alpha > 2$, then they are unstable with respect to an arbitrary small chemical potential - a one-body on-site interaction.*

Proof. Let $X \in \Omega$ be a ground-state configuration. That is for any Y which differs from X on a finite number of sites, the relative Hamiltonian is not smaller than 0, $H(Y|X) \geq 0$. Let us assume that the system does not satisfy the strict boundary condition. It means that for any $C > 0$, there are two segments of consecutive lattice sites, $S_1, S_2 \subset \mathbb{Z}$ of the lenght L, such that $n(X(\Lambda_2)) - n(X(\Lambda_1)) > C$, where $n(X(\Lambda_i))$ is the number of 1's in X on Λ_i, $i = 1, 2$. We construct a finite excitation Y of X, that is a configuration Y which differs from X only on a finite number of lattice sites. Namely, let $Y = X$ outside S_1 and Y on Λ_1 is equal to X on Λ_2. Let us now introduce an on-site interaction - a chemical potential which favors the presence of particles - it assigns to each particle a negative energy $-\mu$ for some small $\mu > 0$. We will show that for $\alpha > 2$, $H(Y|X) < 0$ that is by a finite change of X one can decrease the energy hence X is not a ground-state configuration for a perturbed Hamiltonian.

Obviously, the chemical potential decreases the energy by $C\mu$. Now we have to bound appropriately the possible increase of the energy associated with a two-body original interactions. The increase of the energy can be divided into two parts: E_1 associated with interactions between particles at a distance smaller than L and E_2 associated with interactions between particles at a distance equal or bigger than L. Now we have,

$$E_1 \leq 2\sum_{r=1}^{L} \frac{r}{r^{\alpha}} < \int_{x=1}^{L} \frac{2}{x^{(\alpha-1)}} dx + 2 = \frac{2}{(2-\alpha)} L^{2-\alpha} + 2 - \frac{2}{2-\alpha}, \tag{2.1}$$

$$E_2 \leq 2L\sum_{r=L}^{\infty} \frac{1}{r^{\alpha}} < 2L(\int_{L}^{\infty} \frac{1}{x^{\alpha}} dx + \frac{1}{L^{\alpha}}) = \frac{2}{\alpha - 1} L^{2-\alpha} + 2L^{1-\alpha}. \tag{2.2}$$

It follows that for $\alpha > 2$, $E_1 + E_2 < C\mu$. Therefore for any arbitrarily small μ there exists sufficiently large L such that $H(Y|X) < 0$ hence X is not a ground-state configuration for the perturbed interaction. $\square$

3 Toeplitz period-doubling ground state

We construct Toeplitz configurations [22] (also known as period-doubling configurations) in the following way. We place -1 (a symbol corresponding to the absence of a particle) on a sublattice $L_1 \subset \mathbb{Z}$ of period 2. Then we place 1 (a symbol corresponding to a particle) on a sublattice $L_2 \subset \mathbb{Z}$ of period 4 which is disjoint from L_1. We repeat this procedure ad infinitum and get a Toeplitz configuration $X \in \Omega = \{-1,1\}^{\mathbb{Z}}$ such that $X(i) = (-1)^j$ if $i \in L_j$. X is obviously non-periodic, there are particles on every sublattice L_j for even $j's$.

Let T be the translation operator, i.e., $T : \Omega \to \Omega, (T(X))(i) = X(i-1), X \in \Omega$. It is easy to see that the closure of the orbit of X by T supports the unique translation-invariant measure, we denote it by ρ_{To}. In this way we have constructed a uniquely ergodic dynamical system (Ω, ρ_{To}, T). The density of particles (1's) in X is equal to 1/3.

Now we will show that X does not satisfy the strict boundary condition. Let us look at particles on sublattices $L_j, j \leq m$. One can find a segment $W \subset \mathbb{Z}$ such that $i \in L_2$ is the first site of W, $X(i) = 1$, $i+2 \in L_4$ so $X(i+2) = 1$, $i+2+8 \in L_6$ so $X(i+2+8) = 1$, ... , $i + 2 + 8 + ... + 2 \times 4^{(m/2)-1} \in L_{(m/2)-1}$ so $X(i + 2 + 8 + ... + 2 \times 4^{(m/2)-1}) = 1$. Let the length of W be equal to $2(2 + 8 + ... + 2 \times 4^{(m/2)-1}) = \frac{4}{3}(4^{m/2} - 1)$.

Hence the number of particles in X on sublattices $L_j, j \leq m$ is equal to $n_1+...+n_{m/2}$, where $n_{m/2} = 2$ and $n_{k-1} = 4n_k - 2, k = m/2, m/2-1, m/2-2, ..., 1$.

Let $V = T_a(W)$, where T_a is a shift operator to the right by $a = 4\sum_{k=0}^{(m/2)-2} 4^k = \frac{4}{3}4^{(m/2)-1}$. We see that the number of particles in $X(V)$ is smaller by $m/2$ with respect to $X(W)$, one particle for each $L_j, j \leq 2k, k = 1, ..., m/2$.

Obviously, the strict boundary condition is not satisfied. Moreover, we can apply the above procedure for any $m \leq |V|$ so the fluctuations of the number of particles on the $V \subset Z$ can be of the order $\log_4 |V|$ where $|V|$ is the length of V.

Let us assume that the Toeplitz measure ρ_{To} is the unique ground state of some two-body interactions decaying as $1/r^{\alpha}$, where r is the distance between particles.

Theorem 3.1. *The Toeplitz ground state ρ_{To} is unstable against an arbitrarily small chemical potential which favors the presence of particles.*

Proof. To prove the instability of the ground state ρ_{To} we introduce a chemical potential μ favoring the presence of particles and a local perturbation Y of X such that $H(Y|X) < 0$. Y is constructed as follows: we take $X(W)$, described above, and place it on V, a certain shift of W, such that sublattices $L_j, j \leq n$ (for some even n to be chosen later) agree in Y. In this way we introduced an extra 1 on all sublattices $L_j, n < j < \log_4 |V|$ with even $j's$.

Now we will construct an upper bound for $H(Y|X)$. Denote by H_{4^n} the energy of interactions of the first particle on the left side of L_{n+2} with the particles on the part of the complement of V left to the particle. We get

$$H_{4^n} < \sum_{k=4^n}^{\infty} \frac{1}{k^\alpha} < 2 \int_{4^n}^{\infty} \frac{1}{x^\alpha} dx = \frac{2}{(\alpha - 1)4^{n(\alpha-1)}}. \tag{3.1}$$

It follows that

$$H(Y|X) < 2 \sum_{k=1}^{|V|/4^n} H_{k4^n} < 2 \sum_{k=1}^{|V|/4^n} \frac{1}{(k4^n)^{(\alpha-1)}} < \frac{4|V|^{2-\alpha}}{(\alpha-1)(2-\alpha)4^n} \tag{3.2}$$

for $1 < \alpha < 2$.

Let n be a minimal even number such that

$$\frac{4|V|^{2-\alpha}}{(\alpha-1)(2-\alpha)4^n} < 1 \tag{3.3}$$

so $n < (2-\alpha)\log_4 |V| - \log_4(\alpha-1)(2-\alpha) + 3$.

It follows that the two-body interaction energy is bounded (independent on $|V|$) and the number of excessive $1's$ is bigger than

$$\log_4 |V| - (2-\alpha)\log_4 |V| + \log_4(\alpha-1)(2-\alpha) - 3 > \frac{\epsilon}{2} \log_4 |V| \tag{3.4}$$

for any $\alpha = 1 + \epsilon$ and a sufficiently big ϵ dependent on V.

It shows that $H(Y|X) < 0$ so X is not a ground-state configuration for any arbitrarily small μ.

□

4 Thue-Morse ground state

We prove here that the Thue-Morse ground state is unstable with respect to arbitrarily small four-body interactions.

We begin by constructing a one-sided Thue-Morse sequence. We put 1 at the origin and perform successively the substitution S: $1 \rightarrow 10, 0 \rightarrow 01$. In this way we get a one-sided sequence 1001011001101001..., $\{X_{TM}(i)\}, i \geq 0$. We define $X_{TM} \in \Omega = \{0,1\}^{\mathbb{Z}}$ by setting $X_{TM}(i) = X_{TM}(-i-1)$ for $i < 0$. Let G_{TM} be the closure (in the product topology of the discrete topology on $0,1)$ of the orbit of X_{TM} by the translation operator T, i.e., $G_{TM} = \{T^n(X_{TM}), n \geq 0\}^{cl}$. It can be shown [18] that G_{TM} supports exactly one translation-invariant probability measure μ_{TM} on Ω.

Let us identify now 1 with $+1$ and 0 with -1, so particles are represented by spins up and the empty spaces by spins down. It was shown in [17] that ρ_{TM} is the only ground state of the following exponentially decaying four-spin interactions,

$$H_{TM} = \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} H_{r,p}, \tag{4.1}$$

where

$$H_{r,p} = \sum_{i \in \mathbb{Z}} J(r,p)(\sigma_i + \sigma_{i+2^r})^2 (\sigma_{i+(2p+1)2^r} + \sigma_{i+(2p+2)2^r})^2 \tag{4.2}$$

and $\sigma_i(X) = X(i) \in \{+1, -1\}$.

Theorem 4.1. *The Thue-Morse ground state ρ_{TM} is unstable against an arbitrarily small chemical potential which favors the presence of molecules consisting of two up or down spins.*

Proof. Let $X \in \{+1,-1\}^{\mathbb{Z}}$ be any Thue-Morse sequence and let $Y(i) = X(i)X(i+1)$. It is easy to see that Y is a Toeplitz sequence. Let us now consider 11 and 00 as A and 10 and 01 as B type molecules, respectively. A and B molecules form a Toeplitz sequence and therefore their numbers on the segment V may fluctuate on the order of $\log_4 |V|$. Now we introduce a chemical potential h favoring A-type molecules. It follows (in the same way as in the Toeplitz case) that if we take into account interactions $J(0,p)$ such that $J(0,p)$ decays as $1/p^{\alpha}$, then the Toeplitz ground state is unstable with respect to any arbitrarily small μ for any $\alpha > 1$.

Now we have to take care of $J(r,p)$ for all $r \geq 1$. We assume that $J(r,p)$ decays as $1/[(2p+2)2^{r}]^{\alpha}$. We will use the fact that Thue-Morse sequences are self-similar. Namely, we can group two successive symbols, 10 and 01, and replace them by 1 and 0 respectively and in this way we get again a Thue-Morse sequence. One can do analogous groupings on every scale, for example $1001 \rightarrow 1$, $0110 \rightarrow 0$. We will also use the structure of interactions. First we notice that for any sequence of successive blocks of 10 and 01, no two pairs of either 11 or 00 are at an odd distance, therefore interactions in the Hamiltonian for $r = 0$ and any p attains the zero value. It follows from the self-similarity of Thue-Morse sequences that for any sequence of successive blocks of 1001 and 0110, no two pairs of either two 1's or two 0's at a distance 2 are at a distance $2(2p+1)$ for any $p > 0$ so the Hamiltonian for $r = 1$ and any p attains the zero value. In general, for any sequence of successive blocks of $S^{r}(1)$ and $S^{r}(0)$, the Hamiltonian associated with any $r \geq 0$ and any p attains the zero value, $H_{r,p}(Y|X) = 0$.

Now, to prove the instability of the Thue-Morse ground state we mimic the procedure used in the Toeplitz case. Namely, let X be a Thue-Morse sequence. We construct Y, a local perturbation of X, in the following way. Let $V \subset \mathbb{Z}$ be a segment of $\mathbb{Z}$, then $Y = X$ on the complement of V and we put on V an appropriate block of X such that Y consists of successive blocks of $S^{r}(1)$ and $S^{r}(0)$ for any $r < r^{*}$ (to be chosen later) so $H_{r,p}(Y|X) = 0$ for any p and $r < r^{*}$ and the number of excessive A-molecules is bigger than $\log_{4}|V| - r^{*}$.

Now fix $r \geq r^{*}$ and consider interactions $J(r,p), p > 0$. Similar calculations as in the Toeplitz case show that for

$$H_{r} = \sum_{p=0}^{\infty} H_{r,p} \tag{4.3}$$

we have

$$H_{r}(Y|X) < \frac{|V|^{2-\alpha}}{(\alpha-1)(2-\alpha)2^{r}} \tag{4.4}$$

for $1 < \alpha < 2$ and therefore

$$H(Y|X) = \sum_{r \geq r^*} H_r(Y|X) < \frac{|V|^{2-\alpha}}{(\alpha-1)(2-\alpha)2^{r^*-1}}. \tag{4.5}$$

We choose a minimal r^* such that $\frac{|V|^{2-\alpha}}{(\alpha-1)(2-\alpha)2^{r^*-1}} < 1$.

Hence we get that $H(Y|X) < 0$ which shows the instability of the Thue-Morse ground state. □

5 Sturmian ground states

We will identify the circle C with $R/\mathbb{Z}$ and consider an irrational rotation by φ (which is given by translation on $R/\mathbb{Z}$ by $\varphi \mod 1$).

Definition 5.1. *Given an irrational $\varphi \in C$ we say that $X \in \{0,1\}^{\mathbb{Z}}$ is generated by φ if it is of the following form:*

$$X(n) = \begin{cases} 0 & \textit{when } x + n\varphi \in P \\ 1 & \textit{otherwise} \end{cases}$$

where $x \in C$ and $P = [0, \varphi)$.

We call such X a Sturmian sequence corresponding to φ and x. Let G_{St} be the closure of the orbit of X by translations. It can be shown that G_{St} supports exactly one translation-invariant probability measure ρ_{St} on Ω.

From now on we will consider only rotations by badly approximable numbers.

Definition 5.2. *We say that a number φ is* ***badly approximable*** *if there exists $c > 0$ such that*

$$\left|\varphi - \frac{p}{q}\right| > \frac{c}{q^2}$$

for all rationals $\frac{p}{q}$.

Sturmian sequences can be characterized by absence of certain finite patterns.

Theorem 5.3. [19, Theorem 4.1] *Let $\varphi \in (\frac{1}{2}, 1)$ be irrational. Then there exist a natural number m and a set $F \subseteq \mathbb{N}$ of forbidden distances such that Sturmian sequences generated by φ are uniquely determined by the absence of the following patterns: m consecutive 0's and two 1's separated by a distance from F.*

To characterize Sturmian sequences generated by irrationals from $(0, \frac{1}{2})$ we have to change the roles of 0's and 1's. Let us note that F from Theorem 5.3 can also be described by rotations. Namely

$$k \notin F \iff \exists y \in [\varphi, 1)\ y + k\varphi \in [\varphi, 1).$$

The following two characterizations of F are equivalent.

Proposition 5.4.

$$k \notin F \iff \exists y \in [\varphi, 1)\ y + k\varphi \in [\varphi, 1),$$

$$k \in F \iff k\varphi \in [1 - \varphi, \varphi].$$

Proof. Let $z \in C$, then

$$\neg(\exists y \in [\varphi, 1)\, y+z \in [\varphi, 1)) \iff \forall y \in [\varphi, 1)\, y+z \in [0, \varphi) \iff \{y+z : y \in [\varphi, 1)\} \subseteq [0, \varphi).$$

The arc $\{y + z : y \in [\varphi, 1)\}$ is contained in $[0, \varphi)$ if and only if its endpoints are in $[0, \varphi]$ which means that $z \in [1 - \varphi, \varphi]$. We set $z = k\varphi$ to finish the proof. □

Given a set of forbidden distances we may easily construct non-frustrated Hamiltonians for which the unique ground-state consists exactly of Sturmian sequences generated by φ. We simply need to assign positive energies to all forbidden patterns and zero otherwise (for more details see [19, Theorem 5.2]).

Sturmian sequences have the following property.

Lemma 5.5. *Let $\varphi \in (0, 1)$ be irrational. Then for each $m \in \mathbb{N}$ there is $k \geq m$ and a finite word $w \in \{0, 1\}^{\{0,1,\ldots,k-1\}}$ of lenght k such that*

- *w is a subword of a Sturmian sequence generated by φ,*
- *doubling of w, $\widetilde{w} \in \{0, 1\}^{\{0,1,\ldots,2k-1\}}$ given by*

$$\widetilde{w}(i) = \widetilde{w}(i + k) = w(i) \text{ for } i < k$$

  also is a subword of a Sturmian sequence generated by φ.

Proof. Fix $n \in \mathbb{N}$. Let $\varepsilon > 0$ be such that $i\varphi \notin [1-\varepsilon, 1)$ for $i = -1, 0, 1, 2, \ldots, n-1$. Let k be the smallest natural number such that $k\varphi \in [1-\varepsilon, 1)$ and X be a Sturmian sequence generated by φ with the initial point ε. We define $\widetilde{w}(i) = \widetilde{w}(i+k) = X(i)$ (and $w(i) = X(i)$). We need to show that $X(i+k) = X(i)$ for $0 \leq i \leq k-1$.

By definition, if $X(i) = 0$ then $i\varphi + \varepsilon \in [0, \varphi)$ and by assumption $i\varphi \notin [1-\varepsilon, 1)$ so $i\varphi + \varepsilon \in [\varepsilon, \varphi)$. Hence $i\varphi + k\varphi + \varepsilon \in [0, \varphi)$ which means that $X(i+k) = 0$. If $X(i) = 1$ then $i\varphi + \varepsilon \in [\varphi, 1)$, and since $(i-1)\varphi \notin [1-\varepsilon, 1]$ we get that $i\varphi + \varepsilon \in [\varphi + \varepsilon, 1)$. This gives $i\varphi + k\varphi + \varepsilon \in [\varphi, 1)$ which means that $X(i+k) = 1$. □

Theorem 5.6. *Assume that $\varphi \in (0,1)$ is badly approximable and the interaction energy decays as $1/r^\alpha$ with $\alpha > 3$. Then Sturmian ground-state configurations generated by φ are unstable.*

Proof. Let X be a Sturmian sequence generated by φ. Let us now introduce a small chemical potential μ which favors the presence of particles (occurrence of 1's). For every $m \in \mathbb{N}$ pick $k_m \geq m$ and a word w_m of length k_m whose doubling is a Sturmian subword as in Lemma 5.5. Let Y_m be the periodic sequence with the period k_m given by w_m. For each i such that $Y_m(i) = 1$ the energy from pairs of 1's containing $Y_m(i)$ is not greater than

$$2\sum_{i=k_m+1}^{\infty} \frac{1}{i^\alpha} \leq 2\int_{k_m}^{\infty} \frac{dx}{x^\alpha} = \frac{2}{\alpha-1} k_m^{-\alpha+1}$$

since there are no forbidden pairs of 1's in Y_m at a distance less than k_m. Hence the difference of energies coming from pairs of 1's between Y_m and X involving particles in the interval $[-l, l]$ may be estimated as follows.

$$E_1(Y_m([-l,l])) - E_1(X([-l,l])) = E_1(Y_m([-l,l])) \leq (2l+1)\frac{2}{\alpha-1} k_m^{-\alpha+1}. \tag{5.1}$$

Denote by $\xi(X)$ and $\xi(Y_m)$ the frequency of 1's in X and Y_m respectively. We have $\xi(X) = 1-\varphi$ and $\xi(Y_m) = \frac{n(w_m)}{k_m}$ (where $n(w_m)$ is the number of 1's in w_m). Since φ is badly approximable, there exists $c > 0$ independent of m such that

$$|\xi(Y_m) - \xi(X)| = \left| \frac{n(w_m) - k_m}{k_m} + \varphi \right| > \frac{c}{k_m^2}.$$

This shows that

$$|E_2(Y_m([-l,l])) - E_2(X([-l,l]))| \geq (2l+1)\frac{|\mu| c}{2k_m^2},$$

where E_2 is the energy of 1's in the interval $[-l, l]$ due to the chemical potential μ.

We replace μ by $-\mu$ if necessary and get

$$E_2(Y_m([-l,l])) - E_2(X([-l,l])) \leq -(2l+1)\frac{|\mu|c}{2k_m^2}. \tag{5.2}$$

We combine (5.1) and (5.2) and get

$$E(Y_m([-l,l])) - E(X([-l,l])) \leq (2l+1)\frac{2}{\alpha-1}k_m^{-\alpha+1} - (2l+1)\frac{|\mu|c}{2}k_m^{-2}.$$

We divide the above inequality by $2l+1$, take the limit $l \to \infty$ and get

$$\rho(Y_m) - \rho(X) \leq \frac{2}{\alpha-1}k_m^{-\alpha+1} - \frac{|\mu|c}{2}k_m^{-2},$$

where $\rho(Y_m) and \rho(X)$ denote energy densities of Y_m and X respectively.

Since $\alpha > 3$, $\frac{2}{\alpha-1}k_m^{-\alpha+1}$ tends to 0 faster than $\frac{|\mu|c}{2}k_m^{-2}$ when $m \to \infty$, for large enough m we have

$$\rho(Y_m) < \rho(X)$$

which completes the proof. □

6 Discussion

We studied stability of one-dimensional non-periodic ground-state configurations with respect to finite-range perturbations of interactions decaying as $1/r^\alpha$ in classical lattice-gas models with interactions. We showed that the Thue-Morse ground state is unstable for any $\alpha > 1$ and the Sturmian ground states are unstable for $\alpha > 3$.

It is an important problem to construct one-dimensional lattice-gas model with the unique non-periodic ground state which is stable with respect to finite-range perturbations of interactions. We conjecture that Sturmian ground states generated by rotations on the circle by badly approximable irrationals are stable for some small values of $\alpha > 1$.

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References

[1] D. Shechtman, I. Blech, D. Gratias, and J.W. Cahn, Metallic phase with long-range orientational order and no translational symmetry, Phys. Rev. Lett. 53: 1951 (1984).

[2] C. Radin, Crystals and quasicrystals: A lattice gas model. Phys. Lett. 14A: 381–383 (1986).

[3] J. Miękisz, Many phases in systems without periodic ground states, Commun. Math. Phys. 107: 577–586 (1986).

[4] J. Miękisz, Stable quasicrystalline ground states, J. Stat. Phys. 88: 691–711 (1997).

[5] J. Miękisz, Classical lattice-gas models of quasicrystals, J. Stat. Phys. 97: 835–850 (1999).

[6] H. Wang, Proving theorems by pattern recognition — II. Bell System Tech. J. 40, 1–41 (1961).

[7] R. M. Robinson, Undecidability and nonperiodicity for tilings of the plane, Invent. Math. 12: 177–209 (1971).

[8] J. Miękisz and C. Radin, The unstable chemical structure of quasicrystalline alloys, Phys. Lett. 119A: 133-134 (1986).

[9] M. Bundaru, N. Angelescu, and G. Nenciu, On the ground state of Ising chains with finite range interactions, Phys. Lett. 43A: 5–6 (1973).

[10] C. Radin and L. Schulman, Periodicity of classical ground states, Phys. Rev. Lett. 51: 621–622 (1983).

[11] J. Miękisz and C. Radin, The Third Law of thermodynamics, Mod. Phys. Lett. B1: 61-65 (1987).

[12] P. Bak and R. Bruinsma, One-dimensional Ising model and the complete Devil's staircase, Phys. Rev. Lett. 49: 249–151 (1982).

[13] S. Aubry, Complete devil's staircase in the one-dimensional lattice gas, J. Phys. Lettres. 44: L247–L250 (1983).

[14] S. Aubry, Exact models with a complete Devil's staircase, J. Phys C: Solid State Phys. 16: 2497–2508 (1983).

[15] J. Jędrzejewski and J. Miękisz, Devil's staircase for non-convex interactions, Europhysics Letters 50: 307–311 (2000).

[16] J. Jędrzejewski and J. Miękisz, Ground states of lattice gases with "almost" convex repulsive interactions, J. Stat. Phys. 98: 589–620 (2000).

[17] C. Gardner, J. Miękisz, C. Radin, and A. van Enter, Fractal symmetry in an Ising model, J. Phys. A.: Math. Gen. 22: L1019–L1023 (1989).

[18] M. Keane, Generalized Morse sequences, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 10: 335–353 (1968).

[19] A. van Enter, H. Koivusalo, and J. Miękisz, Sturmian ground states in classical lattice–gas models, J. Stat. Phys. 178: 832–844 (2020).

[20] J. Peyriere, Frequency of patterns in certain graphs and in Penrose tilings, J. Phys. Colloq. 47: C 41–62 (1986).

[21] J. Aliste-Prieto, D. Coronel, and J.-M. Gambaudo, Rapid convergence to frequency for substitution tilings of the plane, Commun. Math. Phys. 306: 365–380 (2011).

[22] K. Jacob and M. Keane, 0-1 sequences of Toeplitz type, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 13: 123–131 (1969).

[23] J-P. Allouche and J. Shallit, Automatic Sequences: Theory, Applications, Generalizations, Cambridge University Press (2003).