On graphs coverable with k shortest paths

Maël Dumas

Joint work with: Florent Foucaud[2], Anthony Perez[1], Ioan Todinca[1]

[1]LIFO, Université d'Orléans, France
[2]LIMOS, Université Clermont Auvergne, France

Graphs coverable by k shortest paths

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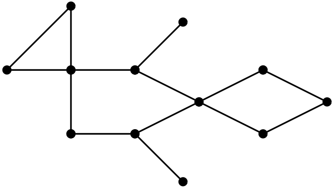

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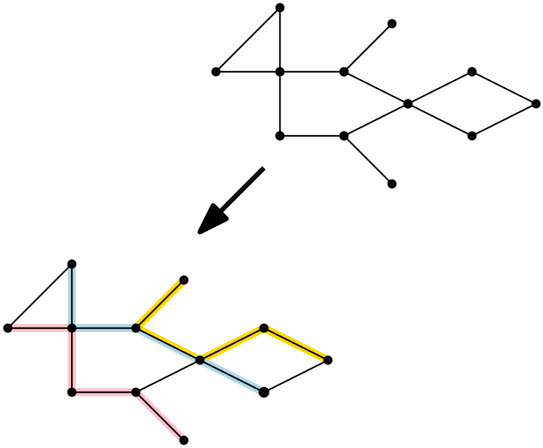
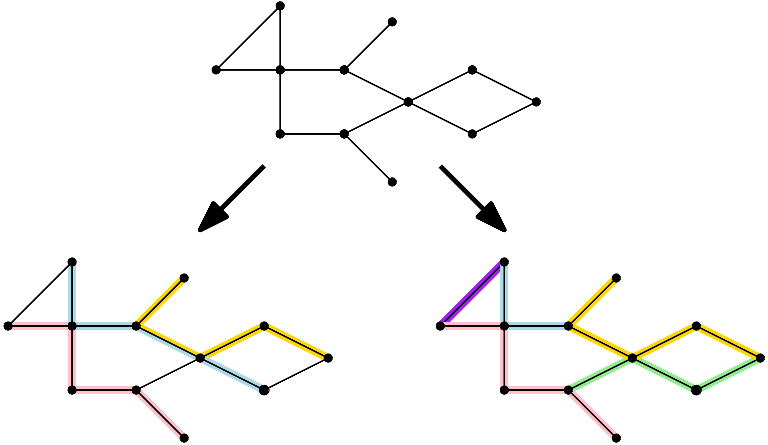


vertex-coverable

Graphs coverable by k shortest paths



vertex-coverable edge-coverable

Combinatorial results

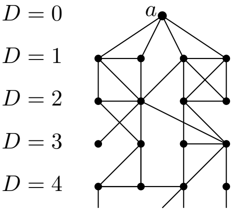
Theorem 1

If G is **coverable by** k **shortest paths** then, for any vertex a and fixed distance D, the number of vertices at distance exactly D from a is upper bounded by some function $g(k)$.

- Edge-coverable : $g(k) = O(3^k)$.
- Vertex-coverable : $g(k) = O(k \cdot 3^k)$.

Corollary 1

G is of **pathwidth** at most $2 \cdot g(k) - 1$.



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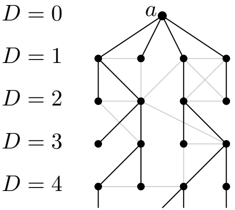

Path decomposition :

- Do a a breadth-first search (BFS) from a vertex a.

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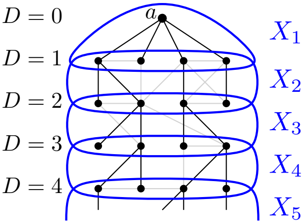
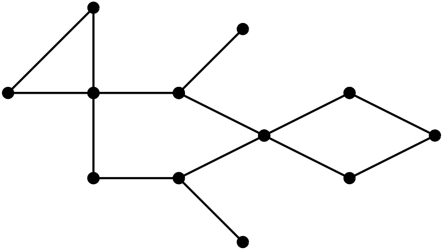


Path decomposition :

- Do a a breadth-first search (BFS) from a vertex a.
- Each bag : two consecutive layers.

Algorithmic Consequences

Problems



Isometric Path Cover (IPC)

Input : A graph G and an integer k.
Question : Does there exists a set of k shortest paths of G, such that each vertex of G belongs to at least one of the shortest paths ?

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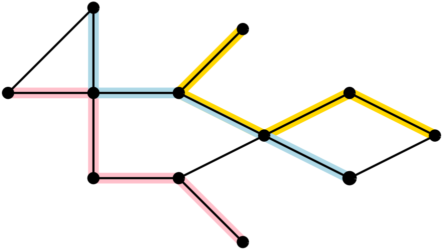

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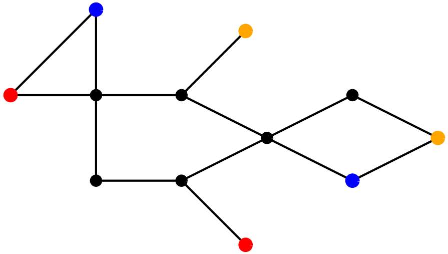

Isometric Path Cover with Terminals (IPC-WT)

Input :A graph G, and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$, the **terminals**.
Question : Does there exists a set of k shortest paths of G, the ith path being an s_i-t_i shortest path, such that each vertex of G belongs to at least one of the shortest paths ?

Context

Isometric number [Fisher and Fitzpatrick 2001]

Problem introduced in the context of cops and robber game :

$$\text{cop number of } G \leq \text{Isometric number of } G$$

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Complexity

- IPC is NP-Complete even on split graphs
  [Chakraborty, Dailly, Das, Foucaud, Gahlawat, and Ghosh, 2022]
  [Ordyniak, Chakraborty, Müller, Panolan and Rychlicki, 2024]
- IPC is polynomial in :
  - chain graphs, cographs [OCMPR'24]
  - block graphs [Pan and Chang, 2005]
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Question

Are problems IPC and IPC-WT **FPT** ? Or at least **XP** ?

FPT : running time $f(k) \cdot n^{O(1)}$

XP : running time $O(n^{h(k)})$

Isometric Path Cover with Terminals is FPT

Theorem [Courcelle. 1990]

Every problem expressible in **monadic second order logic** (MSO_2) can be solved in $f(w) \cdot n$ time on graphs of treewidth at most w.

Isometric Path Cover with Terminals is FPT

Theorem [Arnborg, Lagergren, Seese. 1991]

Every problem expressible as an $\mathbf{EMSO_2}$ problem can be solved in $f(w) \cdot n$ time on graphs of treewidth at most w.

Extended MSO_2 problem :

- MSO_2 formula $\varphi(X_1, \ldots, X_l)$ and an linear function $h(|X_1|, \ldots, |X_l|)$
- Find an assignation of $X_1, \ldots, X_l$ that satisfies $\varphi(X_1, \ldots, X_l)$ and maximize/minimize $h(|X_1|, \ldots, |X_l|)$

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FPT Algortihm :

1. Compute a tree decomposition by BFS. If width $> 2g(k)$ return `false`.

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$$\varphi(E_1, \dots, E_k) = \exists\ V_1, \dots, V_k, \text{Cover}(V_1, \dots, V_k) \bigwedge_{1 \le i \le k} \text{Path}(V_i, E_i, s_i, t_i)$$

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3. **If** $\forall i, |E_i| = dist(s_i, t_i)$ **then** answer `true`, **else** answer `false`.

Algorithmic Consequences

Theorem

Isometric Path Cover with Terminals is **FPT**

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Corollary

Isometric Path Cover is in **XP**

Idea : Brute-force all combination of k pairs of terminals with the FPT algortihm.

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Isometric Path Cover is in **XP**

Idea : Brute-force all combination of k pairs of terminals with the FPT algortihm.

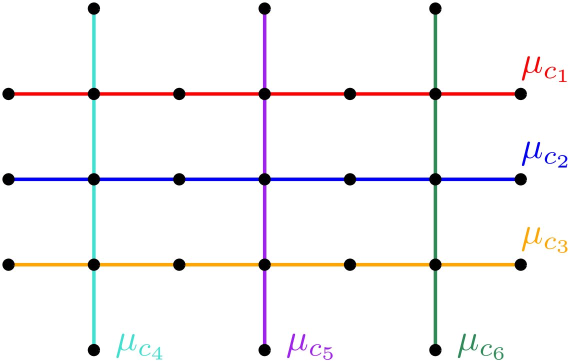
$\longrightarrow$ These results can be generalized to :

- Edge-covering
- Edge/vertex partitioning

Graphs edge-coverable by k shortest paths

Colouring base path

Assign a color to each shortest path covering the graph.



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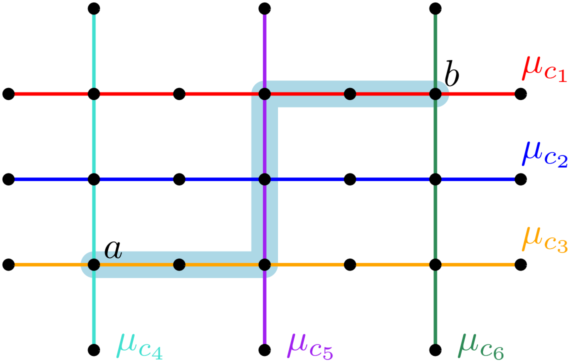

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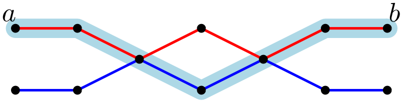

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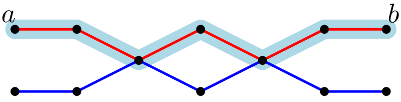

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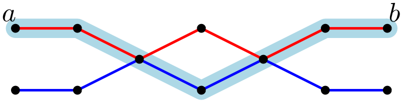

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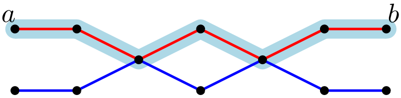

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Good colouring Lemma

For every pair of vertices *a*, *b*, there exists a shortest path from *a* to *b* that is well coloured.

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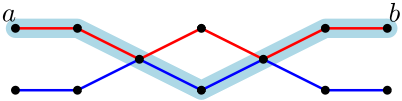

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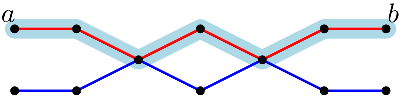

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For every pair of vertices a, b, there exists a shortest path from a to b that is well coloured.

- Take a shortest path P between a and b.

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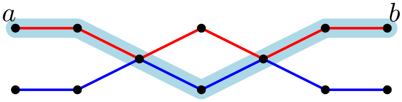

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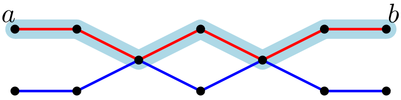

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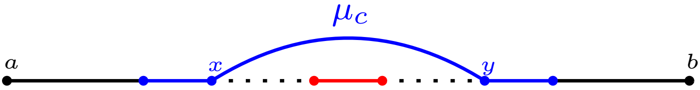

- Take a shortest path P between a and b.
- Replace $P[x, y]$ by $\mu_c[x, y]$.

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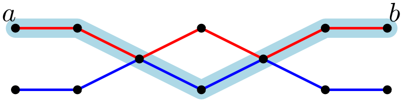

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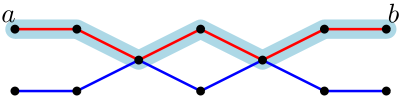

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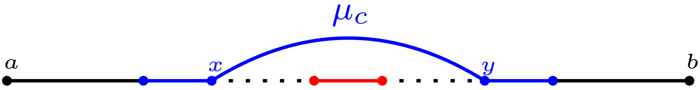
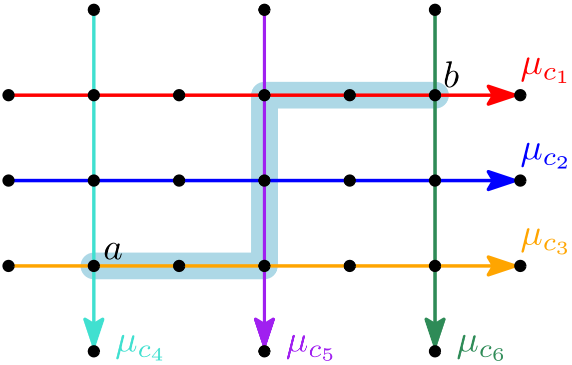


- Take a shortest path P between a and b.
- Replace $P[x, y]$ by $\mu_c[x, y]$.
- Repeat until the path is well colored.

Colour-signs word



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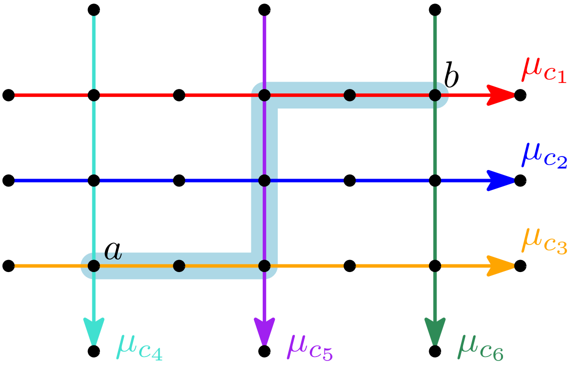

$$\omega = ((c_3, +), (c_5, -), (c_1, +))$$

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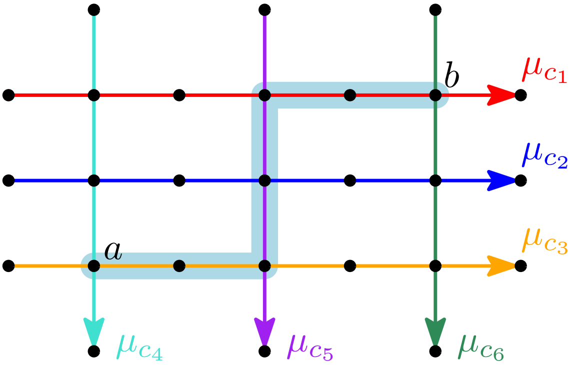

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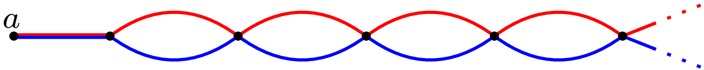
Number of colour-signs words possible for all well coloured paths :

$$\sum_{\ell=1}^{k} 2^{\ell} \cdot \ell! \cdot \binom{k}{\ell} = O(k^k)$$

A first bound

Multiple shortest paths of same length may have the same colours-signs word :



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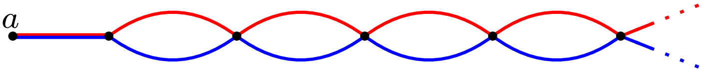

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The shortest paths starting at a vertex a, of length D and colours-signs word ω all ends at the same vertex b.

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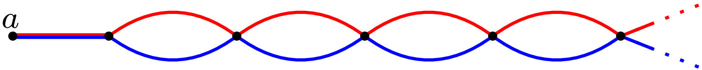

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Theorem

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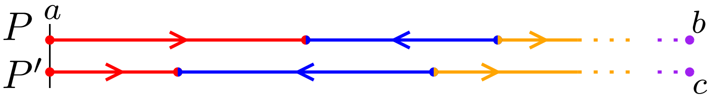
Proof of the Colours-signs word Lemma

Let b and c be vertices at same distance from a vertex a of G.
Let (P, col), (P', col') be a well-coloured shortest a-b and a-c paths.

Claim : If they have the same colours-signs word, then $b = c$.



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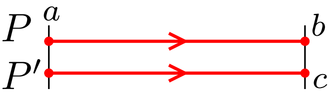

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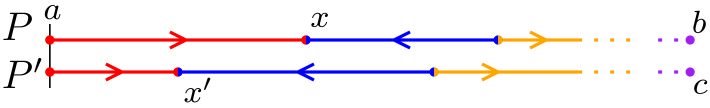

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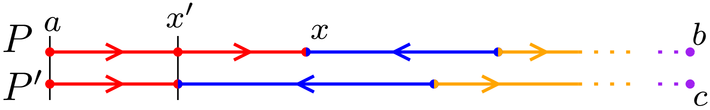

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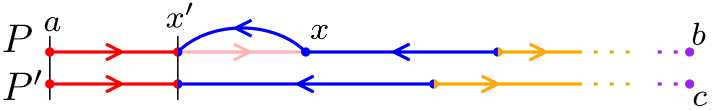

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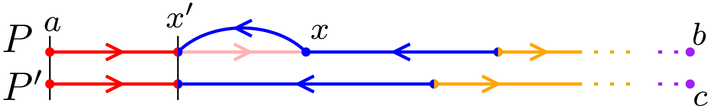

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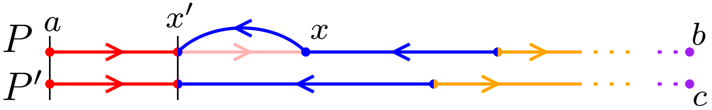

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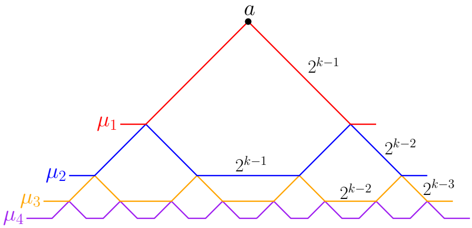
(weak) Theorem 1

For any vertex a and any fixed distance D, the number of vertices at distance exactly D from a is upper bounded by $O(k^k)$ (number of colours-signs words).

A better bound ?

We have shown the upper bound : $O(k^k)$

Lower bound : $O(2^k)$

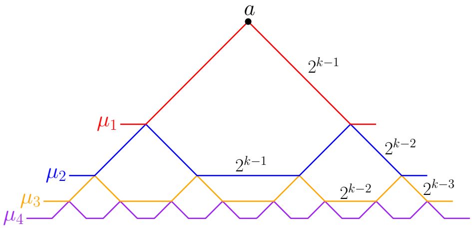


Goal : Single exponential bound.

A better bound ?

We have shown the upper bound : $O(k^k)$

Lower bound : $O(2^k)$



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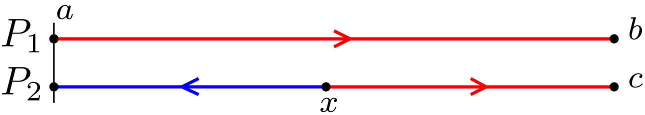

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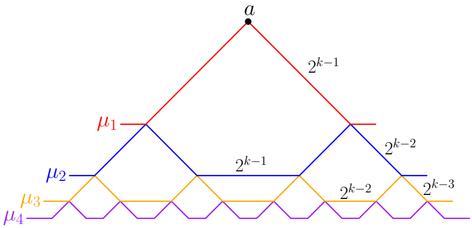

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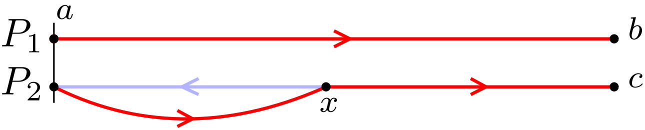
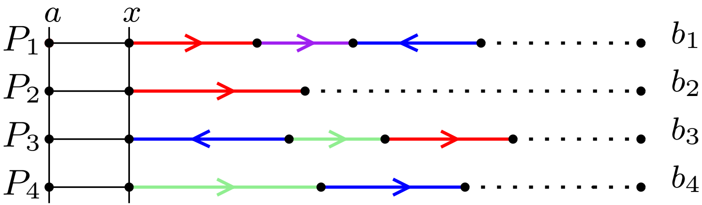


Branched colouring

Idea : Generalize the previous observation recursively.

Take a set of paths from *a* to the vertices at distance *D* from *a*.



The colours red, blue and green does not appear in the dotted subpaths.

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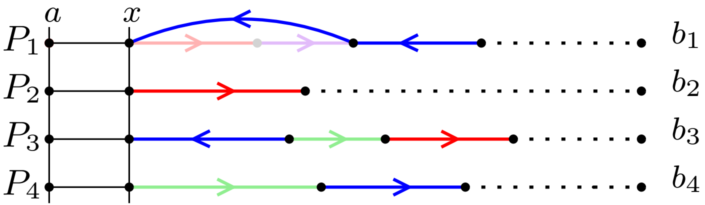

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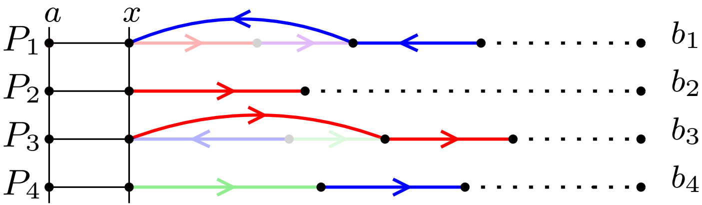

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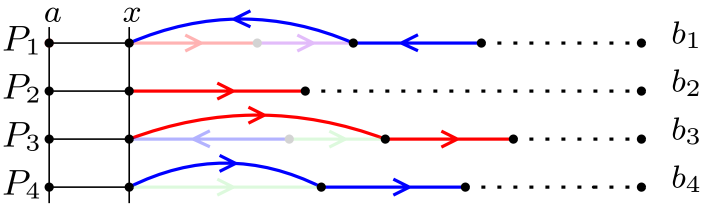

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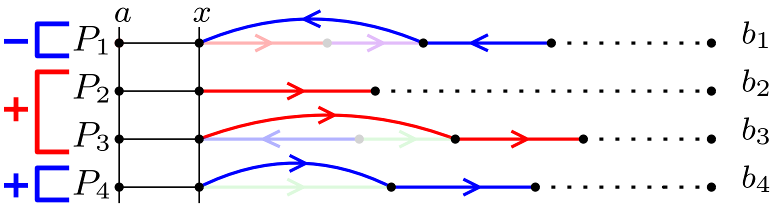

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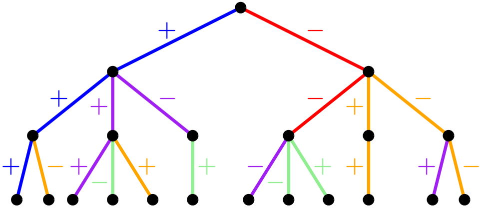
⇒ Apply recursively this process on each subset of paths independently.

Branched colouring

Structure of the paths at the end of the recursive process :



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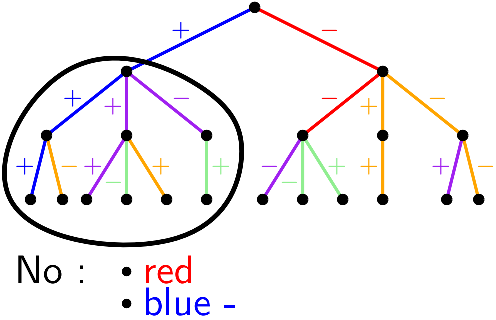

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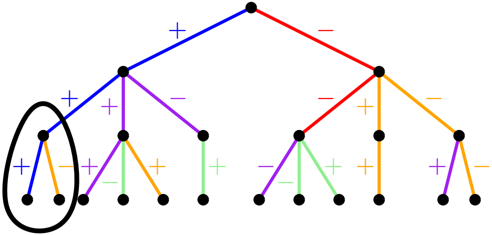

No :
- red
- blue -
- purple

Branched colouring

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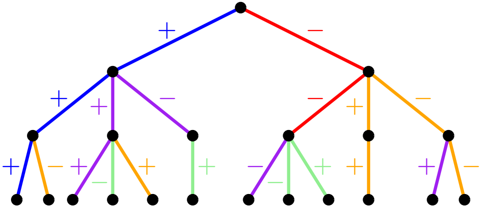

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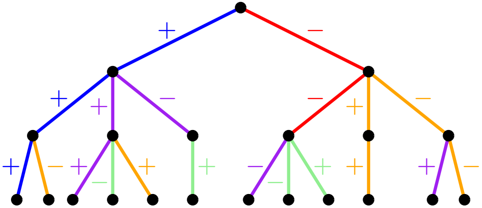

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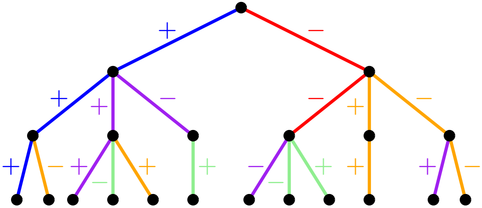

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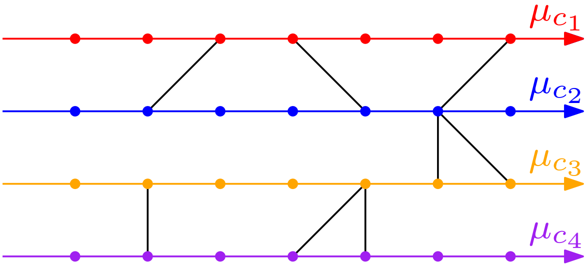
Theorem 1 :

$O(3^k)$ vertices at a given distance of an arbitrary vertex.

Graphs vertex-coverable by k shortest paths

Colouring of a path

A colour and a direction given to each base path.



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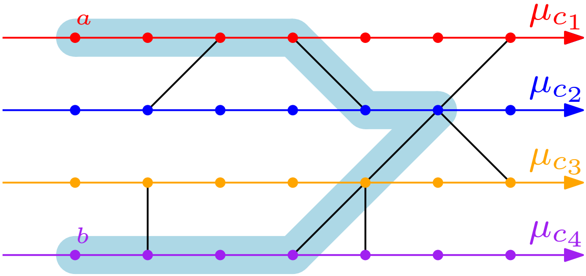

Colours-signs words are defined in a similar way as in the edge case.
Here : $\omega = ((c_1, +), (c_2, +), (c_3, +), (c_4, -),)$

Bound for the vertex case

- Good colouring Lemma works the same way.

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Colours-signs word Lemma

The shortest paths starting in a vertex a, of length D and colours-signs word ω all ends in the same vertex b.

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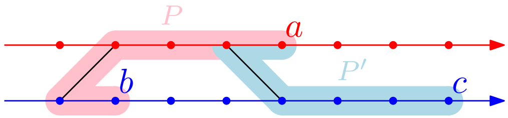

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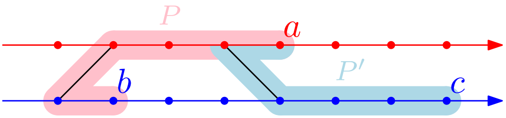

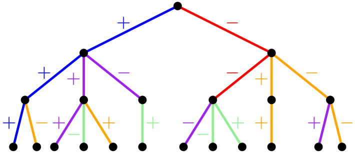

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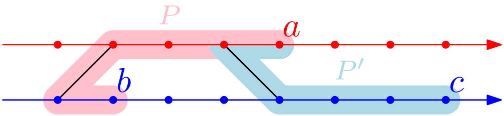

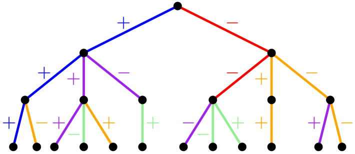

- **Branched colouring** can be adapted to the vertex case, but ~~1~~ $O(k)$ paths may share the same colours-signs word.
- There is at most $g(k) = O(k \cdot 3^k)$ vertices at a given distance of a.

Conclusion

Results

- In graphs vertex/edge-coverable by k shortest paths, the number of vertices at same distance of a source is upper bounded by $g(k) = O^*(3^k)$.
  $\longrightarrow$ Implies a $O^*(3^k)$ upper bound on the pathwidth.
- Isometric Path Cover with Terminals is FPT.
- Isometric Path Cover IPC is in XP.

Questions

- Polynomial bound on the treewidth/pathwidth ?
- Is Isometric Path Cover FPT ? W-hard ?

Thanks!