An improved kernelization algorithm for Trivially Perfect Editing

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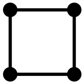
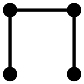
Trivially Perfect Editing

Trivially perfect graphs (quasi-threshold)

- Do not contain P_4 and C_4 as induced subgraphs.
- Every connected induced subgraph admits an universal vertex.



C_4



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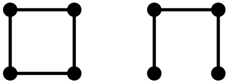
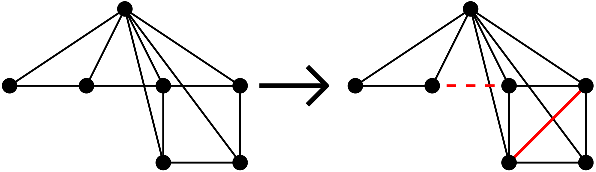


C_4 P_4

Trivially Perfect Editing

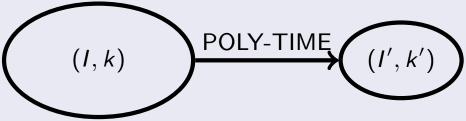
- **Input:** an arbitrary graph $G = (V, E)$, a parameter integer k.
- **Question:** $\exists F \subseteq [V]^2$ of size at most k such that $G \triangle F$ is trivially perfect?
  $G \triangle F = (V, (E \cup F) \backslash (E \cap F))$ — F is an edition of G



Completion (**Deletion**) : only allowed to add (delete) edges.

Kernelization algortihms

Kernel



$|I| = n$

$|I'| \leq f(k)$

$k' \leq k$

(I, k) Yes-instance $\Leftrightarrow$ (I', k') Yes-instance
f polynomial $\Rightarrow$ polynomial kernel

Theorem

A problem $\mathcal{Q}$ is Fixed-Parameter Tractable $\Leftrightarrow$ $\mathcal{Q}$ admits a kernel.

Kernels for trivially perfect edge modification problems

Existing vertex-kernels :

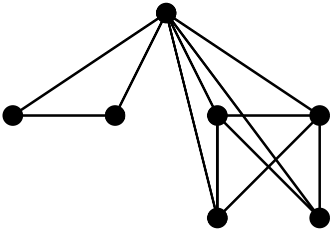
- Completion: $O(k^3)$ (announced, never published) [Guo - ISAAC 2007]
- Editing, deletion, completion: $O(k^7)$ [Drange, Pilipczuk - ESA 2015]
- Editing, deletion, completion: $O(k^3)$ [Dumas, Perez, Todinca - MFCS 2021]
- Completion: $O(k^2)$ [Bathie, Bousquet, Pierron - IPEC 2021] [Cao, Ke - IPEC 2021]

Our result :

Editing, deletion : $O(k^2)$ vertex-kernel

Universal Clique Decomposition (UCD)

Universal clique : the set of universal vertices.
Universal Clique Decomposition : The tree of universal cliques.



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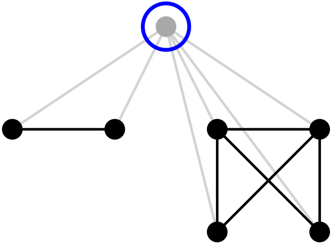

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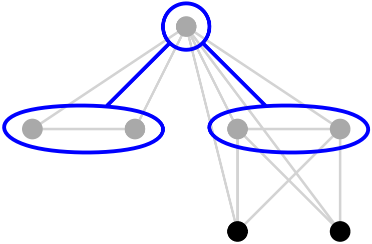

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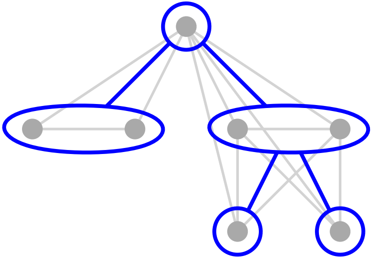

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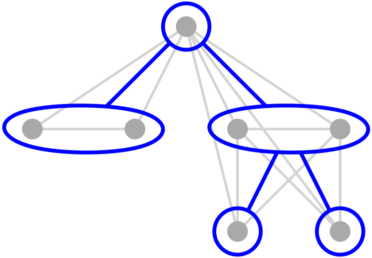
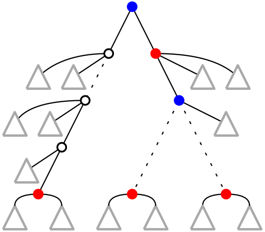


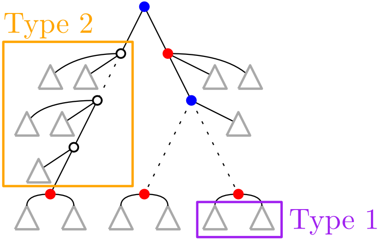
- G is trivially perfect $\Leftrightarrow$ G admits an UCD.
- Bags of the UCD correspond to **critical cliques** (maximal clique modules).
- Rooted subtrees of the UCD correspond to trivially perfect modules.



For an edition F of G, we consider T the UCD of $H = G \triangle F$:

- A the set of nodes that contains vertices affected by F,
- A' lowest common ancestor closure of A,
- $|A| \leq 2k$ and $|A'| \leq 2k$.

Kernel: general approach



Three type of connected component in $T\backslash(A \cup A')$:

- Type 0 : not adjacent to any node of $A \cup A'$,
- Type 1 : adjacent to one node of $A \cup A'$ (modules),
- Type 2 : adjacent to two nodes of $A \cup A'$ (combs),

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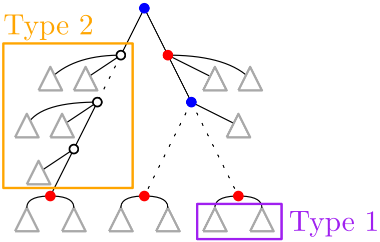

Three type of connected component in $T \backslash (A \cup A')$:

- Type 0 : not adjacent to any node of $A \cup A'$,
- Type 1 : adjacent to one node of $A \cup A'$ (modules), size : $g(k)$,
- Type 2 : adjacent to two nodes of $A \cup A'$ (combs), size : h(k),

Theorem

Trivially Perfect Editing admits a kernel with $O(k \cdot (g(k) + h(k)))$ vertices.

$\Rightarrow$ We show: $g(k) = O(k)$ and $h(k) = O(k)$.

Classical rules

Rule 1

Remove the trivially perfect connected component of G.

$\Rightarrow$ Removes type 0 components.

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Rule 2

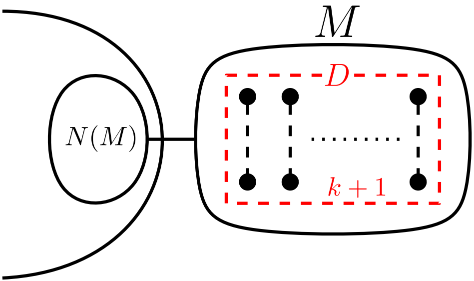
Let $K \subseteq V$ be a critical clique of G such that $|K| > k + 1$. Remove arbitrarily $|K| - (k + 1)$ vertices of K from G.

$\Rightarrow$ Bound the size of nodes of the UCD by $k + 1$.

Bounding the size of trivially perfect modules

Rule 3

Let $M \subseteq V$ be a trivially perfect module of G. If $G[M]$ contains a $(k+1)$-sized anti-matching D, then remove the vertices contained in $M \setminus D$.



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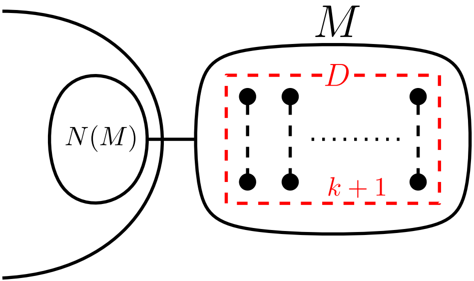
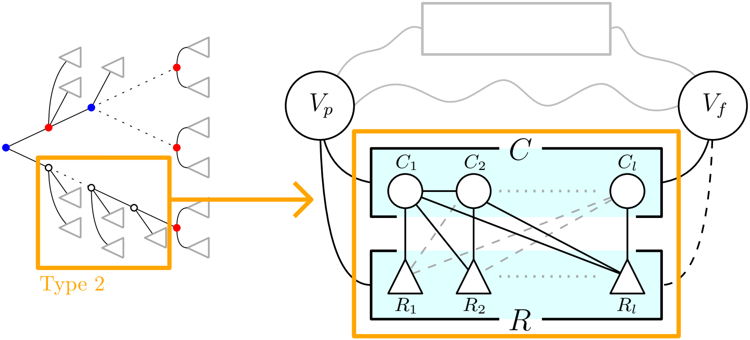


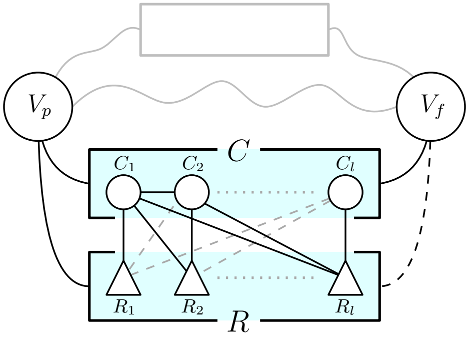
Question : how to bound the size of modules with small anti-matching ?

Combs



A **Comb** is a pair (C, R) of set of vertices such that :

- C is a clique composed of l critical cliques (the **shaft**),
- R is a set of l non-adjacent trivially perfect modules (the **teeth**),
- The induced graph by $G[C \cup R]$ is trivially perfect,
- $N_G(C) \backslash R = V_p \cup V_f$, $N_G(R) \backslash C = V_p$.



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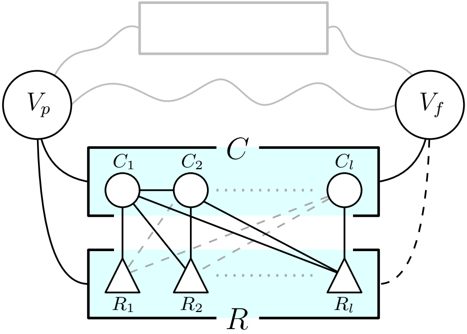
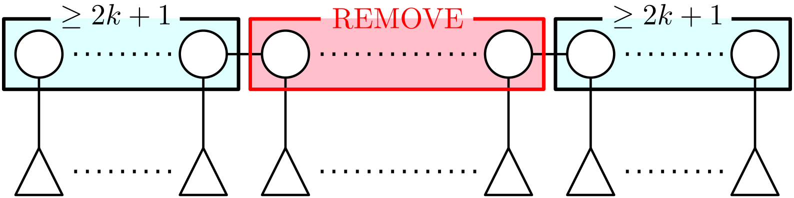


Goal

Bound to $O(k)$ the number of vertices in a comb.

Rule 4

Let (C, R) be a comb of G. Keep at least $2k+1$ vertices at the beginning and the end of the **shaft**, remove the others.



Bound the number of vertices in the shaft

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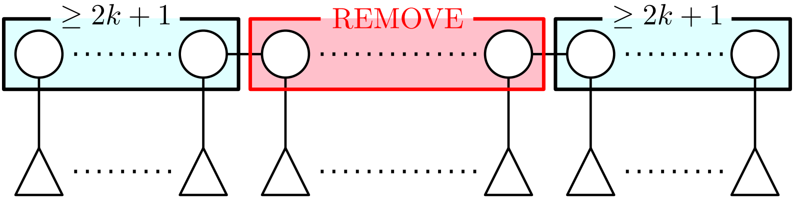
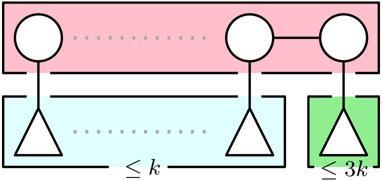


$\Rightarrow$ combs have $O(k)$ vertices in their shaft.

Bound the size of modules with small anti-matching

- Trivially perfect modules can be decomposed into a comb.
- Decomposition of a comb with small anti-matching:



$\Rightarrow$ They can be decomposed in a comb (C, R) such that $|R| = O(k)$.

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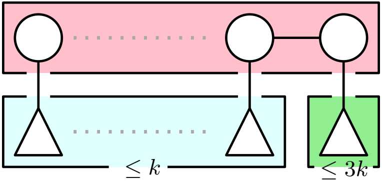

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Conclusion on trivially perfect modules

- large anti-matching : Rule 3 $\Rightarrow \leq 2k + 2$ vertices
- small anti-matching : Rule 4 $\Rightarrow O(k)$ vertices
- Hence $g(k) = O(k)$

Bound the number of vertices in the teeth

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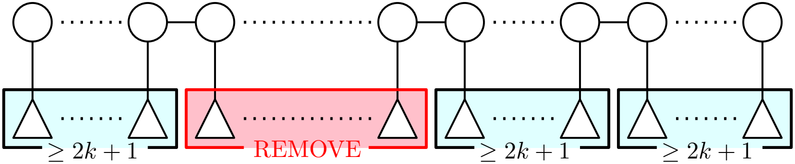
Let (C, R) be a comb of G. Keep at least $2k+1$ vertices at the beginning and two disjoint sets of $2k+1$ vertices at the end of the **teeth**, remove the others.



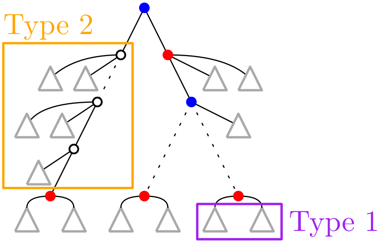
Size of combs

- Combs have $O(k)$ vertices in their teeth.
- Conclusion : combs contains at most $h(k) = O(k)$ vertices.

Reduced instance size

Theorem

Trivially Perfect Editing admits a kernel with $O(k \cdot (g(k) + h(k)))$ vertices.



We showed: $g(k)$ and $h(k)$ are $O(k)$
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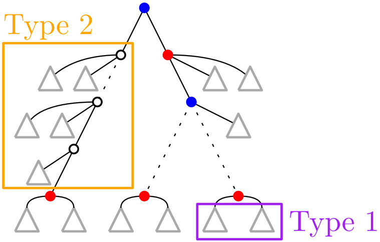

We showed: $g(k)$ and $h(k)$ are $O(k)$
$\Rightarrow$ Trivially Perfect Editing admits a kernel with $O(k^2)$ vertices

$\Rightarrow$ Trivially Perfect Deletion and Completion admit a kernel with $O(k^2)$ vertices

Conclusion

Our result : a $O(k^2)$ vertex-kernel for Trivially Perfect Editing and Deletion.

Questions :

- Can we get a smaller vertex-kernel?
- Several kernels use similar approaches (proper interval, ptolemaic) can we improve them?

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Thank you !