Sufficient conditions for polynomial-time detection of induced minors

Clément Dallard[1], **Maël Dumas**[2], Claire Hilaire[3], Anthony Perez[4]

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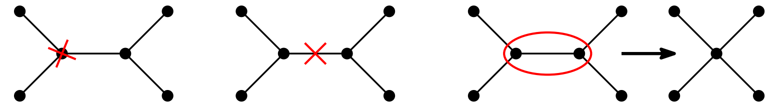
[4]LIFO, Université d'Orléans, France

SOFSEM 2025

Does G contains H ?



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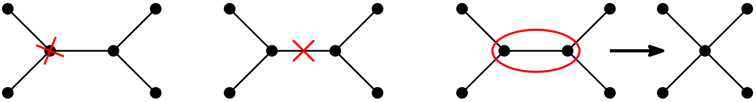

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$\longrightarrow$ NP-hard for most operation sets.

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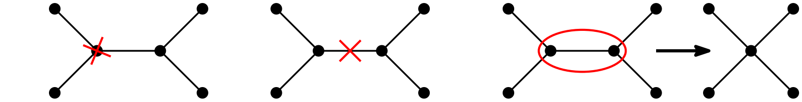

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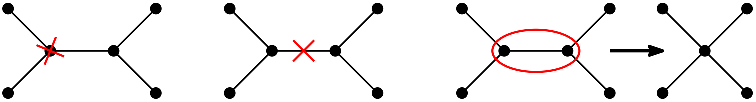

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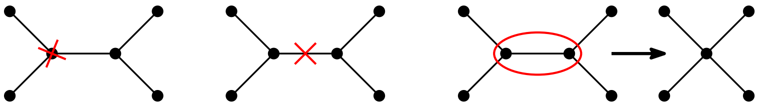

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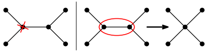

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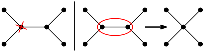

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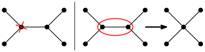

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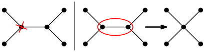

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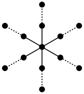
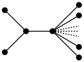
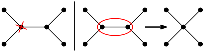
  subdivided stars:  and  [Fiala, Kamiński, Paulusma 12]

Complexity of H-Induced Minor Containment (H-IMC) 

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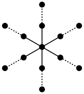
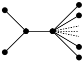
  subdivided stars:  and  [Fiala, Kamiński, Paulusma 12]
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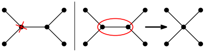

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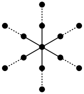
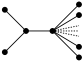
  subdivided stars:  and  [Fiala, Kamiński, Paulusma 12]
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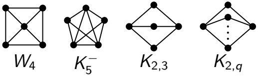
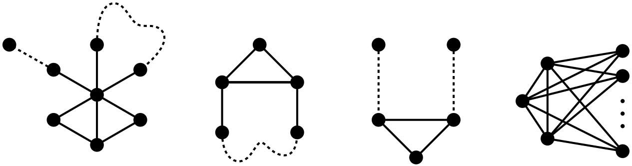
- Some specific graphs 

  [DDHP + Milanic, Trotignon 24]

Contribution: New H for H-IMC in polynomial-time

- Using the structure of induced minor models:



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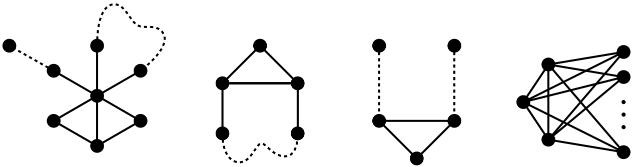

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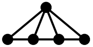
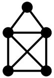
- H-IMC can be solved in polynomial time in graphs without long induced paths.

$\longrightarrow$ Polynomial time algorithm for :

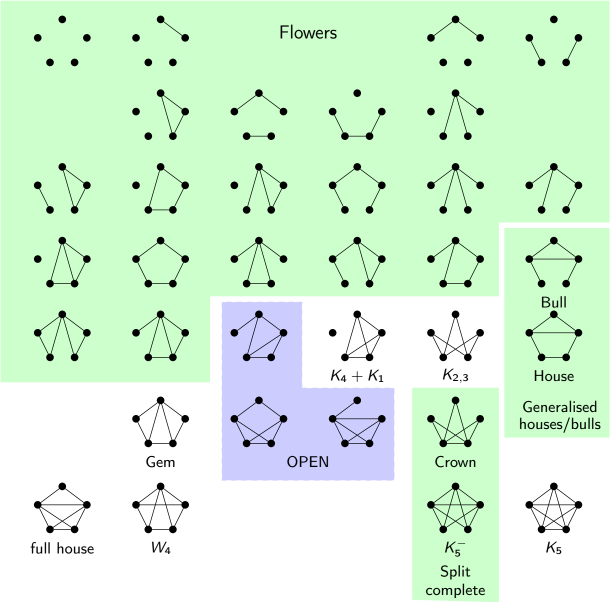


Gem



Full House

Graphs with 5 vertices

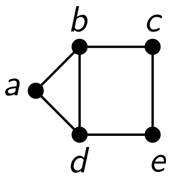


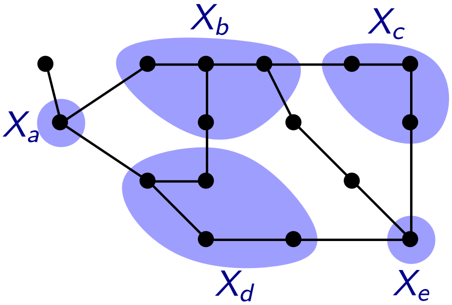
Induced Minor Model

Definition

An **induced minor model** of H in G, is a collection $\mathcal{X}_H = \{X_u\colon u \in V(H)\}$ of pairwise disjoint non-empty subsets of $V(G)$ such that:

- for $u \in V(H)$, $G[X_u]$ is connected, and
- for $u \neq v \in V(H)$, X_u and X_v are adjacent if and only if $uv \in E(H)$.



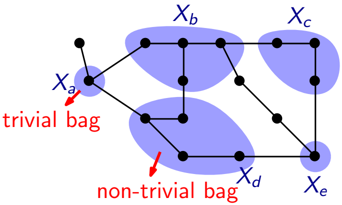


There is a model of H in G $\Leftrightarrow$ $H \subseteq_{im} G$ (G admits H as an induced minor)

Almost trivial models

Definition

A graph H is S-**non-trivial** for $S \subseteq V(H)$ if for all graph G s.t. $H \subseteq_{im} G$, there is a model $\mathcal{X}_H$ in G where only the bags of S are non-trivial.



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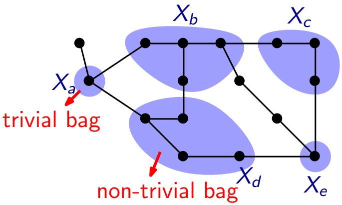

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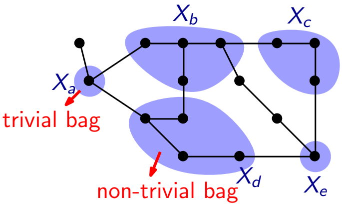

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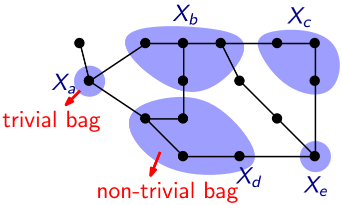

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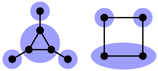
Proof: $\Rightarrow$ Claws and cycles are not $\emptyset$-non-trivial: 

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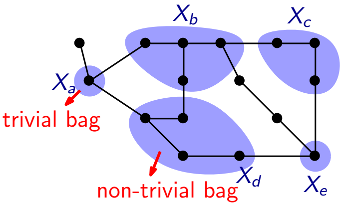

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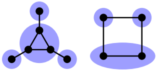
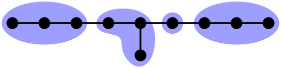


$\Leftarrow$ Given a model of a path:

P_4



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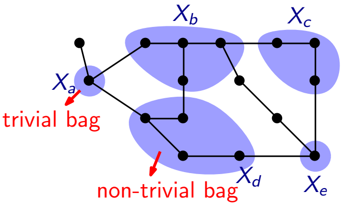

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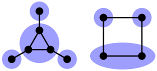

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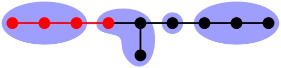

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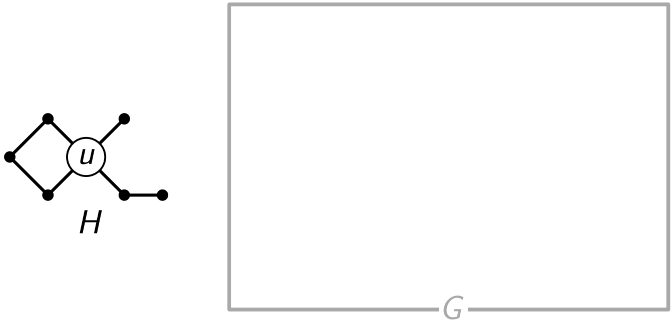

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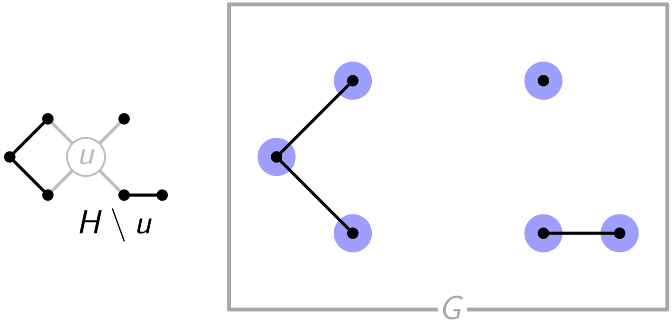

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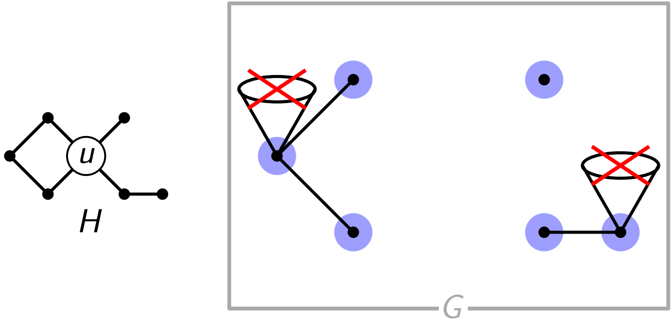

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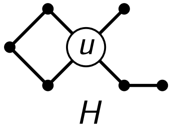

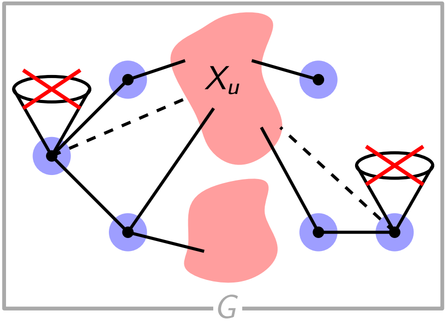

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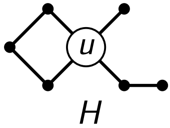
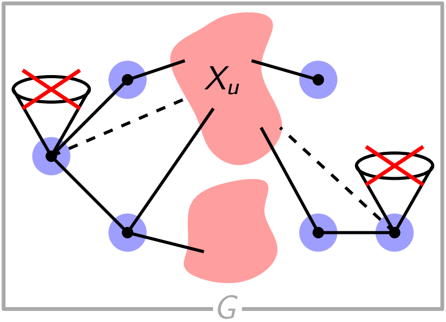
If no model of H, then $H \not\subseteq_{im} G$.





Flower Power

Flowers: H is a flower if $H \setminus u$ is a disjoint union of paths such that for each path P:

- P is connected to u by only 0, 1 or 2 of its endpoints,
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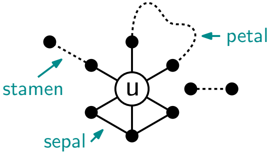

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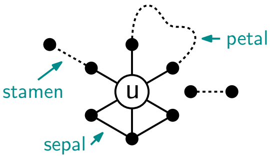


Lemma

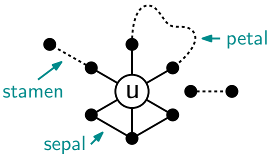
Flowers are $\{u\}$-non-trivial.

$\longrightarrow$ If H is a flower, then H-IMC is polynomial-time solvable.

Flower Power

Flowers: H is a flower if $H \setminus u$ is a disjoint union of paths such that for each path P:

- P is connected to u by only 0, 1 or 2 of its endpoints,
- or $|V(P)| = 3$ and P is complete to u.



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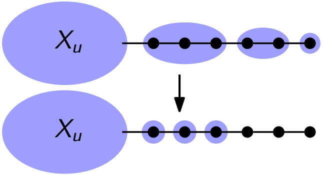

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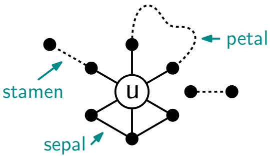

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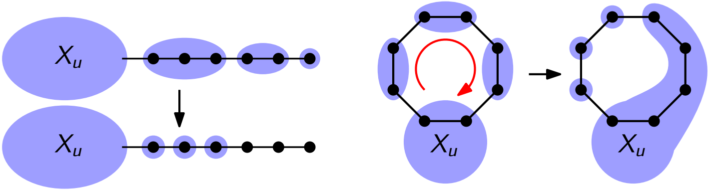

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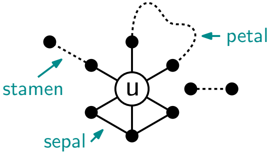

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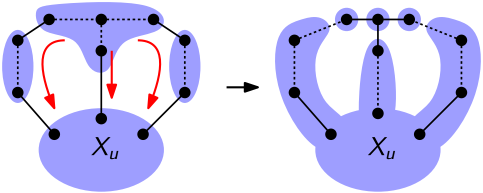
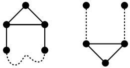
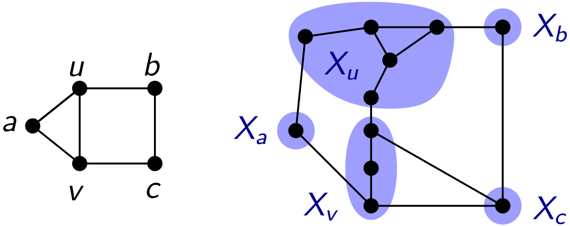


Generalized Houses and Bulls





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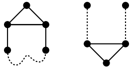

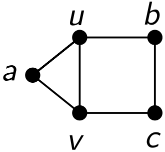

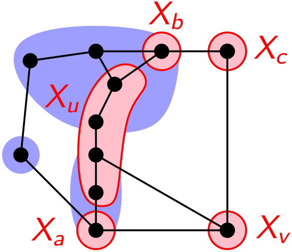

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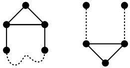

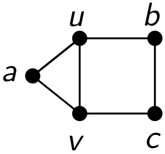

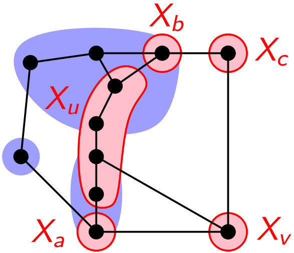

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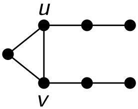
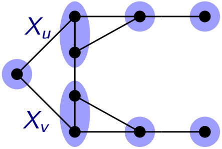
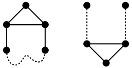
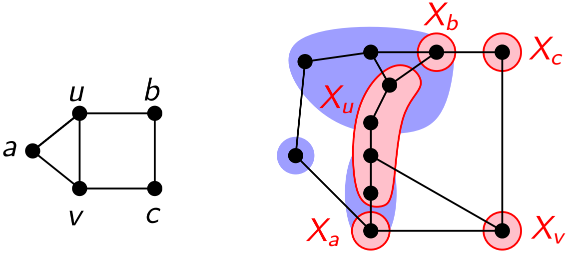




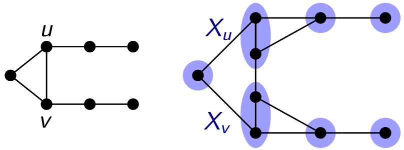
$\longrightarrow$ Generalized Bulls are $\{u, v\}$-non-trivial

Generalized Houses and Bulls





⟶ Generalized Houses are $\{u\}$-non-trivial

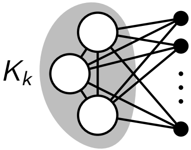


⟶ Generalized Bulls are $\{u, v\}$-non-trivial

Theorem

If H is a generalized bull, then H-IMC can be solved in polynomial time.

Split Complete

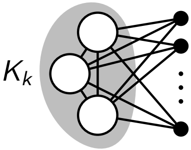


$\longrightarrow S_{k,q}, k \leqslant 3$, is K_k-non-trivial

Theorem

If H is $S_{k,q}, k \leqslant 3$, then H-IMC can be solved in polynomial time.

Split Complete



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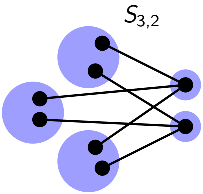


Theorem

If H is $S_{k,q}$, $k \leqslant 3$, then H-IMC can be solved in polynomial time.

Algorithm sketch:

- Fix the independent set
- Put a neighbor of each vertex of the independent set into each bag of the clique
- Try to compute the rooted clique minor

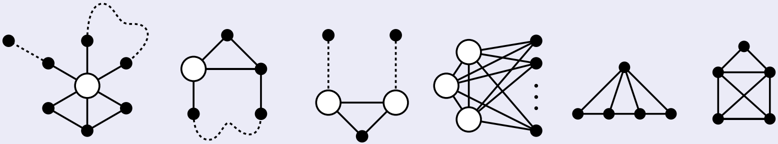
Theorem [Korhonen, Pilipczuk, Stamoulis, 24]

Rooted Minor Containment can be solved in time $O_{H,|X|}((|V(G)| + |E(G)|)^{1+o(1)})$.

Conclusion

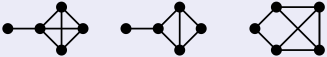
Contributions

H-IMC polynomial-time solvable if *H* is:



$\longrightarrow$ Complexity of *H*-IMC settled for all but 3 graphs of at most 5 vertices.

Open questions

- Complexity of *H*-IMC is open for: 
- Show more polynomial and NP-hard cases for *H*-IMC towards a full dichotomy

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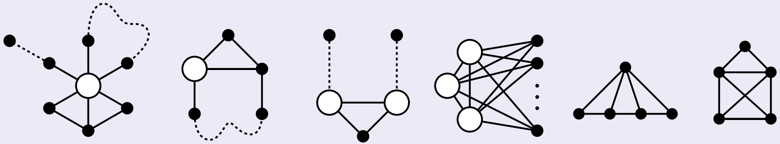

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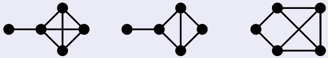
- Complexity of *H*-IMC is open for: 
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Thank you !