Slice knots

Maciej Borodzik
www.mimuw.edu.pl/~mcboro

IMPAN / University of Warsaw

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What is a knot?

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Figure:

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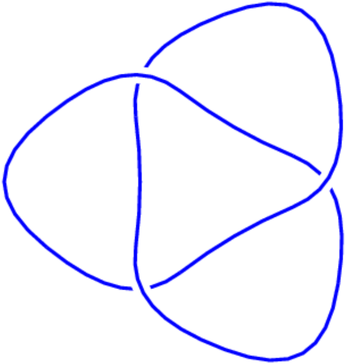

Figure: Trefoil.

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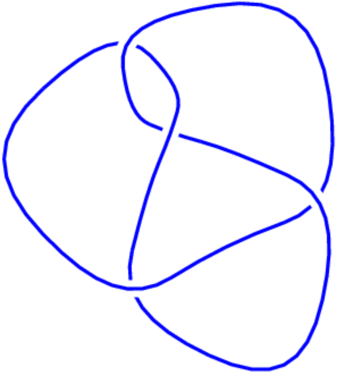

Figure: Figure eight knot.

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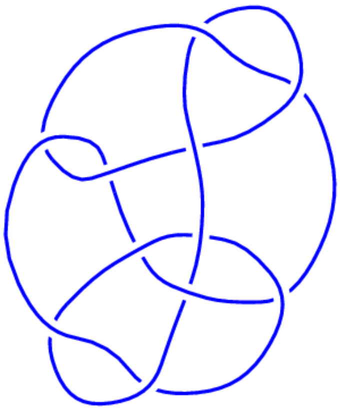

Figure: 11n34.

Slice knots

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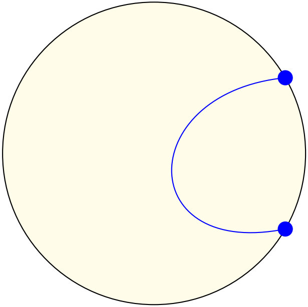

Slice knots

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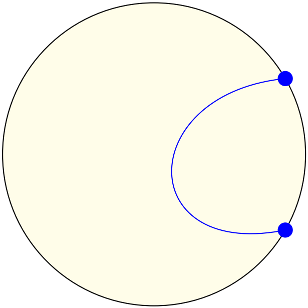
A knot is called smoothly *slice*, if it bounds a smooth disk in B^4.



Slice knots

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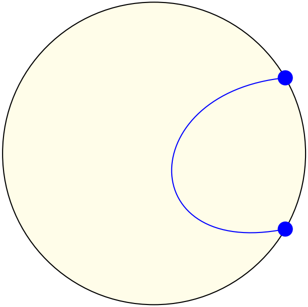
A knot is called topologically *slice*, if it bounds a locally flat disk in B^4.



Concordance

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Two knots K_0, K_1 are called *concordant* if there is an annulus A in $S^3 \times [0,1]$ with $\partial A = K_1 \times \{1\} \sqcup K_0 \times \{0\}$.

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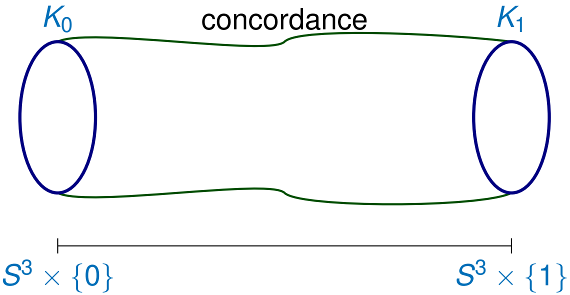

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Remark

A knot is slice if it is concordant to the unknot.

Example. Mirrors

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For any knot K, the connected sum $K\# -K$ is slice.

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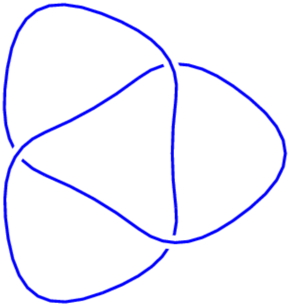

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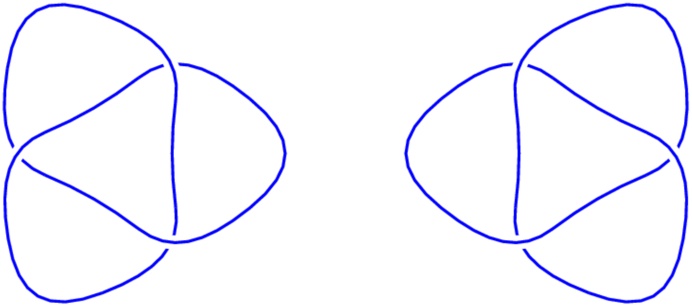

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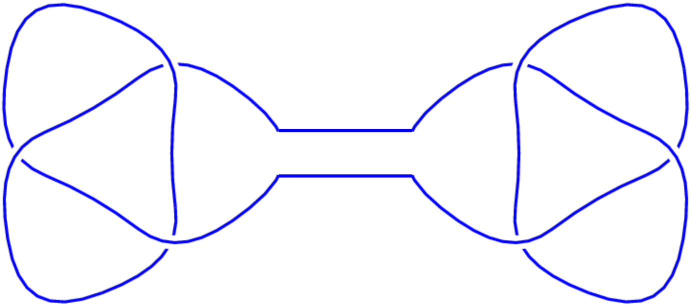

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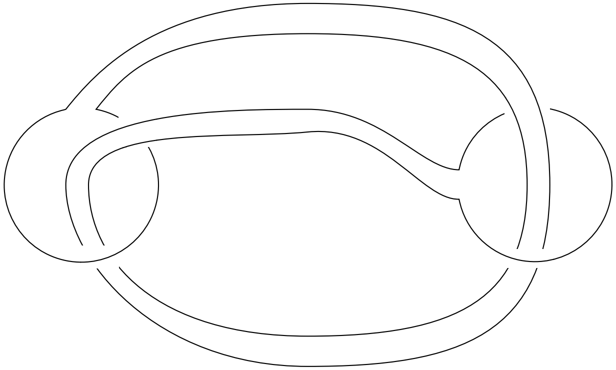

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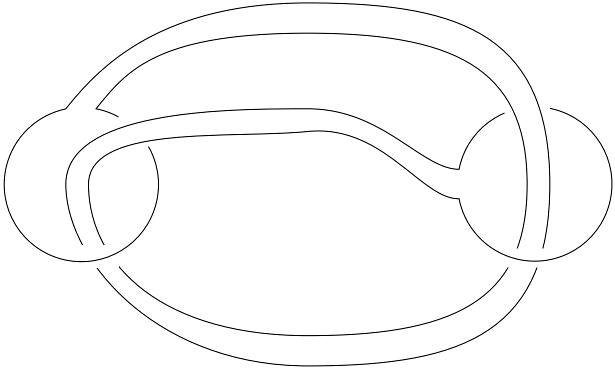
$t = 1$



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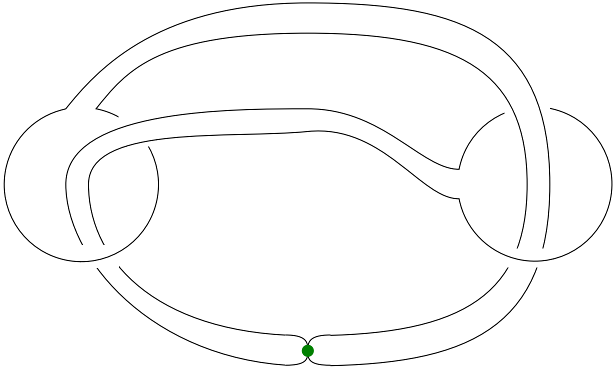
$t = 0.8$



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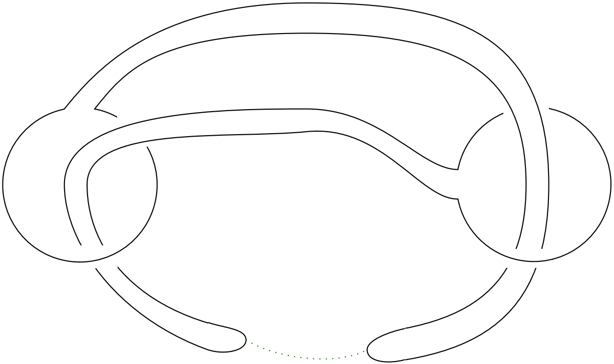
$t = 0.6$



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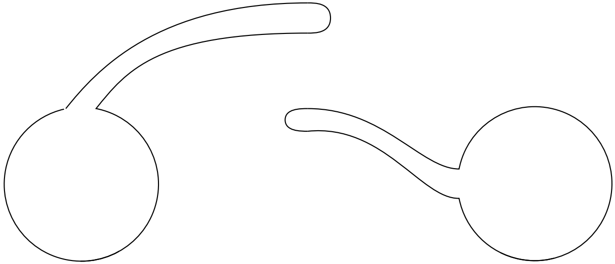
$t = 0.4$



Example. Stevedore's knot

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$t = 0.2$

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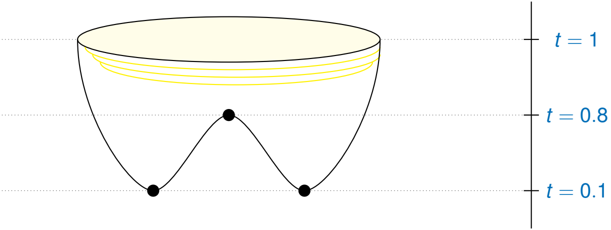

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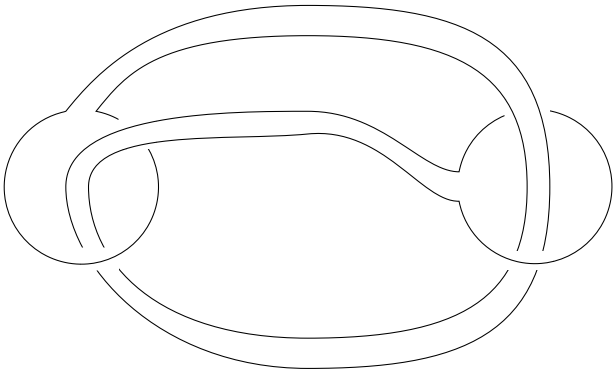

Fox–Milnor

Theorem (Fox, Milnor 1950')

If K *is a slice knot, then* $\Delta(K) = f(t)f(t^{-1})$ *for some* $f \in \mathbb{Z}[t, t^{-1}]$.

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- The figure eight knot has $\Delta = t - 3 + t^{-1}$, it is not slice.
- Figure eight knot is amphichiral, not slice, so $K \# K = 0$.

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Theorem (Hom, Dai–Hom–Stoffregen–Truong, Ozsváth–Stipsicz–Szabó)

The kernel of the map $\mathcal{C} \to \mathcal{C}'$ *contains a* $\mathbb{Z}^\infty$ *summand.*

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Let C be a complex curve in $\mathbb{C}^2$ whose intersection with a large sphere is a knot K_∞, and has finitely many singular points $K_1, \dots, K_n$.

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Theorem (Zajdenberg–Lin, 1970's)

A complex curve as above is equivalent to a curve given by $x^p - y^q = 0$ with $\gcd(p, q) = 1$.

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Any knot is topologically concordant to a $\mathbb{C}$–link.

We believe $\mathbb{C}$–links generate a small subgroup in the smooth concordance group.

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Remark

Unfortunately most obstructions fail to obstruct concordance between K and K' if $M \cong M'$.

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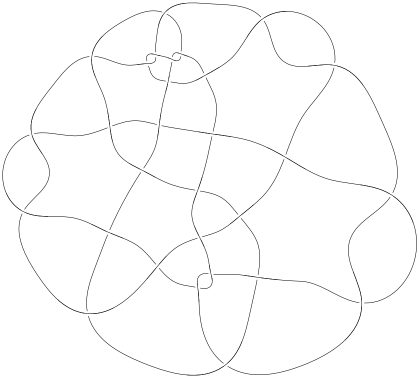

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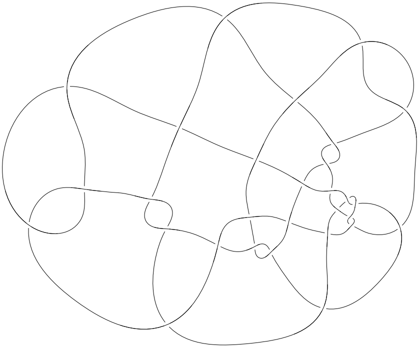

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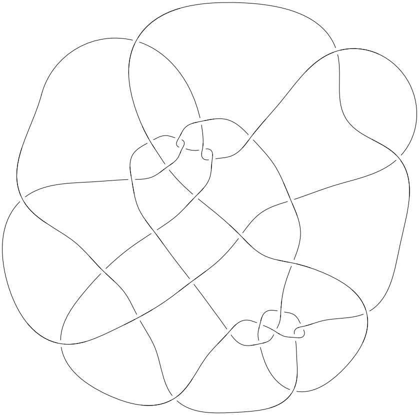

Classical obstructions to sliceness

Theorem (Tristram–Levine, 1968-69)

Let A_0, A_1 be Seifert matrices for K_0 and K_1. If K_0 and K_1 are concordant, then for all but finitely many $z \in S^1$ the signatures of Hermitian matrices $(1-z)A_0 + (1-\overline{z})A_0^T$, $(1-z)A_1 + (1-\overline{z})A_1^T$ are equal.

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Example (Litherland)

Torus knots are independent in the concordance group. But iterated torus knots are not known.

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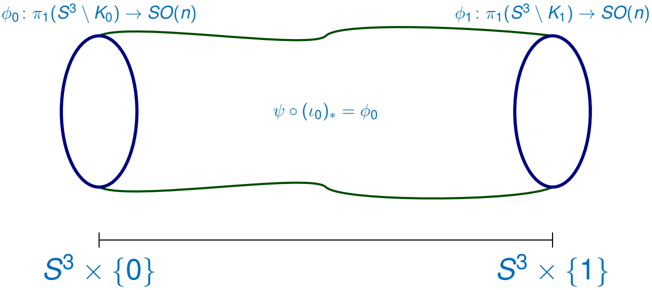

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Example (Hedden–Kirk–Livingston, Conway–Kim–Politarczyk)

The knot $T(2,3;2,13)\#T(2,15)$ is not concordant to $T(2,13)\#T(2,3;2,15)$.

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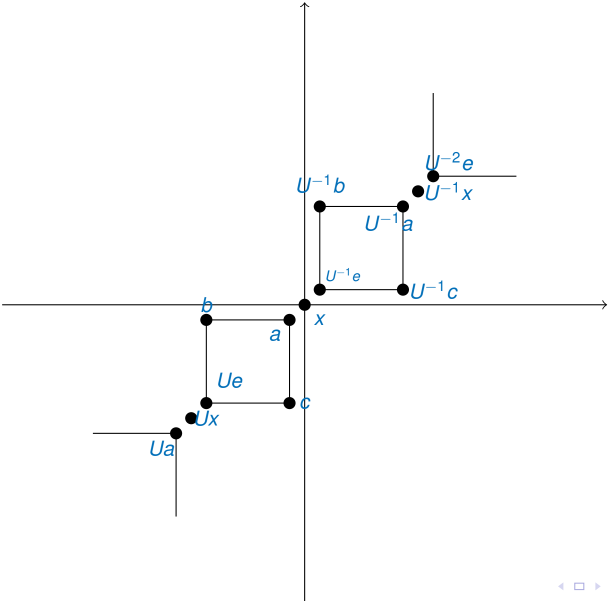
Theorem (Rasmussen 2005)

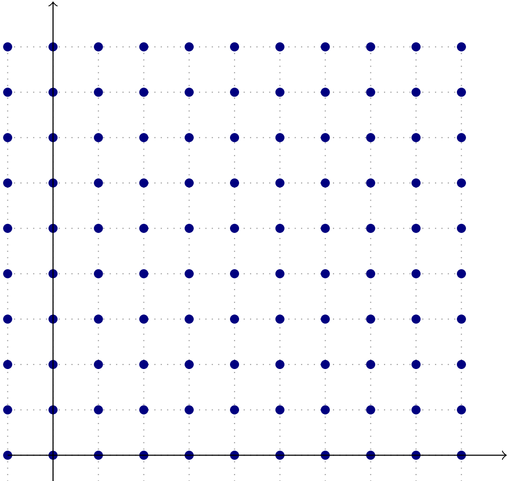
The integers V_i are invariants of smooth concordance.

Example. The figure eight



The torus knot

We consider the $T(4,7)$ torus knot.
$\Delta_{4,7} = t^{18} - t^{17} + t^{14} - t^{13} + t^{11} - t^{9} + t^{7} - t^{5} + t^{4} - t + 1$.



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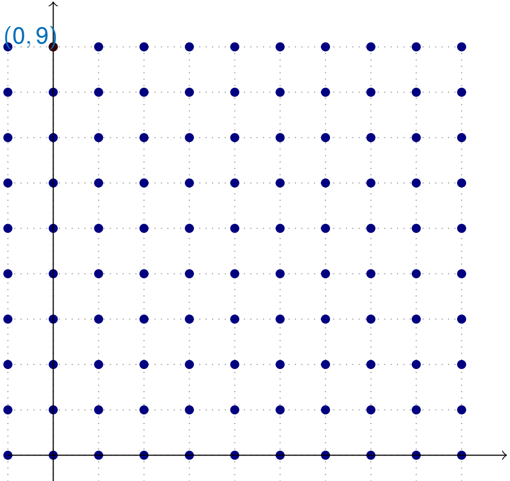

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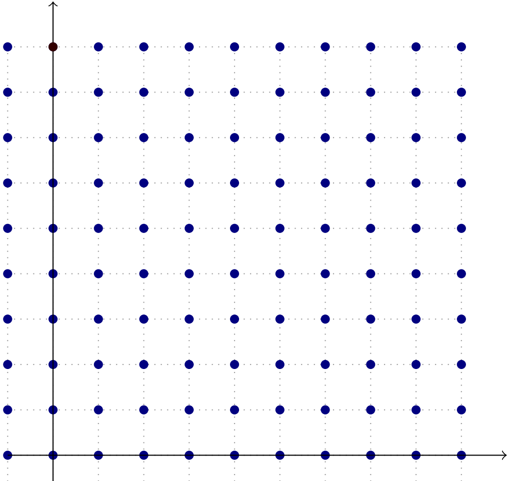

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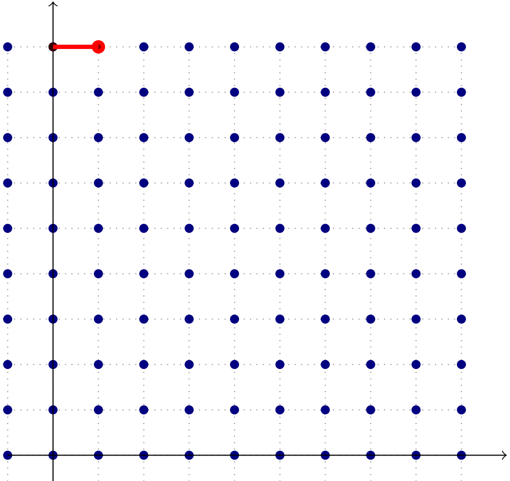

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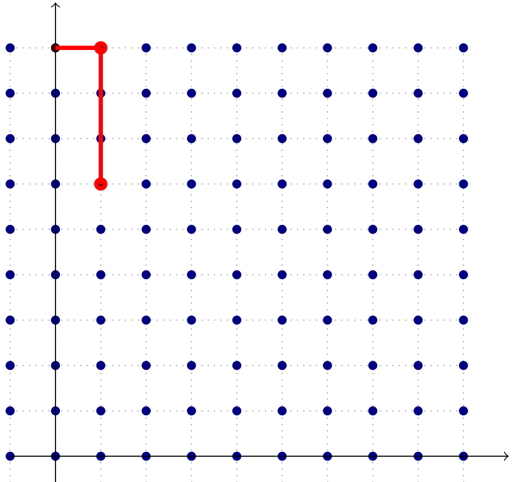

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- $18 - 17 = 1$
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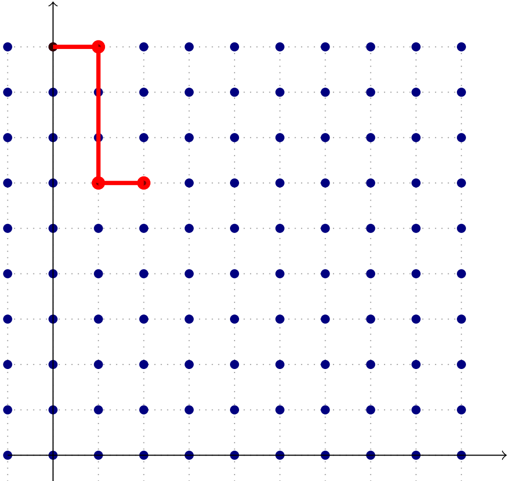

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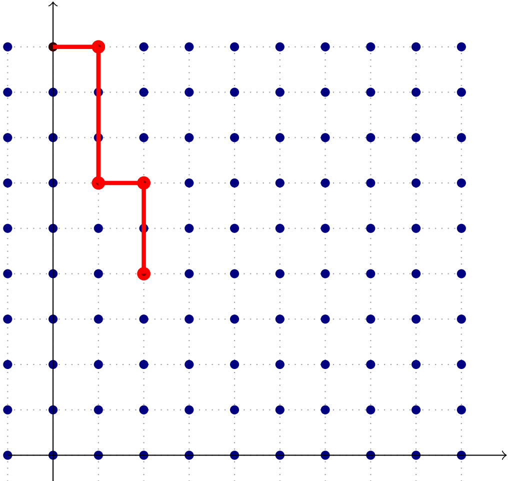

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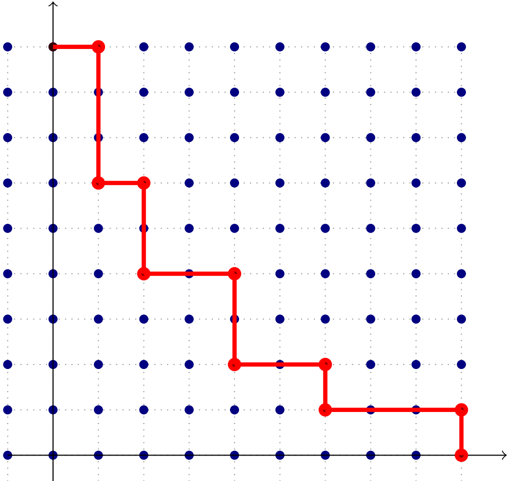

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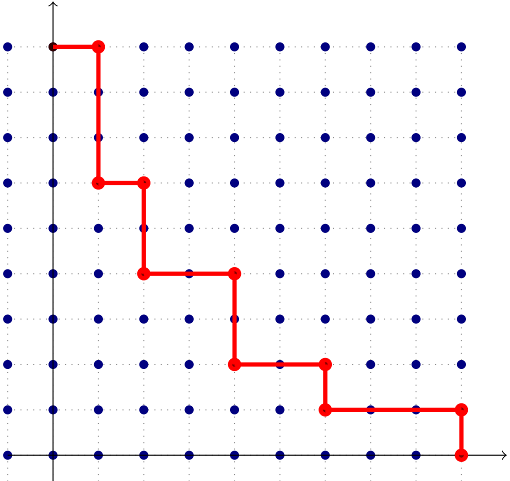

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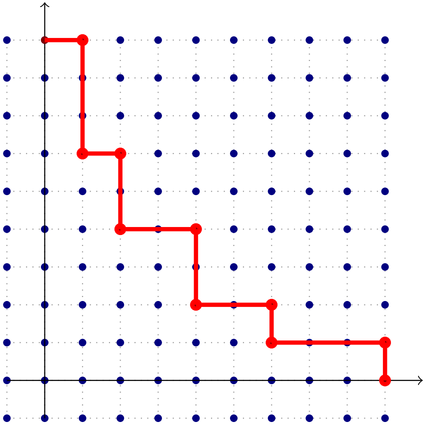
- Symmetry reflects symmetry of Δ

The staircase complex



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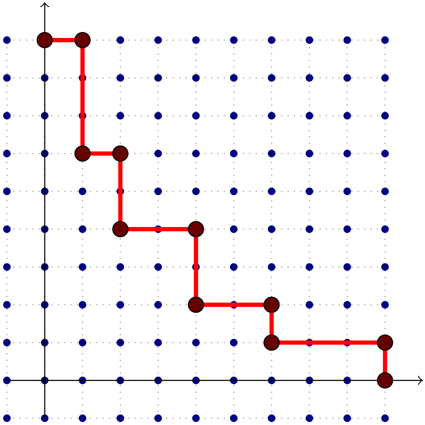

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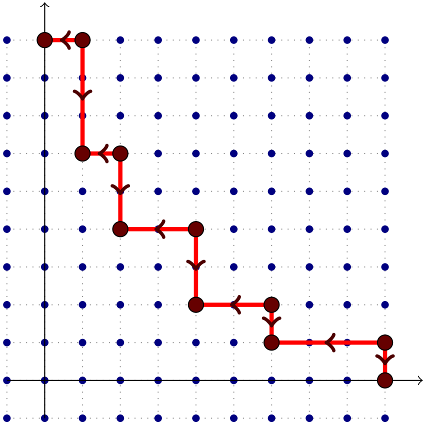

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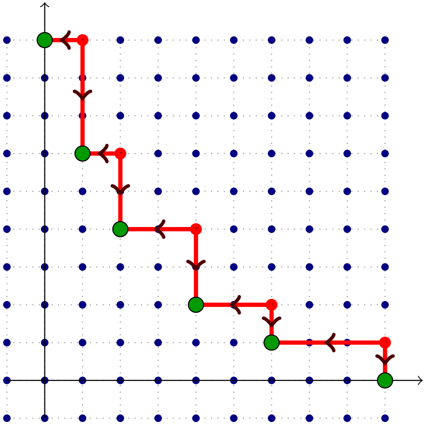

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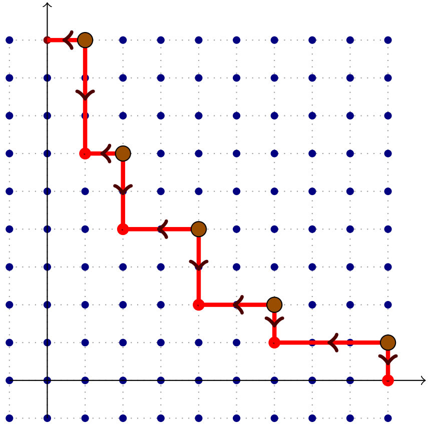

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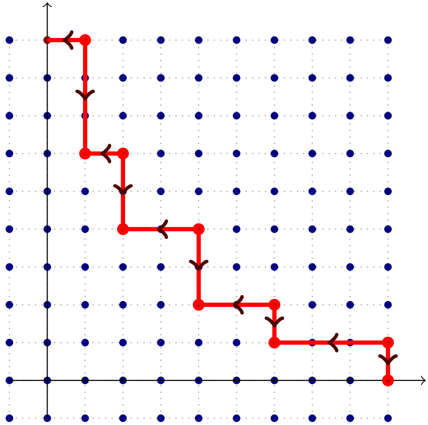

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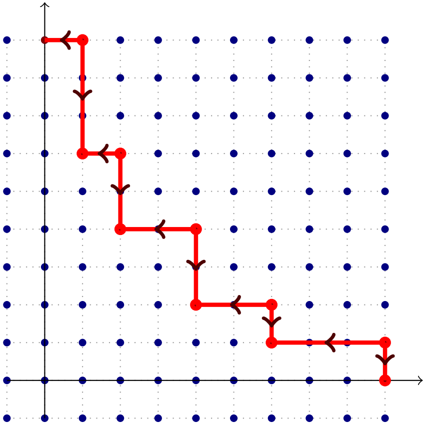

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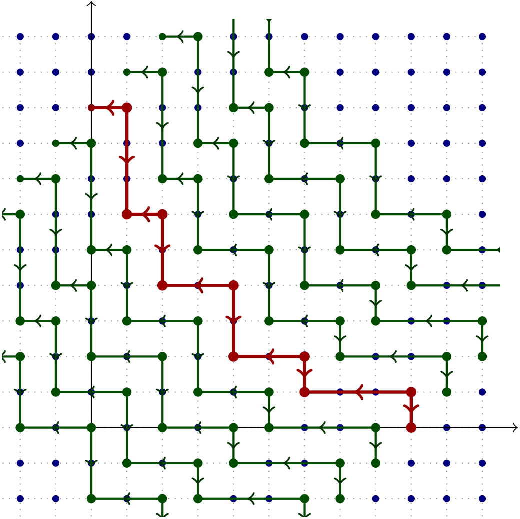
Tensoring



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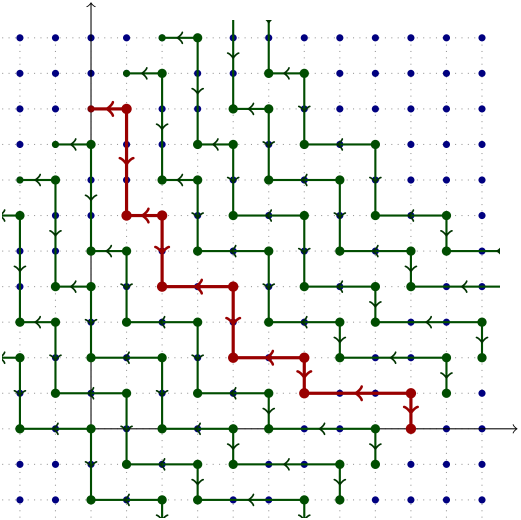

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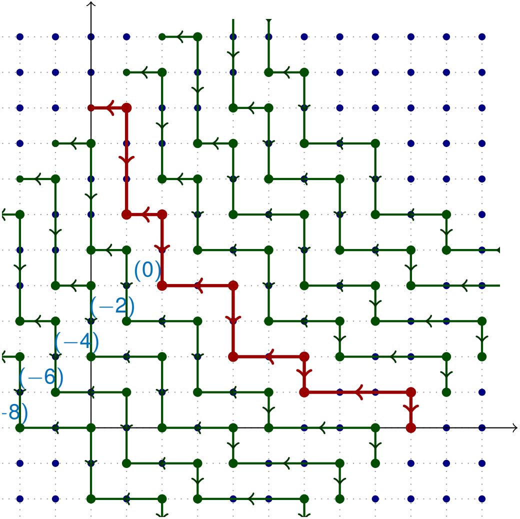

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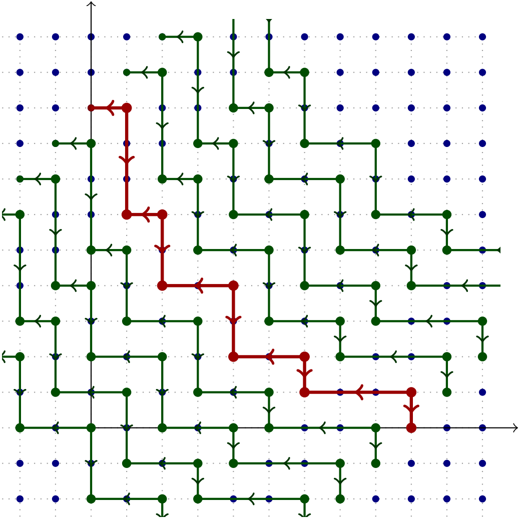

Staircases are everywhere

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Remark

Results on bifiltered modules bear strong resemblance to the paper of Burban and Drozd (2004), Coherent sheaves over rational curves…

Local equivalence

Definition

Let R be a ring. Two complexes C_*, D_* over $R[U]$ are locally equivalent, if there exist maps $f_* : C_* \to D_*$ and $g_* : D_* \to C_*$ such that $g_* f_*$ and $f_* g_*$ are the identity over $R[U, U^{-1}]$.

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- Applied for Khovanov homology (—, Dai, Mallick, Stoffregen, work in progress);

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Theorem (—, Liu, Zemke)

For algebraic links, the link Floer complex is determined by the multivariable Alexander polynomial.