Heegaard Floer Theory and algebraic geometry

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Dictionary

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normal surface singularity	

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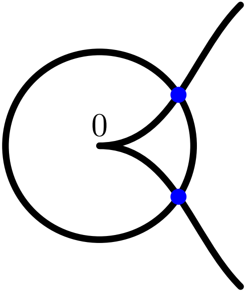

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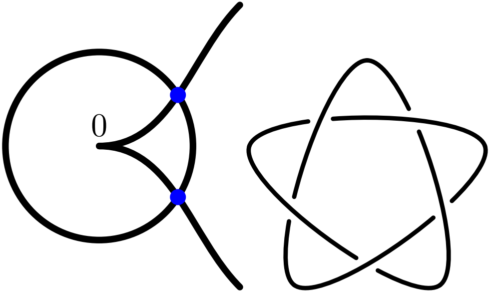

Semigroups

Theorem (Capillo, Delgado, Gussein-Zade, ~2000)

Suppose (C, z) is a unibranched plane curve singularity with semigroup S and link K. Then

$$\Delta_K = 1 + (t-1)\sum_{\substack{g>0 \\ g\notin S}} t^g.$$

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Corollary

Multiplicity of a plane curve singularity is determined by the Alexander polynomial.

Semigroup semicontinuity

Theorem (Gorsky, Némethi 2014)

Suppose C_t is a deformation of a plane curve singularities with central fiber C_0. Let S_t be the semigroup of C_t. Then, for $|t| \ll 1$ and any $x > 0$

$$\#\{\ell \in S_t : \ell < x\} \geq \#\{\ell \in S_0 : \ell < x\}.$$

In particular, multiplicity cannot drop.

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The definition of g_4 is depends on the world we live in.

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Definition

Suppose K_0, K_1 are algebraic knots. We say that there is a topological deformation from K_0 to K_1, if there exists a cobordism (a surface Σ in $S^3 \times [0, 1]$) of genus $g_3(K_0) - g_3(K_1)$ between them.

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Question

Is the multiplicity semicontinuous under such topological deformations?

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We use Heegaard Floer homology techniques.

Heegaard decomposition

Definition

A *Heegaard splitting* of a 3-manifold Y is a presentation of Y as a boundary connected sum, $Y = Y_+ \cup_h Y_-$, where Y_+ and Y_- are handlebodies, $h\colon \partial Y_+ \xrightarrow{\cong} \partial Y_-$ is a diffeomorphism. We denote $\Sigma = \partial Y_+ = \partial Y_-$. It is a genus g surface.

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Example

Two balls glue to a sphere giving a genus 0 decomposition of S^3. Two solid tori can glue to S^3, $S^2 \times S^1$, or a lens space.

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Definition

The tuple $(\Sigma, \alpha, \beta, z)$ is called the *Heegaard data*.

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- Specify a divisor $R_z = \{z\} \times \Sigma^{g-1}$.

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Theorem (Ozsváth–Szabó 2002)

The chain homotopy type of the complex CF^-, ∂ does not depend on the Heegaard data.

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Theorem (Ozsváth–Szabó, 2002)

If $b_1(Y) = 0$, then for any $\mathfrak{s}$, $HF^\infty(Y, \mathfrak{s}) = \mathbb{F}[U, U^{-1}]$.

Spin-c cobordism

Theorem (Ozsváth–Szabó 2002)

Suppose $(W, \mathfrak{t})$ *is a spin-c cobordism between* $(Y_0, \mathfrak{s}_0)$ *and* $(Y_1, \mathfrak{s}_1)$*. Then, there is a map* $F_W^\bullet \colon HF^\bullet(Y_0, \mathfrak{s}_0) \to HF^\bullet(Y_1, \mathfrak{s}_1)$.

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- *If* $b_2^+(W) > 0$, *then* F^∞_W *is a zero map;*
- *There is a consistent choice of grading such that* $\deg F_W = \frac{1}{4}(c_1(\mathfrak{t})^2 - 2\sigma(W) - 3\chi(W))$ *(*U *drops the grading by* 2*).*

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- If Y bounds a $\mathbb{Q}$-homology ball $\mathbb{Q}$-acyclic surface, then $d(Y, \mathfrak{s}) = 0$ for all spin-c structures that extend over balls.

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Corollary

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- Fix a characteristic $\mathfrak{k} \in L^*$ such that $\mathfrak{k}(x) \cong \langle x, x \rangle \bmod 2$ for all x;
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Definition

The lattice chain complex for $L, \mathfrak{k}$ is defined as the complex $\mathbb{F}[U]$ generated by unit cubes, with differential

$$\partial r = \sum \epsilon_{r'} r' U^{\chi_{\mathfrak{k}}(r) - \chi_{\mathfrak{k}}(r')},$$

where r' runs over facets of r and ϵ is a sign choice.

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- Generalizations to relative case due to –, Liu, Zemke (2024, to appear).

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- Can we use LH to prove μ-const conjecture?