Reachability in Symmetric VASS

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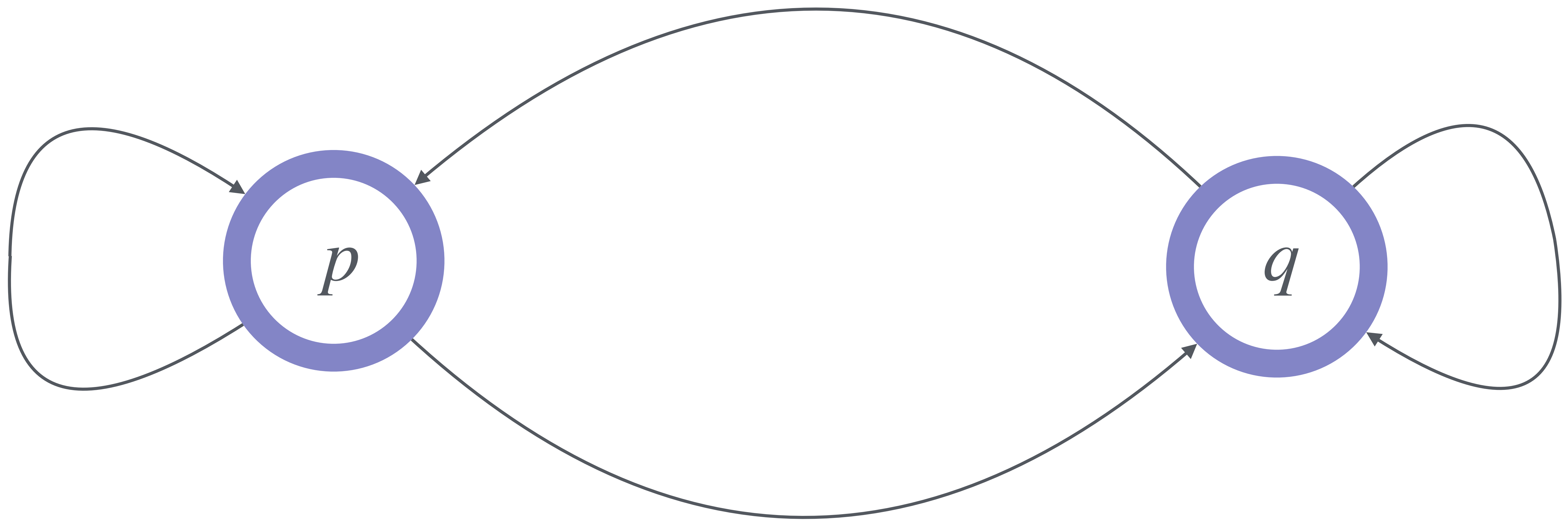
University of Warsaw

MFCS'25

VASS



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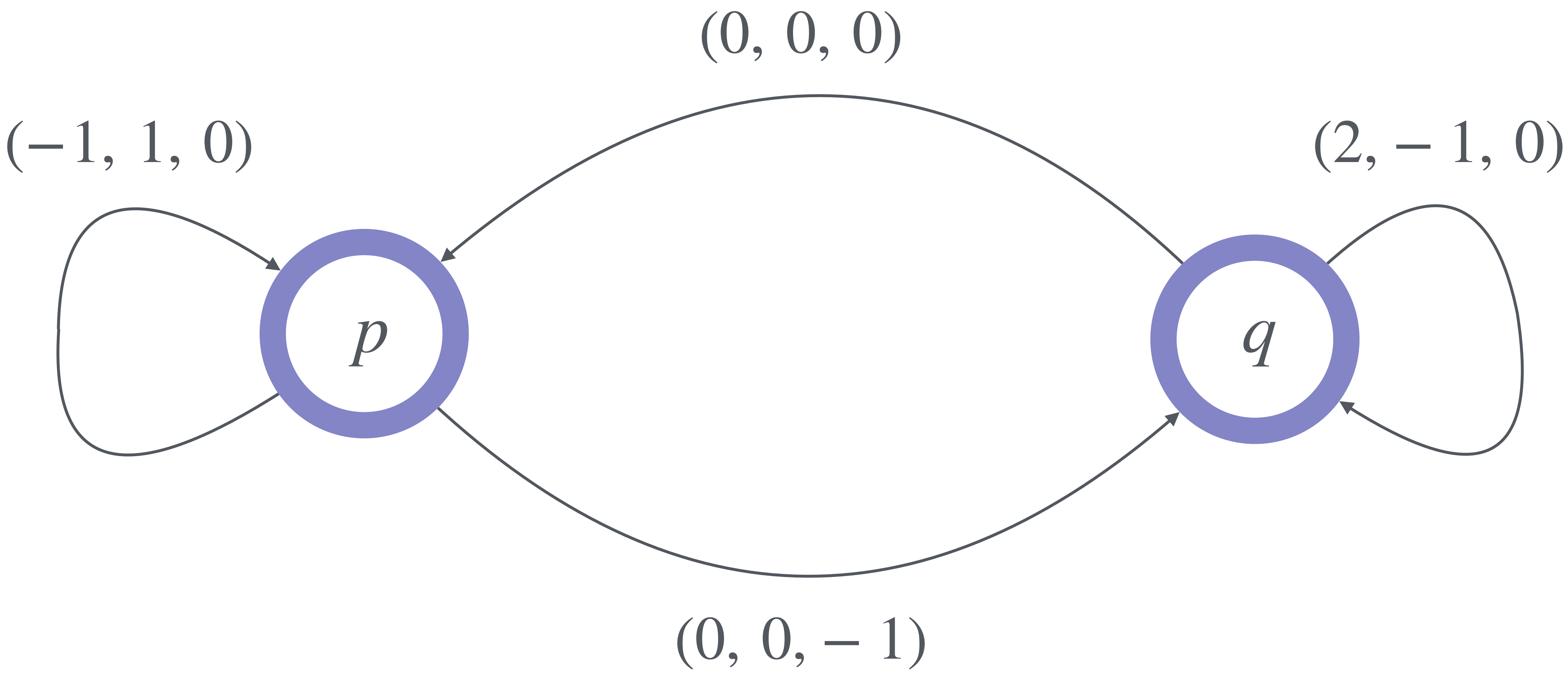

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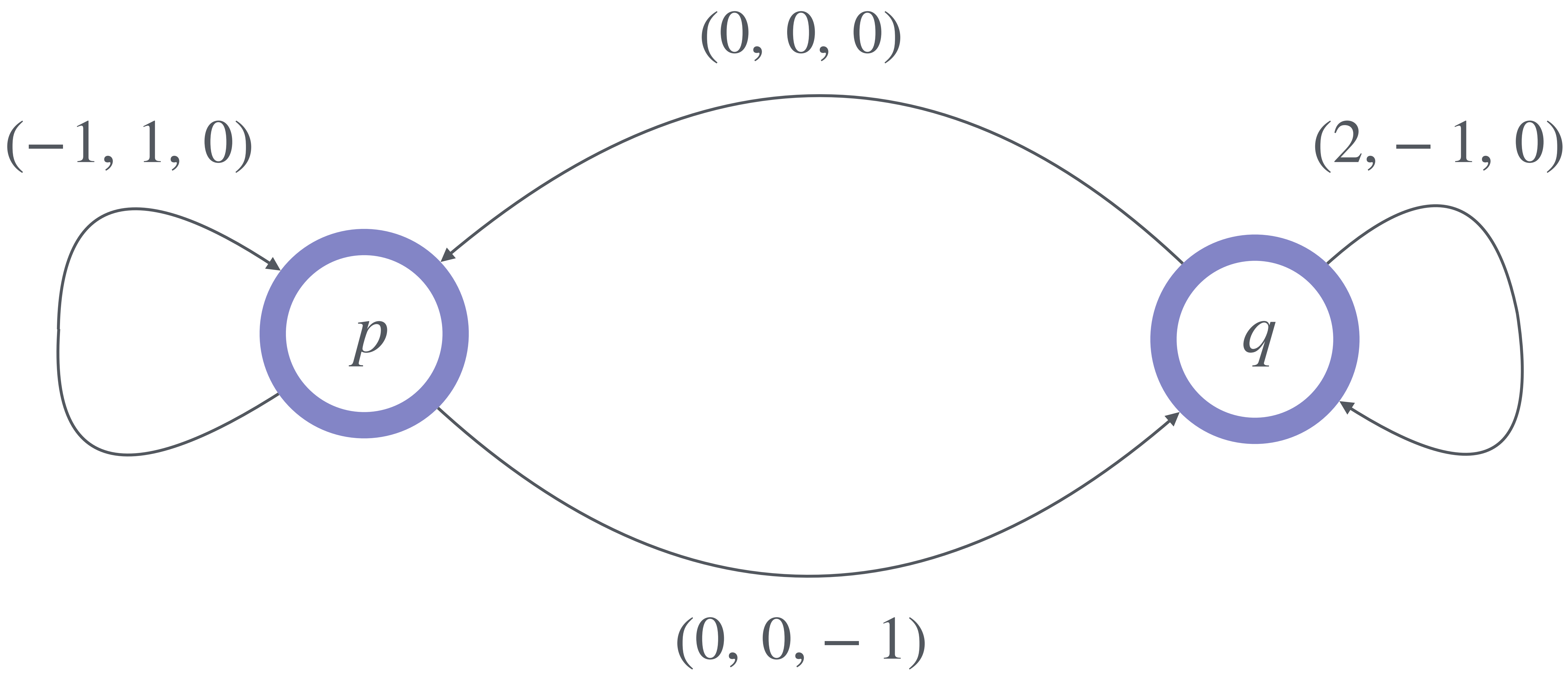
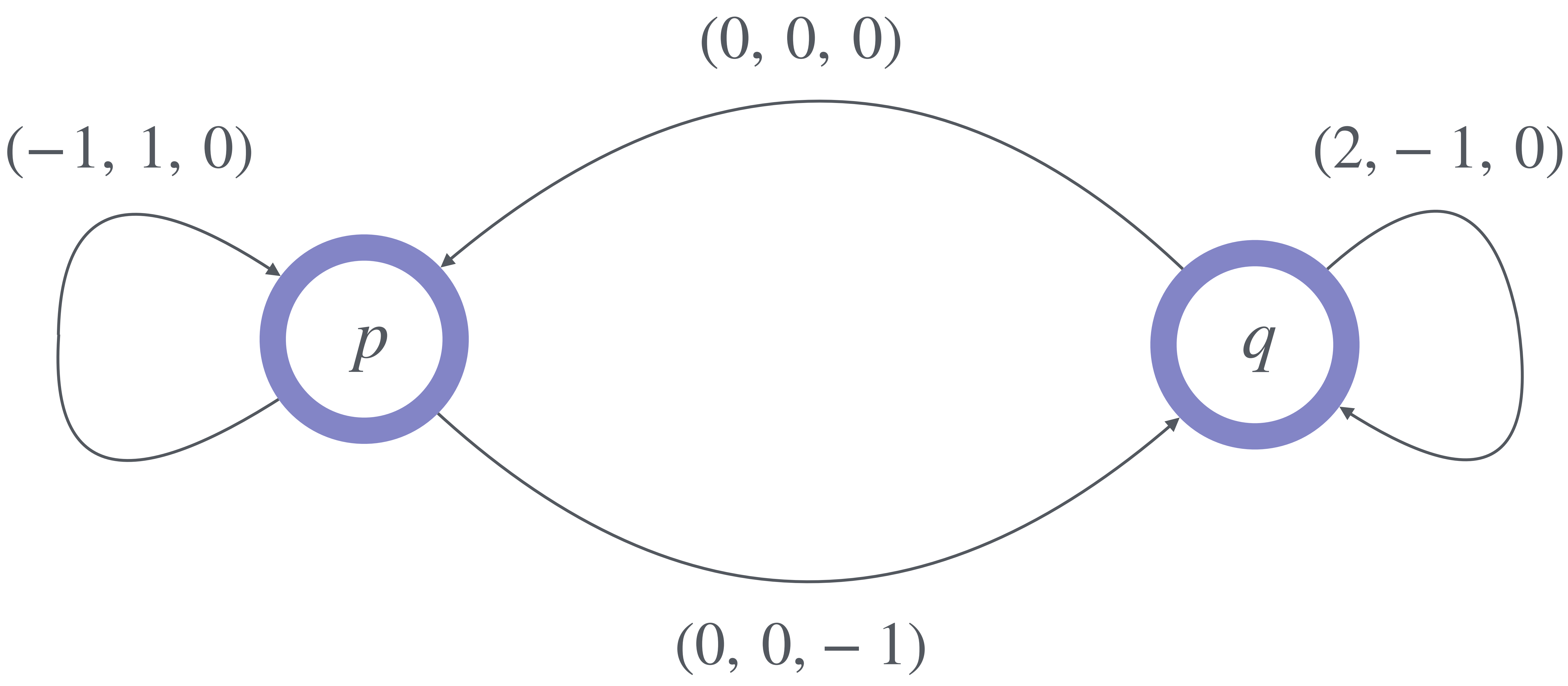


Configuration

$p(\mathbf{v}) \in Q \times \mathbb{N}^d$

VASS



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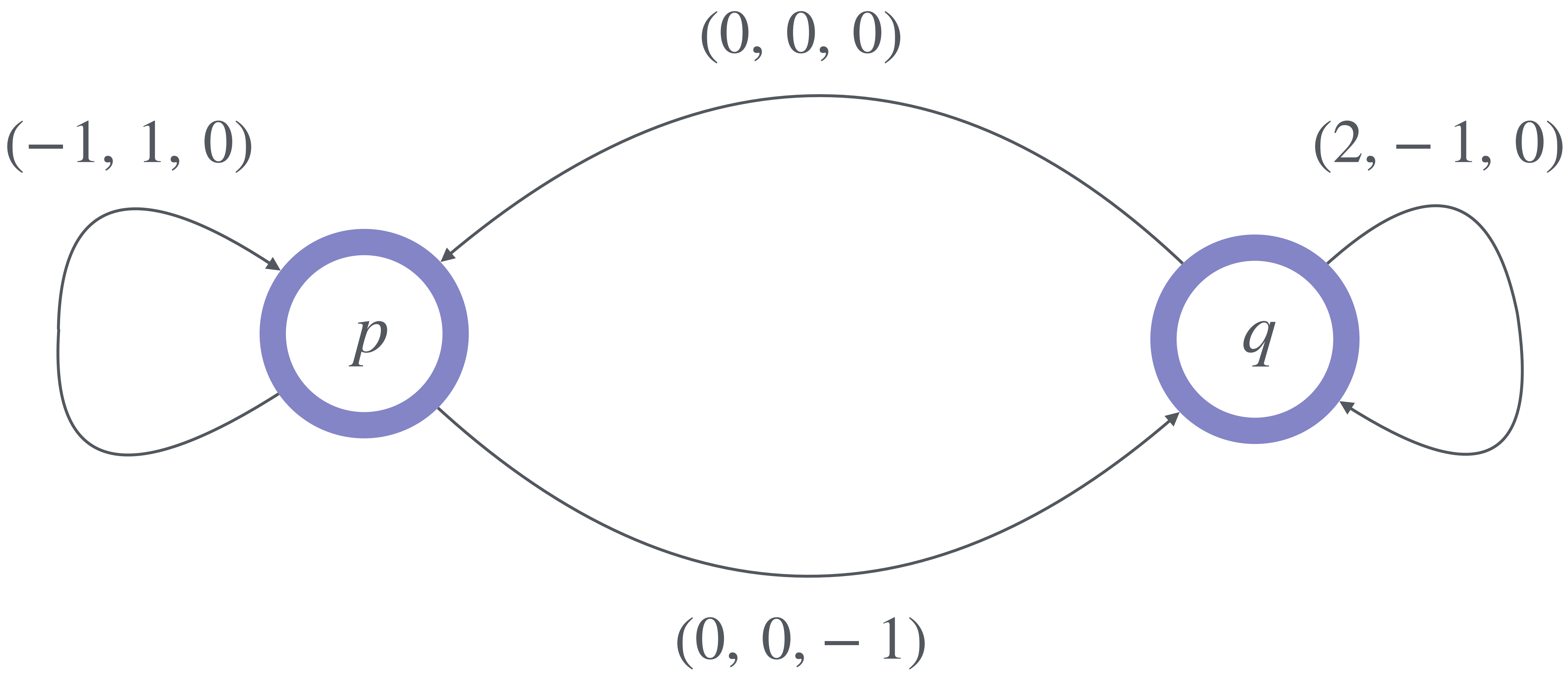
VASS (d-VASS)



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set of states

dimension

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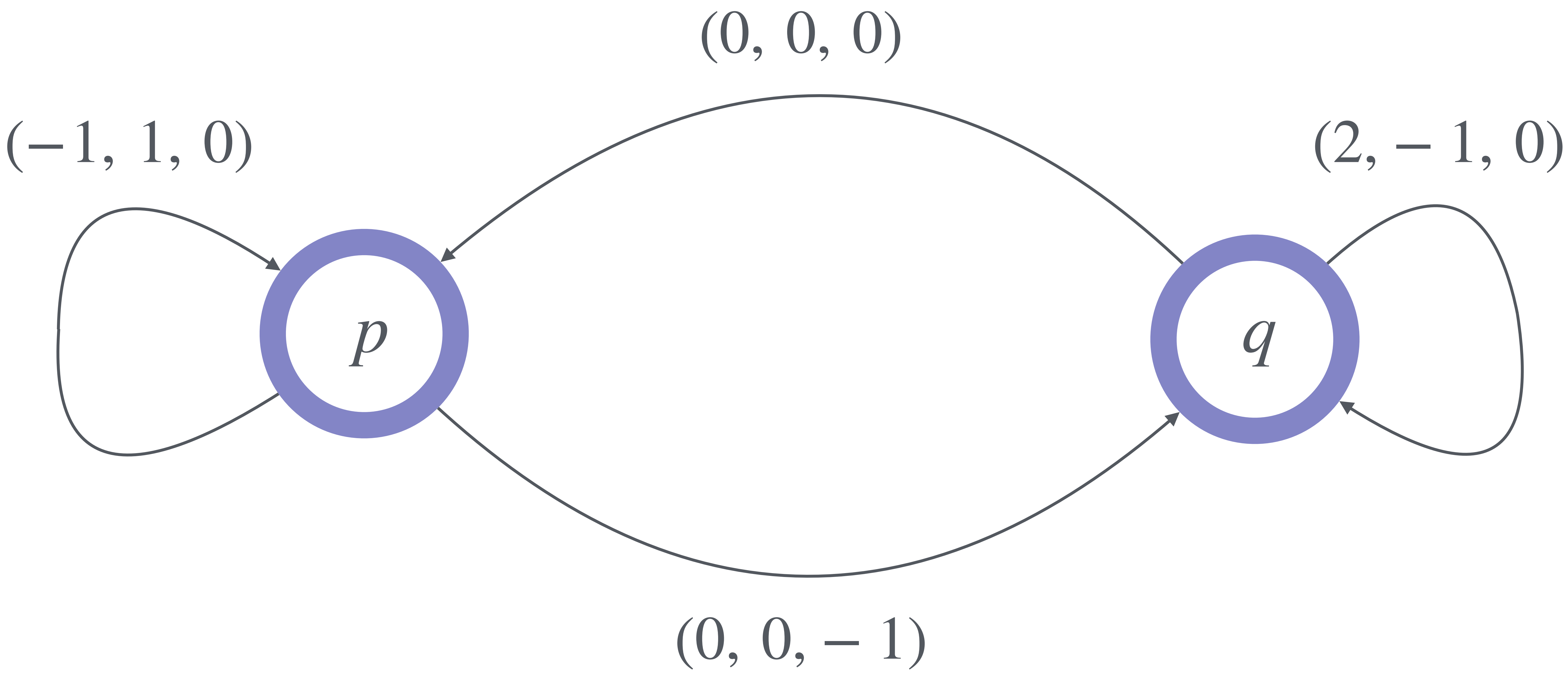

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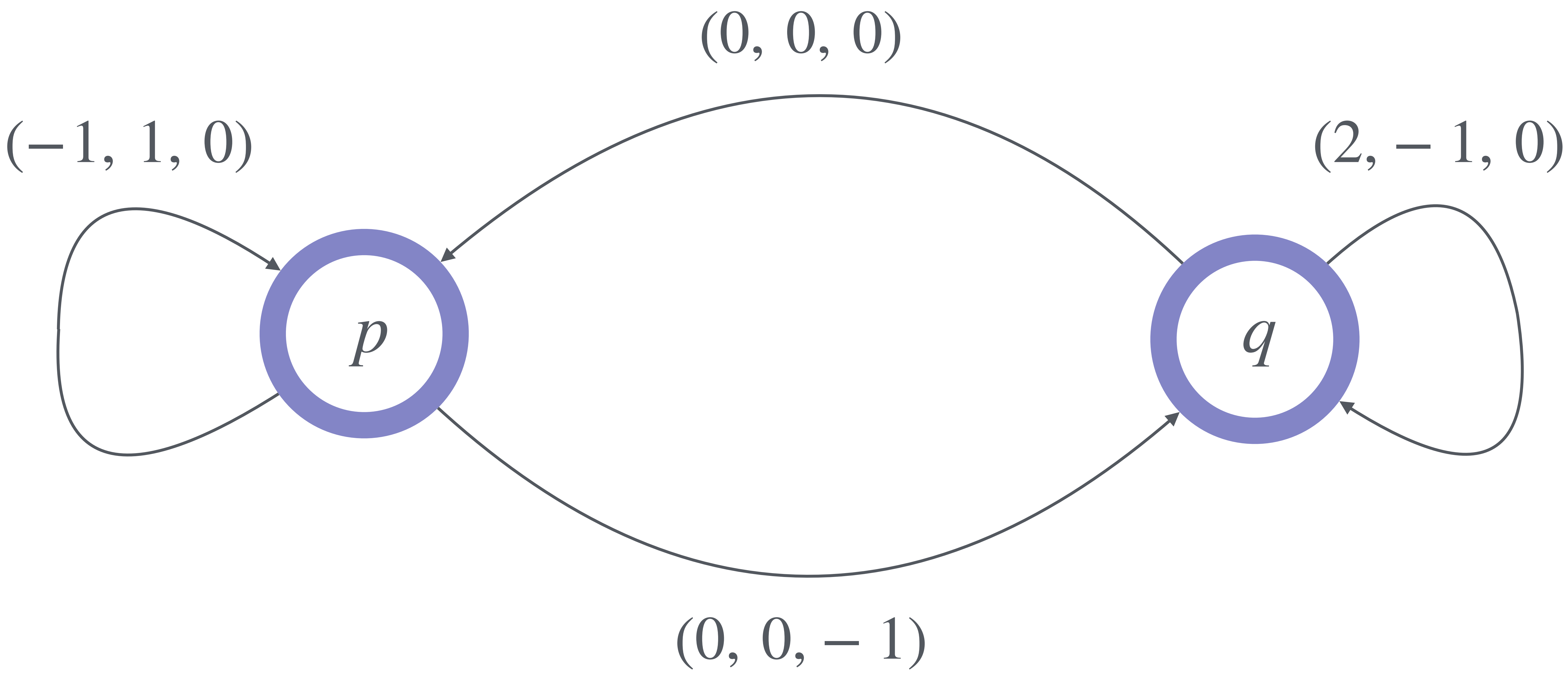

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Run

$p\ (1, 0, 1)$

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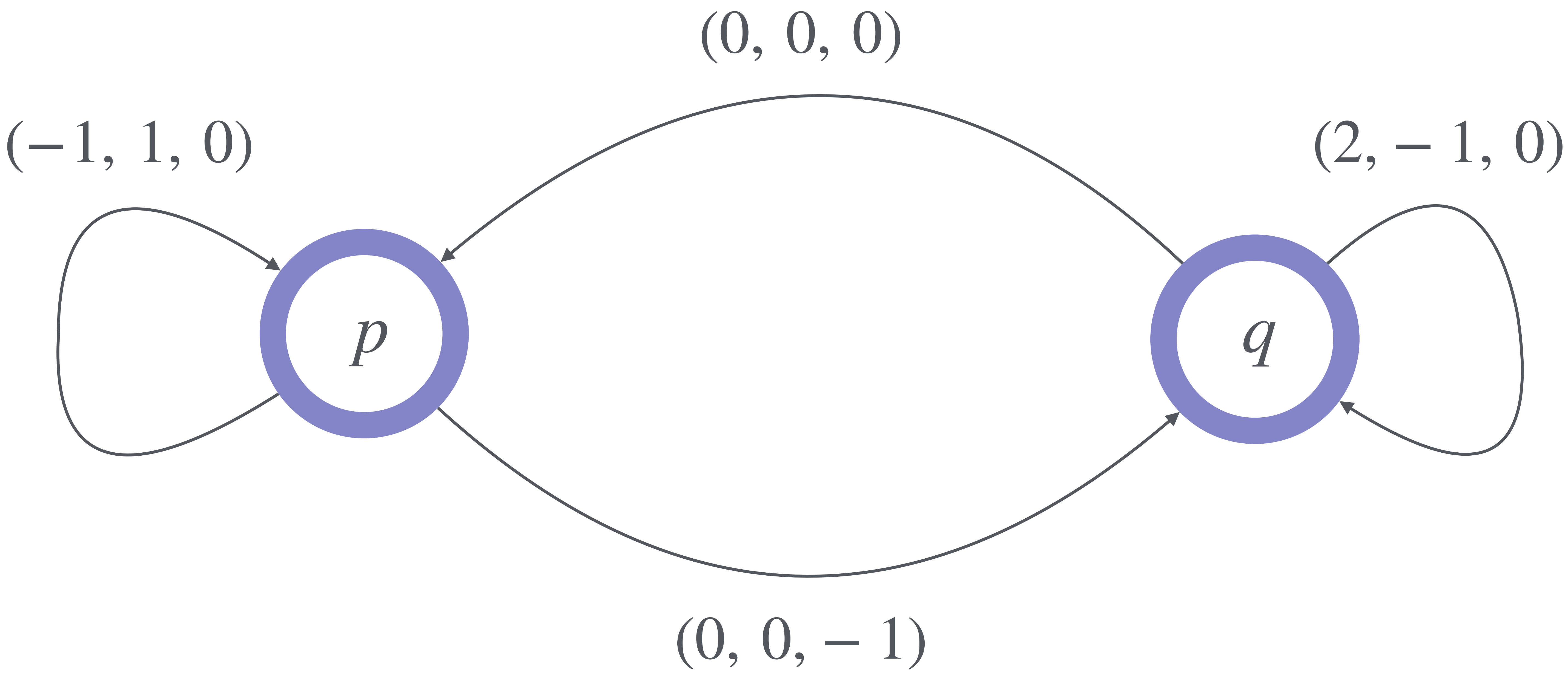

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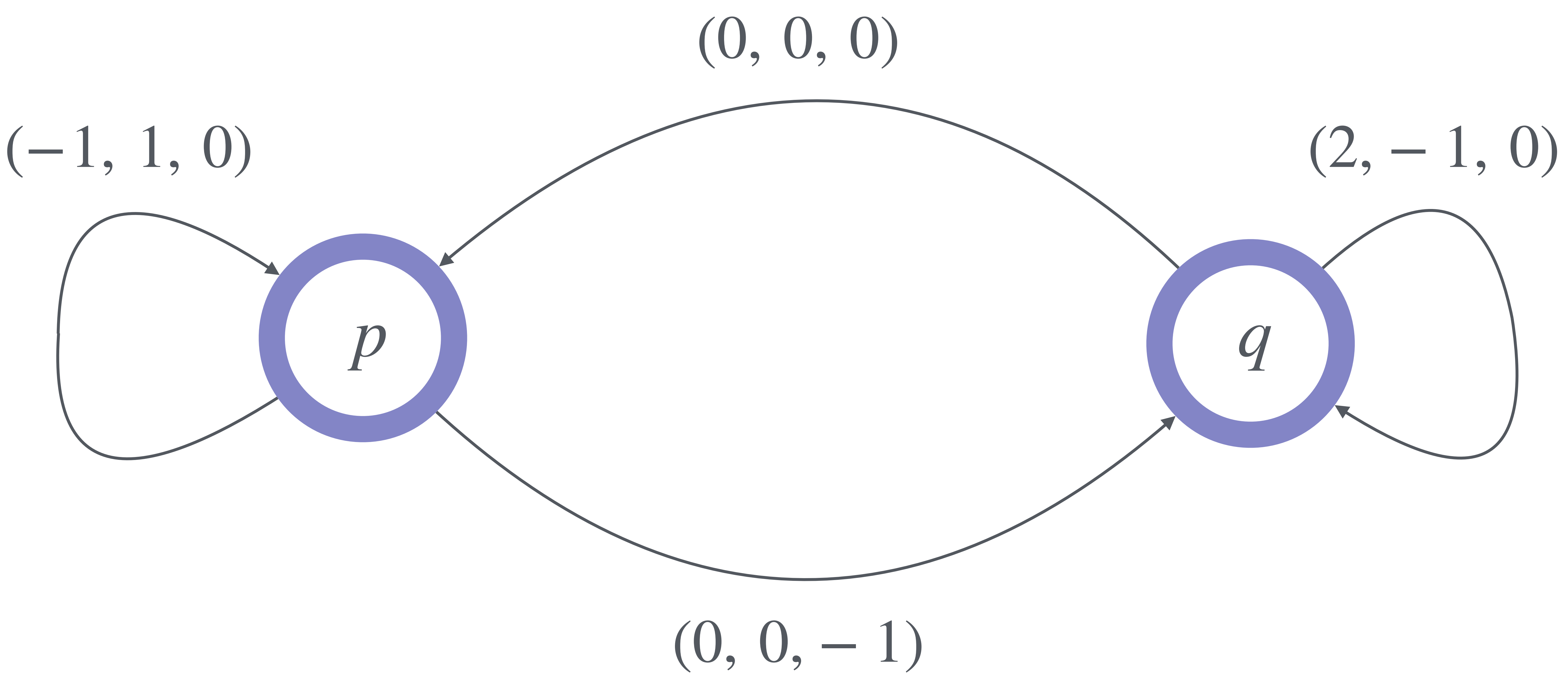

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Run

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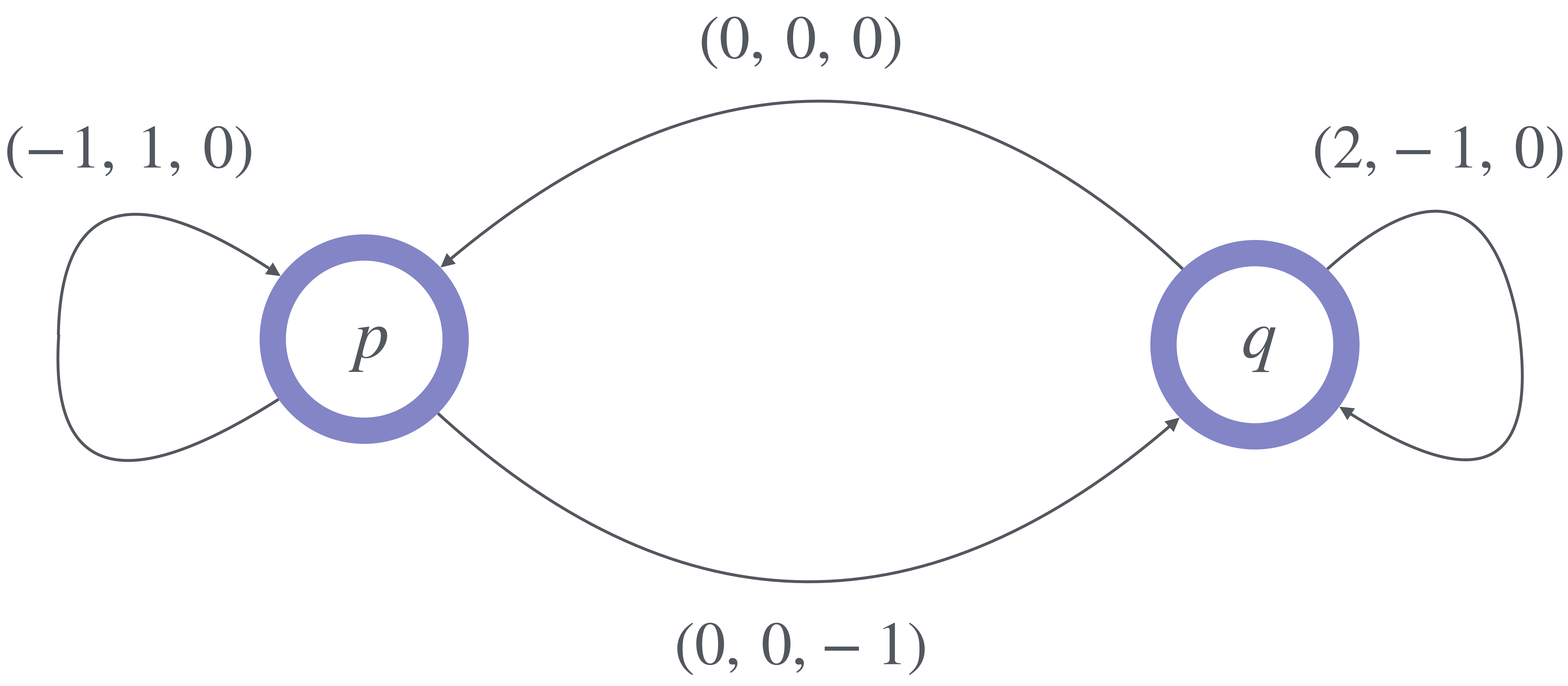
$q\ (0, 1, 0)$

VASS (*d*-VASS)



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set of states

dimension

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$q\ (0, 1, 0)$

$q\ (2, 0, 0)$

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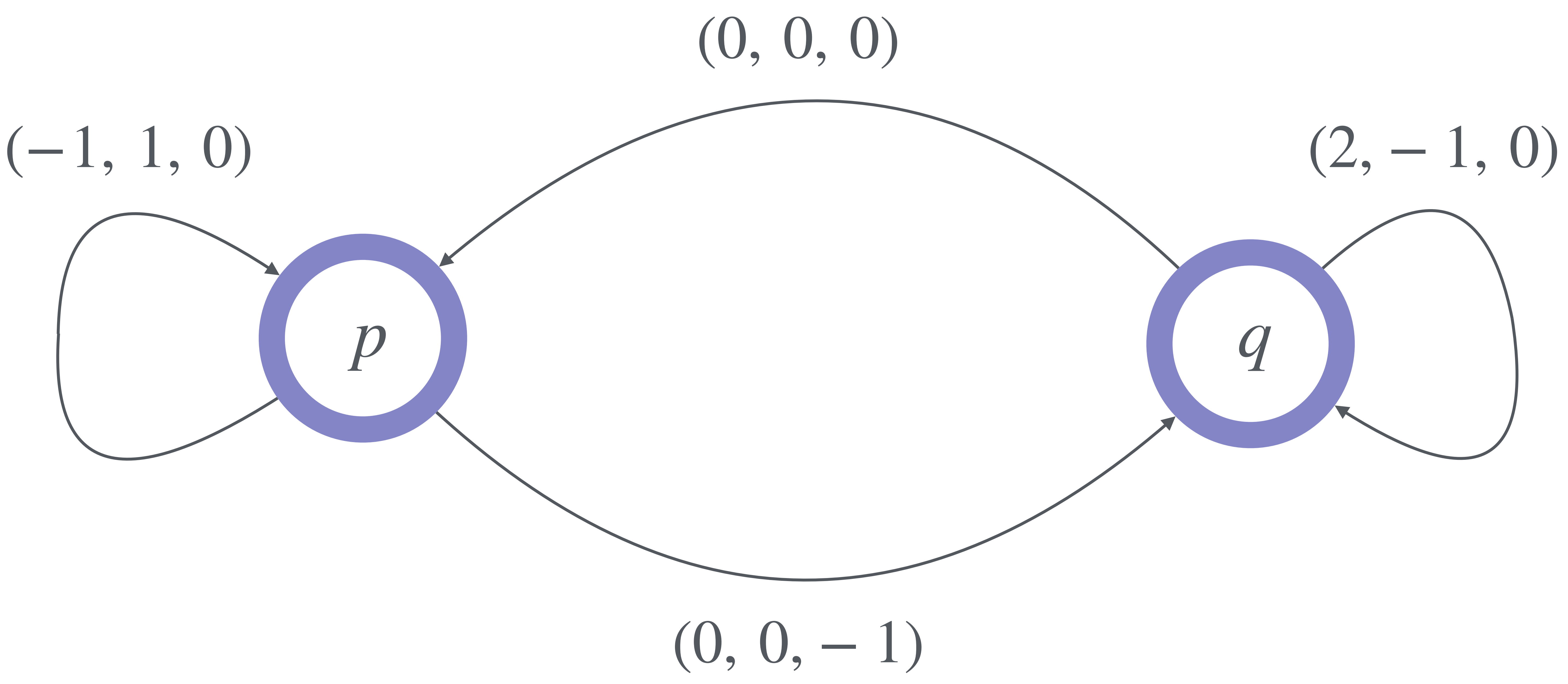

Configuration

$p(\mathbf{v}) \in Q \times \mathbb{N}^d$

set of states

dimension

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$p\ (0, 1, 1)$
$q\ (0, 1, 0)$
$q\ (2, 0, 0)$
$p\ (2, 0, 0)$

Reachability problem

Input: *a VASS, initial and final configurations*

Output: *is there a run from the initial to the final configuration?*

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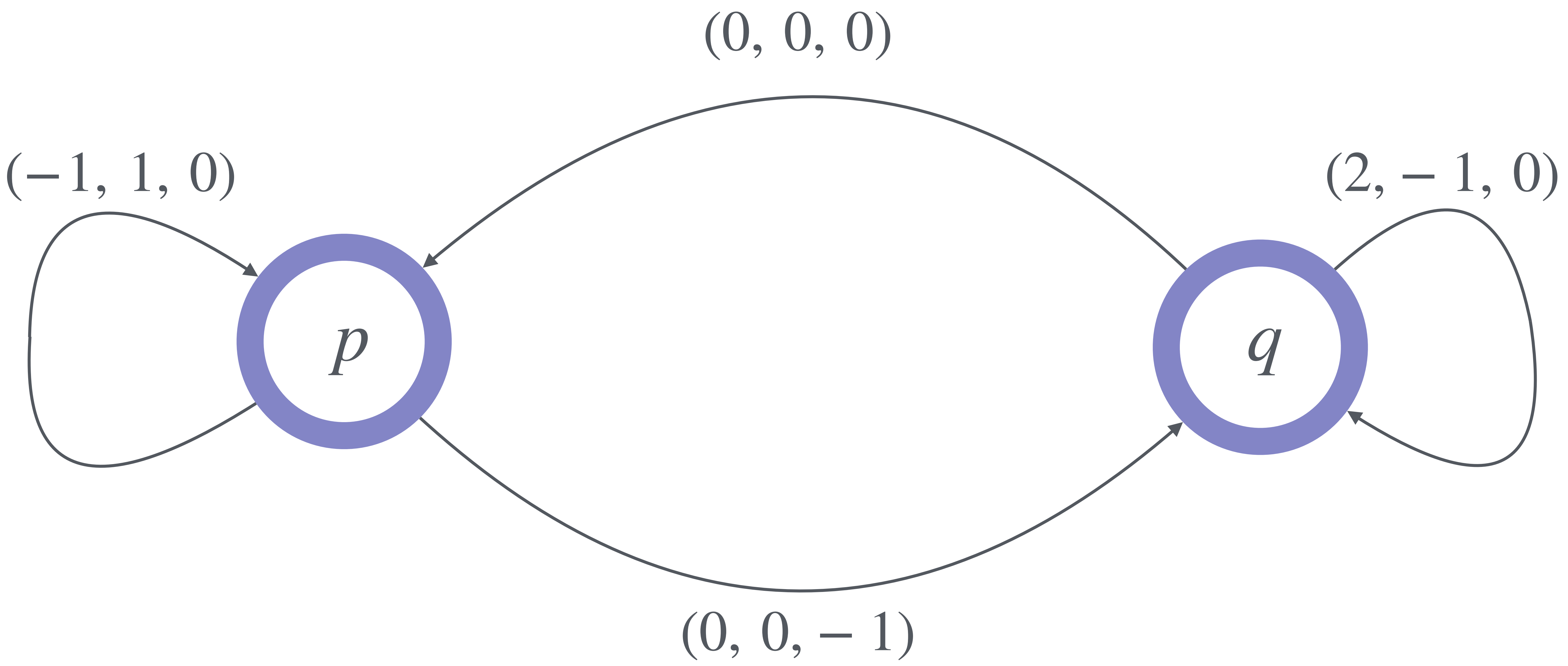
fast-growing hierarchy of complexity classes

Symmetric VASS



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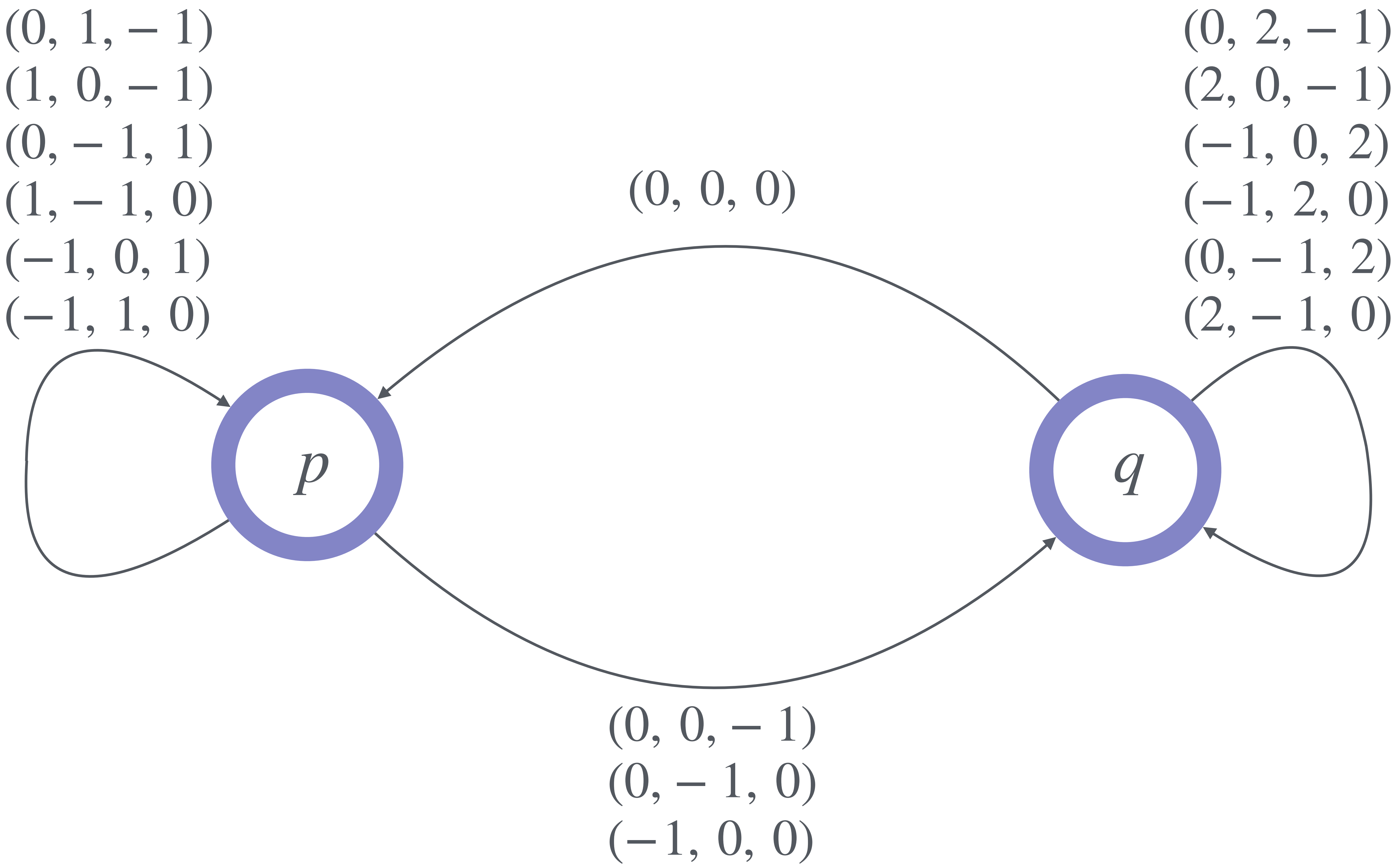
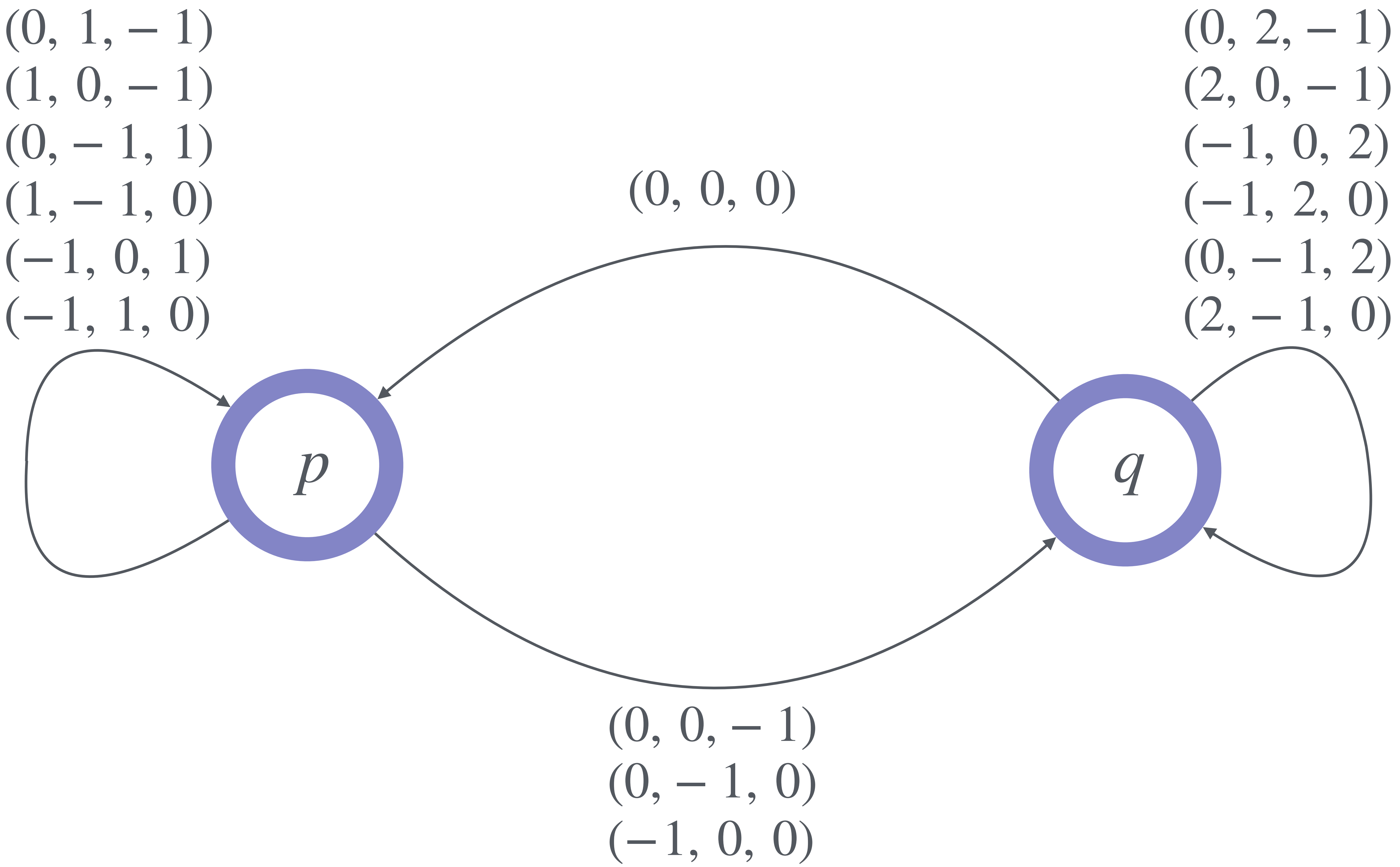


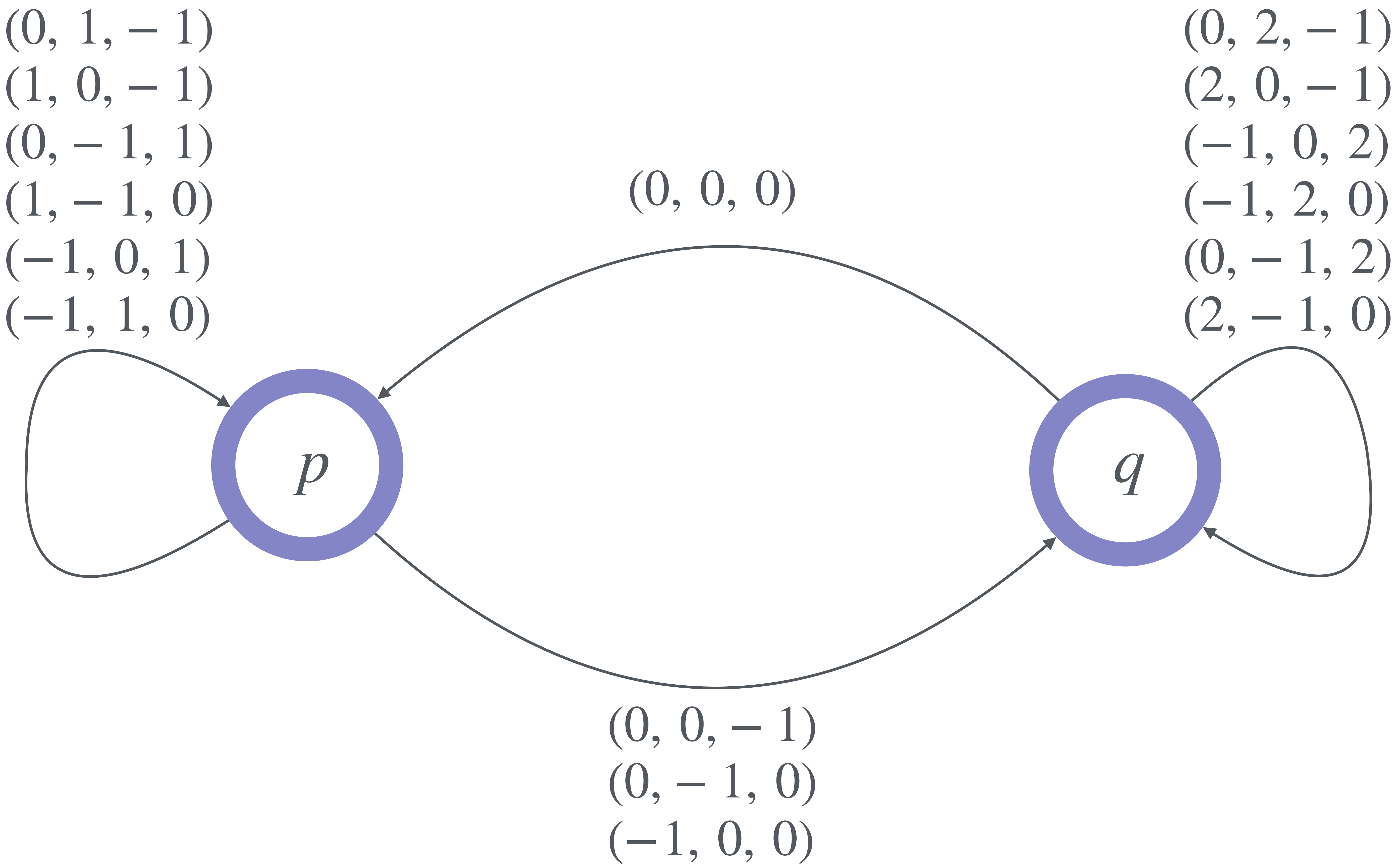
Symmetric VASS S_d-VASS



Symmetric VASS

S_d-VASS

the group of all permutations of *d*-element set



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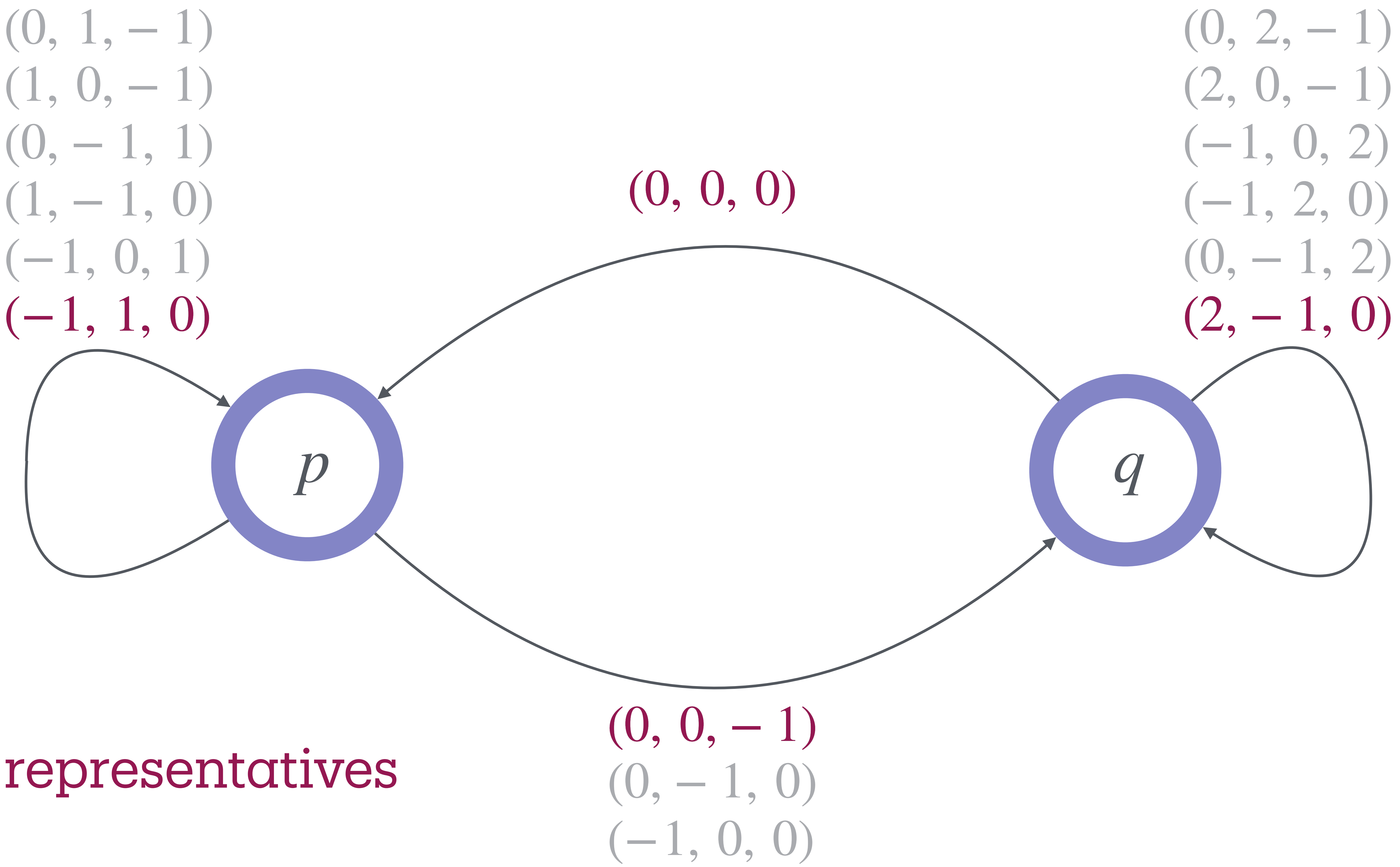

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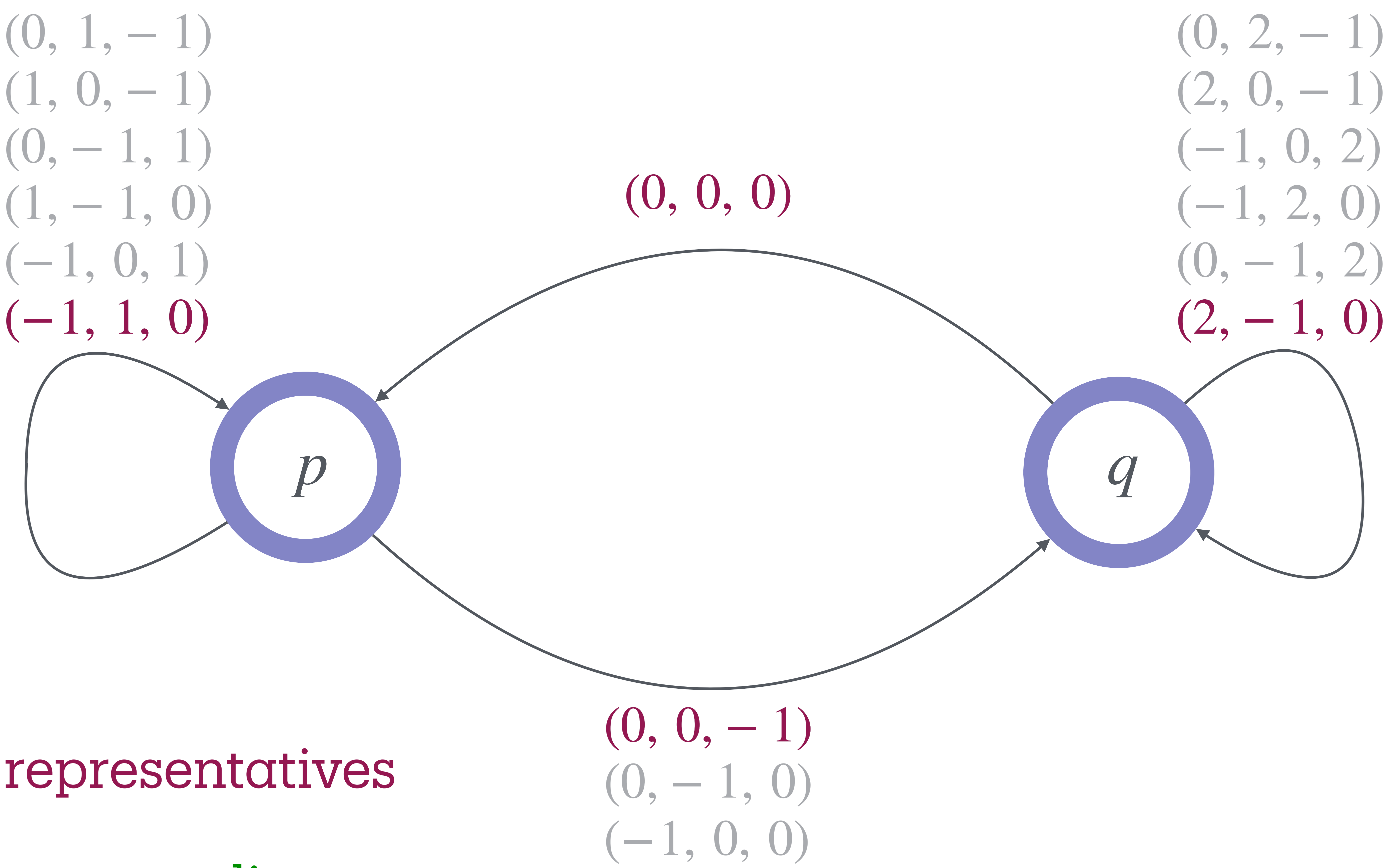

Input contains only representatives

We use binary encoding

Symmetric VASS

S_d -VASS

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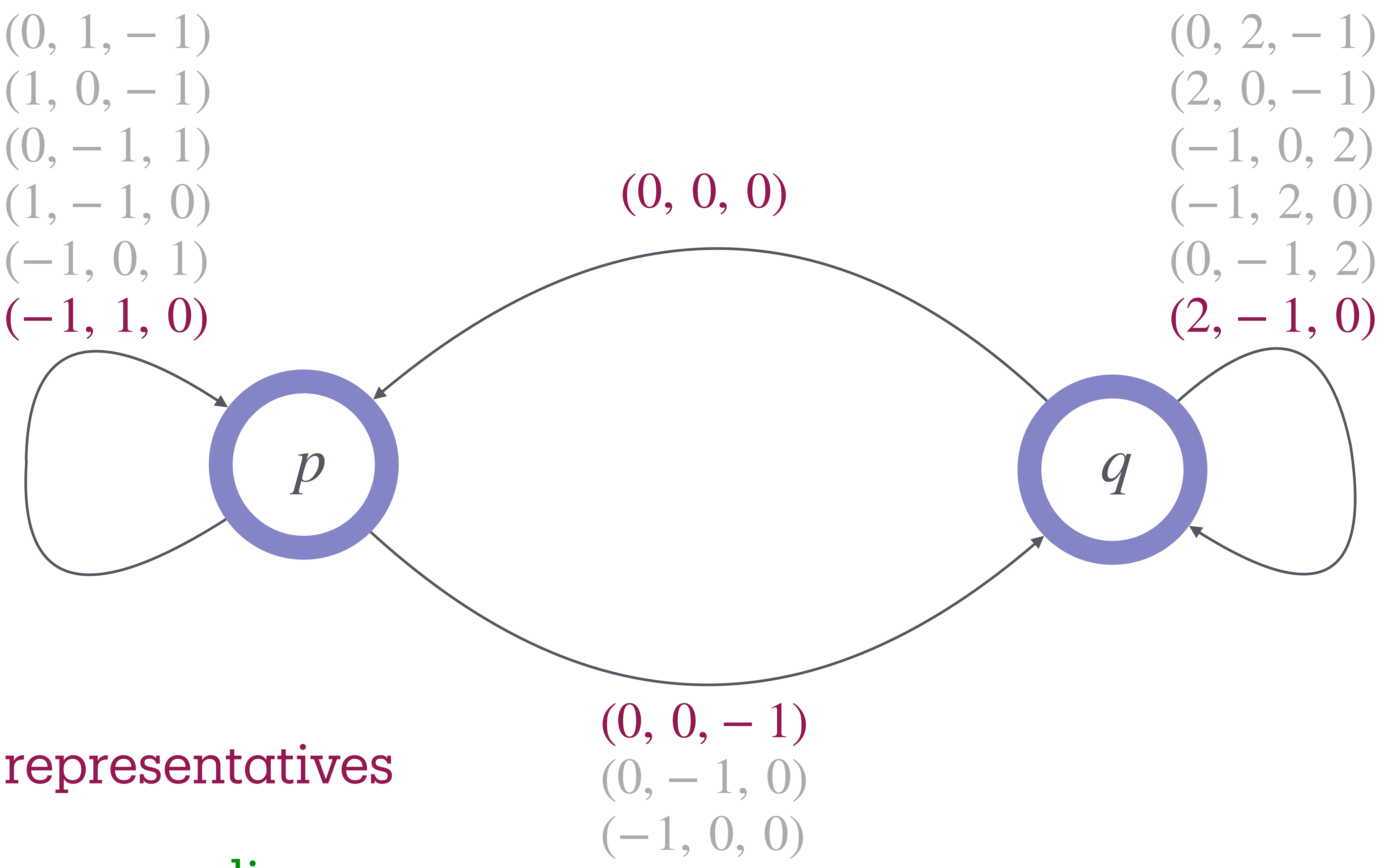
Similarly we define G-VASS for $G \leq S_d$



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Main results

Theorem

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Remark: it improves the PSPACE-hardness of *2-Reach*.

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Remark: for $d = 1$ it is NP-complete.

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Easy

S_d A_d

Hard

Z_d

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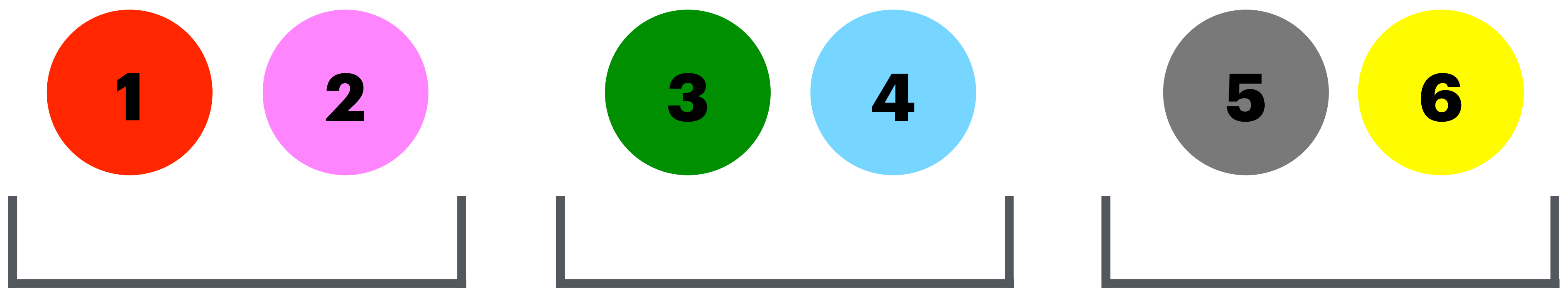

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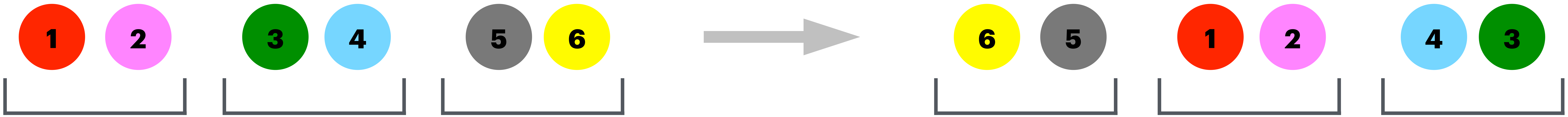

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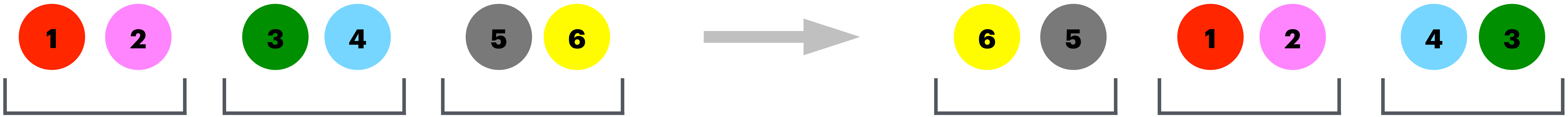

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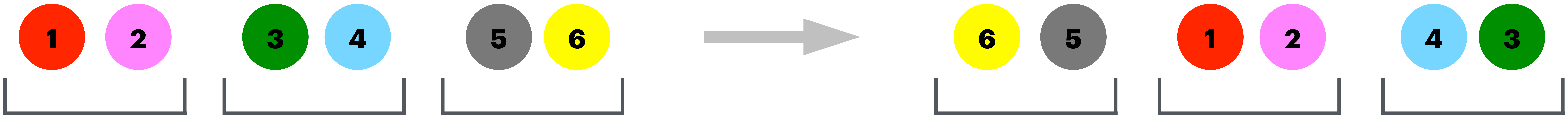

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Trivial group

Theorem

$(d-1)n$-Reach reduces in polynomial time to $(I_d \wr S_n)$-Reach, for $d \geq 2$.

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$(d-1)n$-*Reach reduces in polynomial time to* $(I_d \wr S_n)$-*Reach*, for $d \geq 2$.

Theorem

$(d-1)n$-Reach reduces in polynomial time to $(I_d \wr S_n)$-Reach, for $d \geq 2$.

Theorem

$(S_n \wr I_d)$-Reach reduces in exponential time to d-Reach.

Theorem

$(d-1)n$-Reach reduces in polynomial time to $(I_d \wr S_n)$-Reach, for $d \geq 2$. ✗

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Proofs

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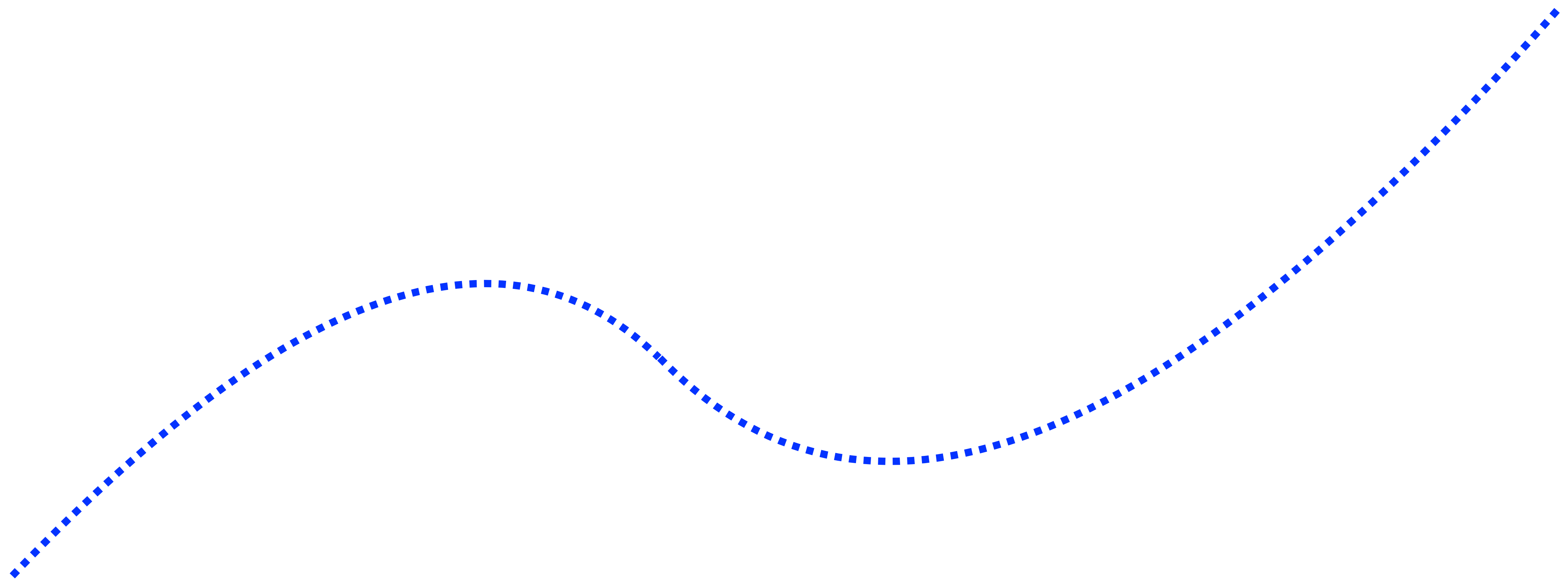

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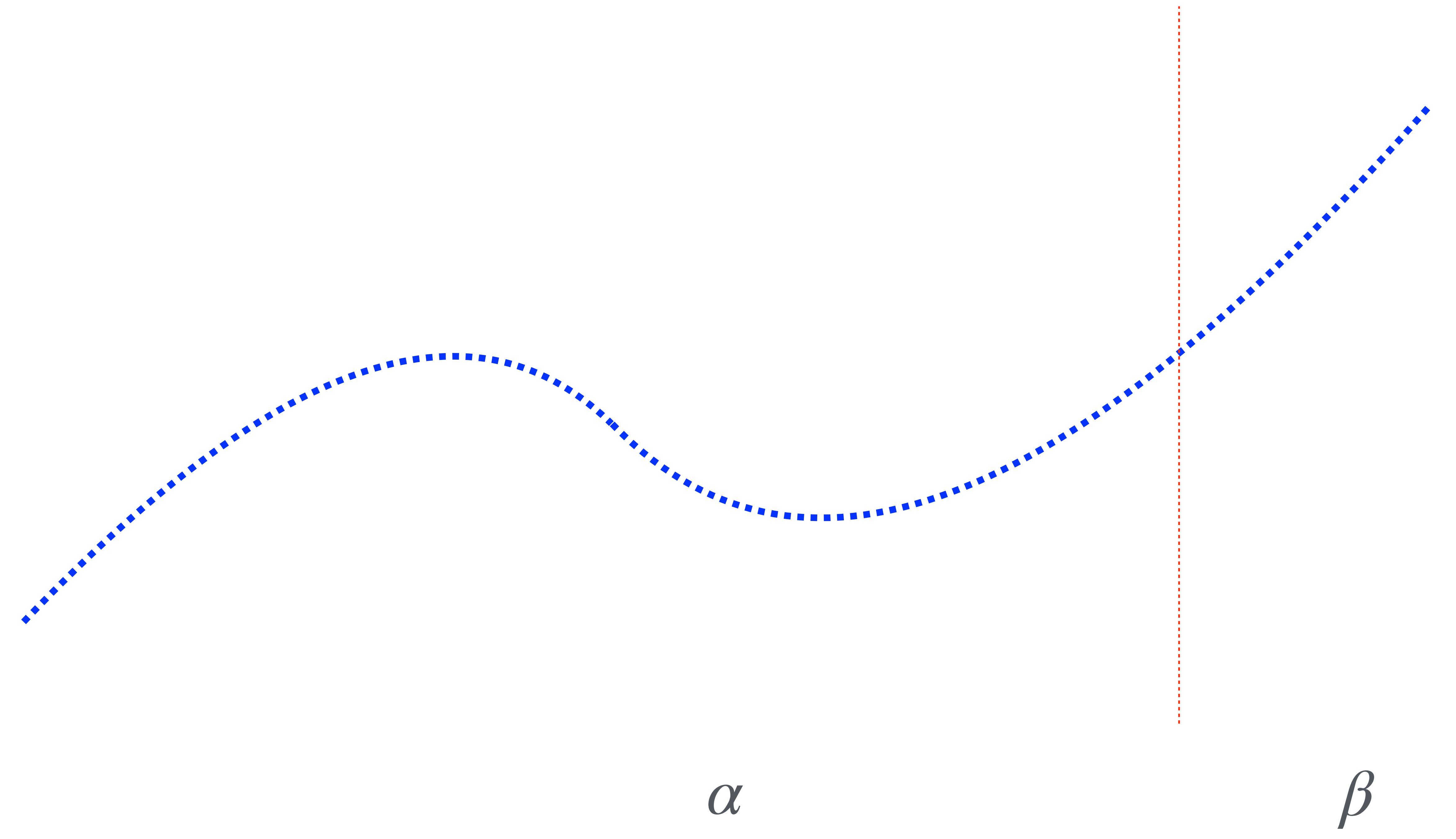

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Key idea for upper bound: *fairness* + *pumping*

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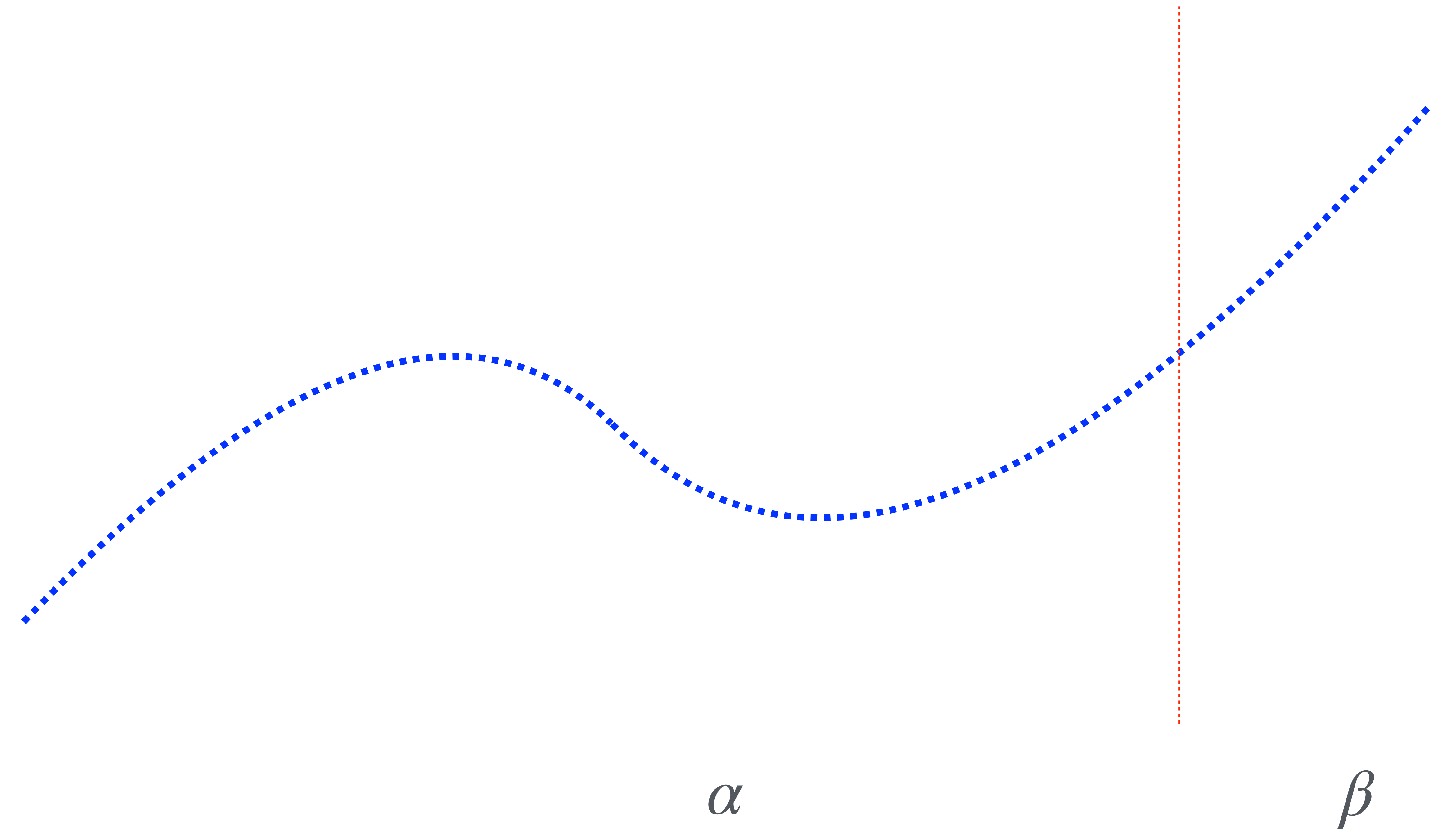

Summary and open problems

Easy			Hard	
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Is there a parameter of a group G which determines the complexity of *G-Reach*?

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The complexity in particular cases.

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- $S_n \wr S_d$

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Thank you!