Set Theory
Homework for March 23, 2016

Ordinal exponentiation α^β, $\alpha \neq 0$, is defined by:

$$\begin{aligned} \alpha^0 &= 1 \\ \alpha^{\beta+1} &= \alpha^\beta \cdot \alpha \\ \alpha^\gamma &= \sup_{\beta<\gamma}(\alpha^\beta) \qquad \text{for } \gamma \in \mathrm{Lim}. \end{aligned}$$

10. Consider the length-lexicographic order on the set $\omega^{<\omega}$ of all finite sequences of natural numbers:

$$s \preccurlyeq t \Leftrightarrow [(\mathrm{lh}(s) < \mathrm{lh}(t)) \vee (\mathrm{lh}(s) = \mathrm{lh}(t) \wedge s \leqslant_{\mathrm{lex}} t)]$$

(lh stands for the length of a sequence). Prove that the order type of $\langle \omega^{<\omega}, \preccurlyeq \rangle$ is ω^ω.

11. What is the order type of the set $\{\alpha \in \omega^\omega : \alpha \text{ is a limit ordinal}\}$?

12. Prove that α^β is the order type of the set of all functions from β to α with finite support (i.e. f is non-zero for only finitely many arguments) ordered antilexicographically:

$$f \preceq g \Leftrightarrow f = g \vee \exists \gamma\, [f(\gamma) < g(\gamma) \wedge \forall \delta > \gamma\, (f(\delta) = g(\delta))].$$

13. Prove that $1^\alpha + 2^\alpha = 3^\alpha$ for all limit α.

14. Prove that there exists α such that $\omega^\alpha = \alpha$.