Some problems worth to be given to students

Recently two students came to my office with a question: how one finds the limit of a sequence (1^n). They surprised me a little bit. Only a little bit because I am an experienced teacher so it is not very easy to ask a completely unexpected question. After a minute or so it turned out that according to them the limit should equal 1^∞ and this is one of indeterminate forms. More or less at the same time a friend of mine told me that his wife who teaches at a high school and also her friends have a problem how to evaluate math problems that were proposed to high school students as a trial before matura, final secondary school examination, they asked whether or not it legal to talk about e.g. $(-4)^5$. Their real problem was: an exponential function a^x is defined for positive a, so not for -4. These questions can arise only because when new notions are introduced the reasons for the definitions or restrictions in the definition are not given. Let me more precise. Everybody knows that id a is any number and n is a positive integer the $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ letters\ a}$ and in this definition it is absolutely irrelevant on whether a is positive, negative or zero. Later on at high school the definition is extended for other exponents. Usually a teacher does not say why it is done in the way he talks about it. For example why we define $a^0 = 1$ at least for $a \neq 0$. I tried to tell some authors of high school text books that they should write about it. They should write that they want to extend the definition of a power in such a way that the crucial properties of powers will not change. By some reason most of do not like this idea. I might say that I do not understand why it is so but since many years I know the reason. It is so because many teachers expect their students to trust what they say and they should not ask too many questions. In a sense they understand well students who expect simple and short rules of solving math problems, they are ready to memorize some formulas even when they are told it is unnecessary or stupid. They do it because they expect that they will forget it after an exam and they will never will need it again. It does not work this way at a university but it is hard to change one's way of thinking quickly. The authors of textbooks should say that we want to define a^x in such a way that the formula $a^{x+y} = a^x \cdot a^y$ will work for all exponents we accept. If one says it then the formula $a^x = a^{x+0} = a^x \cdot a^0$ must hold. Therefore $a^0 = 1$ at least for $a \neq 0$. There is no other choice. for $a = 0$ the situation is not clear at all but one can easily accept this exception. But this problem is not unique. $a^x = a^{x/2+x/2} = a^{x/2} \cdot a^{x/2} \geqslant 0$ and if $a^u = 0$, to $a = a^{1-u} \cdot a^u = 0$. This means that defining a^x for $a \neq 0$ one must avoid the equality $a^x = 0$. Shortly : if one wants to define a^x for all $x \in \mathbb{R}$ (or $x \in \mathbb{Q}$) he must assume that $a > 0$. At the end all powers of a will be positive. It should be said at some point because it explains why we choose powers to be positive.

But there is a problem. We write $\sqrt[3]{-8} = -2$ without any hesitation. For some time in Polish high schools it was not allowed, roots of negative numbers did not exist due to decision made by specialists on teaching mathematics. Fortunately now roots of odd degrees are allowed even in Poland, nonetheless many teachers who are used to previous theory do not allow them any way. Official theory of people in charge of the Polish examination system is that one may write $\sqrt[3]{-8}$ but not $(-8)^{1/3}$ due to problems with the definition of powers of negative numbers explained above. They will be forced to give up these theories because of computers and calculators. One should say that when we extend the definition of a power and allow noninteger exponents the some theorems on powers are no longer true. Of course we use notation $a^{p/q} = \sqrt[q]{a^p}$ provide p, q are relatively prime integers, q is positive and odd. This notation is convenient in many cases and it is used by many people. It should be used but students be warned that not everything is like in the case of positive a. One should tell that once we allow rational numbers in the exponent the we must be more cautious and say why it is so.

There is something strange with powers at least at Polish high schools. They never tell the students explicitly how to define e.g. $2^{\sqrt{3}}$ but they talk about logarithms. It is possible to give a definition saying that if $\frac{p}{q} < \sqrt{3} < \frac{r}{s}$, $p, q, r, s \in \mathbb{Z}$ and $q, s > 0$ then $2^{p/q} < 2^{\sqrt{3}} < 2^{r/s}$ and say that this defines $2^{\sqrt{3}}$ uniquely. We can do it because it is not hard to prove that id $a > 1$ then the function $x \mapsto a^x$ is strictly increasing on rational numbers. The uniqueness may be proven. The existence is strictly related to R. Dedekind axiom of continuity. This is not a very long story. Artificial treatment of powers leads to questions like the one I talked at the beginning.

By the way. At universities people like to use complex numbers. Also to use powers with complex exponents, at least e^z. One cannot speak of monotonicity because there is no inequality in $\mathbb{C}$. It should be replaced with another condition. The condition is differentiability e.g. at 0 i.e. $\lim_{z\to 0} \frac{e^z-1}{z} = 1$. This condition together with $e^{z+w} = e^z \cdot e^w$ for all $w, z \in \mathbb{C}$ defines e^z uniquely. This could be in fact a problem for good students. Also at the beginning one may ask the students what condition could replace monotonicity, but if we talk to freshmen it may be very hard. A proof may be based on the following statement: if $\lim_{n\to\infty} nz_n = 0$ then $\lim_{n\to\infty}(1 + z_n)^n = 1$. The power e^z may be defined either as $\lim_{n\to\infty}(1 + \frac{z}{n})^n$ or as $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. It is easy to see that $e^{\bar{z}} = \overline{e^z}$. This immediately implies that if $t \in \mathbb{R}$ then the following equality holds $\overline{e^{it}} = e^{\overline{it}} = e^{-it}$ and therefore $e^{it} \cdot \overline{e^{it}} = e^{it}e^{-it} = e^0 = 1$. Once we know this we may tell our students that the map $t \mapsto e^{it}$ describes the motion along the unit circle with constant speed: $|e^{i(t+h)} - e^{it}| = |e^{ih} - 1|$ for all real numbers t. We may immediately add that the instantanous speed at time t equals to $\lim_{h\to 0} \frac{1}{h}\left(e^{i(t+h)} - e^{it}\right) = ie^{it}$ so this velocity is represented by a vector perpendicular to e^{it} as it should be. I think of telling this story to freshmen so to people who do not know derivatives or integrals. In fact one can think that this should prepare them for the definitions of derivatives and integrals. The distance traveled at time (from 0 to t) is the least upper bound of lengths of the polygonal curve with the vertices $1 = e^{i0}$, e^{it_1}, e^{it_2},..., $e^{it_n} = e^{it}$ where $0 < t_1 < t_2 < \ldots < t_n = t$. This is efficient way of defining sine and cosine. Unfortunately it is not easy to define them precisely using geometric notions only. The problem is the notion of an oriented angle. By the way, one of the first mathematician defining sine and cosine during the course of Mathematical Analysis as sums of power series was Edmund Landau. He was critisized in the following way by Bieberbach.

„Furthermore Bieberbach used psychological typology to justify the expulsion of non-Aryans from academic posts in Germany and gives a justification of the student boycott led by Teichmüller of Landau's calculus course, stating that Landau is an S-type and gives an example of how he defined the trigonometric functions sine and cosine using series and not the geometric definition and π as twice the smallest positive root of the cosine function: „Our nature becomes of itself in the malaise produce by alien ways. There is an example in the manly rejection of a great mathematician, Edmund Landau, by the students of Göttingen. The un-German style of this man in teaching and research proved intolerable to German sensibilities. A people which has understood how alien lust for dominance has gnawed into its vitals must reject teachers of an alien type"[1]

So you may see some connections of some mathematicians and politics. Also as you see I suggest to go one step further: not only to use a series but also complex numbers.

I think that it is worth to show students connections between different mathematical notions or disciplines. Here is quite fresh example of using complex numbers as a tool that allows to solve a geometric problem quite nicely, Polish Mathematical Olympiad, February 9^{-th}, problem 3, the hardest on day 1 of this competition.

The bisector of the segment BC meets a circumcirle of the triangle ABC at points P ang Q so that A and P lie on one side of the line BC. A point R is a perpendicular projection of the point P onto the line AC. A point S is a midpoint of the segment AQ. Prove that the points A, B, R and S lie on one circle.

I am not going to show a geometric solution of the problem, it is not very long but it requires some idea which does not necessarily come to ones mind in a second. I shall

[1] https: // www.gla.ac.uk/ media / media_338601_en.pdf

show a solution using complex numbers which I was going to show to competitors after the the competition. I did not do it because one of the students participating in the event did by himself essentially in the same way I did it. It is necessary to say that out of 655 competitors 87 people solved the problem, so the problem was quite hard but 13,2% is not much. Solutions were different. I do not know how many of the competitors used complex numbers, at least one of them for sure.

I assume that points $A, B, C \ldots$ are complex numbers and I make no distinction between a point on the plane and the complex number associated to it. With no loss of generality I may assume that $|A| = |B| = |C| = |Q| = |P| = 1$ i.e. a circumcircle of the triangle ABC is centered at 0 and its radius is 1 (it is the unit circle). Moreover I assume that $B = -\overline{C}$, i.e. B and C are symmetric relative to the imaginary axis, $P = i$, $Q = -i$. Let us mention that if $K \neq L$ and Z are different complex numbers with $|K| = |L| = 1$ and W is symmetric to Z relative to the line KL then

$$(S) \qquad W = \overline{(Z-K)\cdot\frac{|L-K|}{L-K}}\cdot\frac{L-K}{|L-K|} + K = \frac{(\bar{Z}-\bar{K})(L-K) + K(\bar{L}-\bar{K})}{\bar{L}-\bar{K}} =$$

$$= \frac{\bar{Z}(L-K) + K\bar{L} - \bar{K}L}{\bar{L}-\bar{K}} = \frac{\bar{Z}(L-K) + K\bar{L} - \bar{K}L}{1/L - 1/K} = -KL\bar{Z} + K + L$$

— we apply shift then rotation after it symmetry relative real axis then rotation by the angle opposite to the previous one and at the end the shift, since $|K| = 1 = |L|$ then $\frac{1}{K} = \bar{K}$ and $\frac{1}{L} = \bar{L}$. This formula allows us to find

$$R = \frac{1}{2}(i - AC\bar{i} + A + C) = \frac{1}{2}(i + ACi + A + C)\,.$$

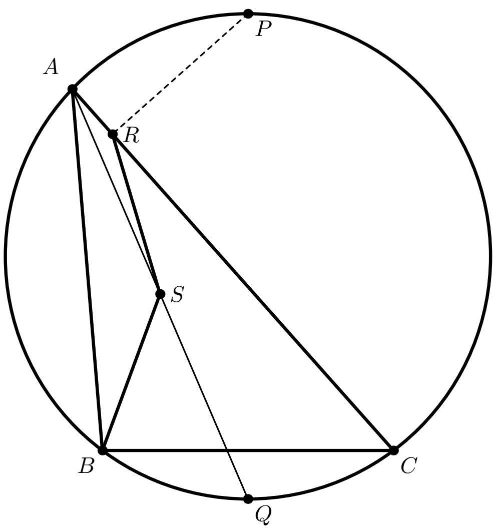
Obviously $S = \frac{1}{2}(A - i)$.

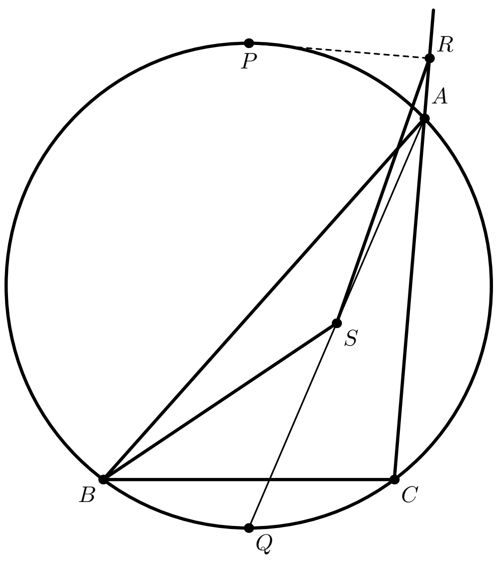
To prove that points A, B, R, S lie on one circle it suffice to notice that they are not colinear and that the number (cross–ratio) $\frac{S-R}{A-R} : \frac{S-R}{A-R}$ is real. This condition does not depend on the order of the points on the circle. It follows immediately from very well known theorems:

a quadrilateral $ABSR$ is cyclic iff $\sphericalangle ABS + \sphericalangle SRA = \pi$

and

if the points AR lie on the same side of the line BS then the points A, B, R, S lie on one circle iff $\sphericalangle SBA = \sphericalangle SRA$.





The easy proof is left to readers. Now we shall show that the number $\frac{S-R}{A-R} : \frac{S-R}{A-R} = \\ = \frac{S-R}{A-R} \cdot \frac{A-R}{S-R}$ is real. We have

$$\frac{A-B}{S-B} \cdot \frac{S-R}{A-R} = 2\frac{A+\bar{C}}{A+2\bar{C}-i} \cdot \frac{ACi+C+2i}{ACi-A+C+i} =$$

$$= 2\frac{|A+\bar{C}|^2}{(A+2\bar{C}-i)(\bar{A}+C)} \cdot \frac{|ACi+C+2i|^2}{(ACi-A+C+i)(-\bar{A}\bar{C}i+\bar{C}-2i)}\,.$$

The numerators are real (and positive) so it is enough to deal with the denominators. We have

$$(A+2\bar{C}-i)\cdot(\bar{A}+C)\cdot(ACi-A+C+i)\cdot(-\bar{A}\bar{C}i+\bar{C}-2i) =$$
$$= (A+2\bar{C}-i)\cdot(\bar{A}+C)\cdot\big(Ai(C+i)+(C+i)\big)\cdot(-\bar{C}i)\cdot(\bar{A}+i+2C) =$$
$$= |A+2\bar{C}-i|^2\cdot(\bar{A}+C)\cdot(Ai+1)\cdot(C+i)\cdot(-\bar{C}i) =$$
$$= |A+2\bar{C}-i|^2\cdot(\bar{A}+C)\cdot(Ai+1)\cdot(\bar{C}-i) =$$
$$= |A+2\bar{C}-i|^2\cdot(\bar{A}+C)\cdot\big(A+\bar{C}+i(A\bar{C}-1)\big) =$$
$$= |A+2\bar{C}-i|^2\cdot\big(|A+\bar{C}|^2+i(\bar{C}+A-\bar{A}-C)\big) =.$$

The number $i(\bar{C}+A-\bar{A}-C)$ is real because the difference of any complex number and its conjugate is pure imaginary. This ends the proof.

Such a problem may be divided into several easy problems. I am convinced that it is worth to show freshmen such methods. The only problem is to which class it should belong. It does not matter at all. When starts talking about complex numbers he is obliged to show his students that they are useful. This necessarily requires solving some problems which are not about complex numbers themselves but as in the case just discussed may be solved easily if one decides to use complex numbers. There exist quite a few such problems.

Problem

For every natural number $n \geqslant 1$ find the smallest value of

$W_n(x) = x^{2n} + 2x^{2n-1} + 3x^{2n-2} + \ldots + (2n-1)x^2 + 2nx$
defined for all reals x.

A solution by a girl who took part in the competition — this almost exactly she wrote.

For $n \geqslant 2$ the polynomial $W_n(x)$ may be written in the way:

$$W_n(x) = x^2 W_{n-1}(x) + (2n-1)x^2 + 2nx \qquad (1).$$

We shall prove that $W_n(-1) = -n$.
$W_n(-1) = 1 - 2 + 3 - 4 + \ldots + 2n - 1 - 2n =$
$(1 + 3 + 5 + \ldots + 2n - 1) - (2 + 4 + \ldots + 2n) =$
$= \frac{1+2n-1}{2} \cdot n - \frac{2+2n}{2} \cdot n = n^2 - (1+n)n = -n.$

Now we are ready to prove that $W_n(x) \geqslant -n$ for all x.
$W_n(x) \geqslant -n$, $W_n(x) + n \geqslant 0$.
I For $n = 1$ we have.
$w_1(x) + 1 \geqslant 0$, $x^2 + 2x + 1 \geqslant 0$, $(x+1)^2 \geqslant 0$.
II Suppose there is $k \in \mathbb{N}_+$ for which the inequality $W_k(x) + k \geqslant 0$ holds for all reals x.
We are going to show that $W_{k+1}(x) + k + 1 \geqslant 0$.
Substitute (1) for $W_{k+1}(x)$
$x^2 W_k(x) + (2k + 2 - 1)x^2 + (2k+2)x + k + 1 \geqslant 0$
$x^2(W_k(x) + (2k+1)) + (2k+2)x + k + 1 \geqslant 0$
$\Delta = (2k+2)^2 - 4(W_k(x) + 2k + 1)(k+1)$
$\Delta = 4(k+1)(k + 1 - W_k(x) - 2k - 1)$
$\Delta = -4(k+1)(W_k(x) + k) \leqslant 0,$
because $k + 1 \geqslant 0$ and $W_k(x) + k \geqslant 0$ — by induction hypothesis, $\Delta \leqslant 0 \implies$ the inequality $W_{k+1}(x) + k + 1 \geqslant 0$ holds.
By I and II , induction step the values of the polynomial $W_n(x)$ are greater than or equal to $-n$. (3).
It follows from (2) i (3) that $-n$ is the smallest value of $W_n(x)$.
I showed this solution some teachers of Mathematical Olympiad finalists and asked them to evaluate this. They were allowed to give 0, 2, 5 or 6 points. As I remember they were ready to give 0 or 2 points None of these teachers wanted to give 6 or 5. The reason was: this is not a quadratic polynomial so we are not allowed to use the discriminant Δ. They were somewhat surprised that opposed their opinion. In fact I immediately showed them that if $a(x) \neq 0$ then
$a(x)x^2 + b(x)x + c(x) = a(x)\left(x + \frac{x}{2a(x)}\right)^2 + c(x) - \frac{b(x)^2}{4a(x)} = a(x)\left(x + \frac{x}{2a(x)}\right)^2 + \frac{-\Delta}{4a(x)}.$
This proves that if $\Delta \leqslant 0$ and $a(x) > 0$ then $a(x)x^2 + b(x)x + c(x) \geqslant \frac{-\Delta}{4a(x)}$.
The girl was given 5 points because she payed no attention to whether or not $a(x) = 0$

or not. When her paper was discussed someone said that we did not know whether or not she was aware of what I explained above. Others said it did not matter. We were able to discuss correctness of her claims not her consciousness.
This is more or less obvious but when a student does something strange or only nonstandard many people want to take off points. I strongly oppose such attitude. Everybody thinks at his own way. Of course in reality many teachers say students they should write this or that because . . . We must be think mainly of completeness of an argument used not of its beauty. If take off points we should be ready to explain the reason in an understandable way. Beauty is not an objective notion. I was many times involved in discussions on evaluating students papers. Many times I forced people to give the precise reason for tasking off a point or more points and they were unable to do it so they could not take off points.

Many years ago ($\approx$ 1965) the following (very easy) problem was given to participants in the Polish Mathematical Olympiad: *Prove that if* $x^3 + 2px + q = 0$ *then* $xq \leqslant p^2$.

At least two students wrote something like this.
If $x \neq 0$ then the equation $x \cdot x^2 + 2px + q = 0$ has a root so
$0 \leqslant \Delta = 4p^2 - 4xq$ therefore $qx \leqslant p^2$. For $x = 0$ we have $0 \cdot q = 0 \leqslant p^2$.

The solution is in the book containing the problems from the Polish Mathematical Olympiad. It is one of possible solutions. Many teachers would say it is incorrect. They force their students to obey their rules. The students do not understand why. This situation is bad. Te result is that especially at the beginning we have to be very precise when we say something to our students, they must start to understand us. It is not very easy in many cases. But it is necessary. Otherwise there arise questions like I showed at the beginning.

A student came to me with 5 problems.

Problem 1. A derangement of size n is a permutation $\pi\colon \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that $\pi(i) \neq i$ for each i. Let D_n be the number of derangements Of size n. Find the number of derangements of size n. (Hint: write a recurrent equation for D_n).

Problem 2. Let $0 < x_1 < x_2 < \ldots < x_n$, $p_i \geqslant 0$ for each index $i \in \{1, 2, \ldots\}$ with $p_1 + p_2 + \ldots + p_n = 1$. Prove that

$$\left(\sum_{=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} p_i x_i^{-1}\right) \leqslant \frac{(x_1 + x_n)^2}{4x_1 x_n}\,.$$

Problem 3. (a) Prove the inequality

$$\sum_{k=1}^{n} \frac{1}{k} \leqslant \prod_{p \leqslant n}\left(1 + \frac{1}{p}\right) \cdot \sum_{k=1}^{n} \frac{1}{k^2}\,.$$

were the product is over all primes p not greater than n.
(b) Prove that

$$\lim a_n = \infty, \qquad \text{where} \quad a_n = \sum_{p \leqslant n} \frac{1}{p}$$

where the summation is over all primes $p \leqslant m$.
Hint. Inequality $e^x \geqslant 1 + x$ or something equivalent to it.

Problem 4. Is the series $\sum\limits_{n=q}^{\infty} \cos^n n$ convergent?

Hint (it was not written by the author of the problems). Consider n real numbers $2\pi - \lfloor 2\pi \rfloor, 4\pi - \lfloor 4\pi \rfloor, 6\pi - \lfloor 6\pi \rfloor, \ldots 2n\pi - \lfloor 2n\pi \rfloor$. All are in the interval $(0,1)$ due to the irrationality of π and 2 are equal. So there are integers $q_1, q_2 \in \{1, 2, \ldots, n\}$, $q_1 \neq q_2$ such that $|2(q_1 - q_2)\pi - p| < \frac{1}{n}$ for some integer p. With no loss of generality we may assume that $q_1 - q_2 > 0$. Let $q = q_1 - q_2$. Then $|2q\pi - p| < \frac{1}{n}$, so $|2\pi - \frac{p}{q}| < \frac{1}{qn} \leqslant \frac{1}{q^2}$. Therefore there exist increasing sequences of positive integers (p_j), (q_j) such that $|2\pi - \frac{p_j}{q_j}| < \frac{1}{q_j^2}$. Obviously $0 < \frac{p_j}{q_j} < 8$. We have $|1 - \cos p_j| = |\cos(2q_j\pi) - \cos p_j| < |2q_j\pi - p_j| < \frac{1}{q_j}$. Therefore $\cos p_j > 1 - \frac{1}{q_j}$, so $\cos^{p_j} p_j > \left(1 - \frac{1}{q_j}\right)^{p_j} > \left(1 - \frac{1}{q_j}\right)^{8q_j} \xrightarrow[j \to \infty]{} \frac{1}{e^8} > 0$. The divergence of the series is now obvious.

Problem 5. Does there exist a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f(x) \notin \mathbb{Q}$ for $x \in \mathbb{Q}$ and $f(x) \in \mathbb{Q}$ for $x \notin \mathbb{Q}$

It turned out during the conversation that these were additional problems for people who did not solve sufficiently many problems during the semester. A student could handle solutions of these problems to the teacher and get quite a few additional points. I explained something and I told the student that it might happen that the teacher could ask her to explain the solutions because the problems are rather hard for freshmen at our university. I said that I would certainly do it because it is hard to imagine that a student who could not solve enough many problems during 3 months is able to solve these. This as I learned did not happened. I was in a difficult situation because I have not asked at the beginning where from the problems were taken and therefore I have not known that I should not solve them. I think that since I forced the student to understand the details it was honest at the end. After all solutions of some of them can be found in the books or even in the internet.

One more problem. There is a theorem that says that given any convergent series $\sum_{n=1}^{\infty} a_n$ with positive terms there exist its subseries convergent to an irrational number. Is the following text a proof of this theorem?

Let a denote a positive irrational number less than $\sum_{n=1}^{\infty} a_n$. We shall construct inductively a subseries with sum a. Let $a_n^{(0)} = a_n$ for all n. A series $\sum a_n^{(k+1)}$ id obtained by throwing away the largest term of the series $\sum a_n^{(k)}$ (possibly there are few largest *terms, then we throw away one of them) such that the sum of the remaining terms of* $\sum a_n^{(k)}$ *will be still greater than* a. *If such term does not exist then* $\sum a_n^{(k)} = a$ *and we are done. If the process does not end, denote by* b_n n^{-th} *term of remaining ones. We shall show that* $\sum_{n=1}^{\infty} b_n = a$. *Obviously* $a \leqslant \sum_{n=1}^{\infty} b_n$. *Suppose there is* $\varepsilon > 0$ *such that* $\sum_{n=1}^{\infty} b_n \geqslant a + \varepsilon$. *Then for some* n_0 *we have* $a_{n_0} < \varepsilon$ *so* $\sum\limits_{n \neq n_0} b_n \geqslant a$. *There are finitely many terms in the convergent series* $\sum_{n=1}^{\infty} a_n$ *which are* $\geqslant b_{n_0}$ *so the term* b_{n_0} *was thrown away at some inductive step. The contradiction ends the proof of the theorem.*

There is an open to students set of Mathematical Analysis problems. Students are told that part of partial exams problems will be from this set. Those who believe in it study them and ask questions. Some of these problems are not easy for them and few are not easy for me, too. This means that I solve them after thinking for some time and sometimes the answer is not obvious. Here is an example.

Let $\sum\limits_{n=2}^{\infty} a_n$ be an arbitrary series with positive terms. Is the series $\sum\limits_{n=2}^{\infty} \frac{\sqrt{a_n}}{\ln n}(n^{a_n} - 1)$ convergent? Justify your answer.

One can write

$$\frac{\sqrt{a_n}}{\ln n}(n^{a_n} - 1) = a_n^{3/2} \cdot \frac{e^{a_n \ln n} - 1}{a_n \ln n}\,.$$

For big n we may write $a_n^{3/2} < a_n < 1$ therefore the series $\sum a_n^{3/2}$ converges. So if one wants to make up a counterexample he must consider a series $\sum a_n$ with $a_{n_k} \ln n_k$ going to ∞ for some sequence (n_k) going to ∞ as $k \to \infty$. We may set $a_{2^{k^3}} = \frac{1}{k^2}$ and $a_n = \frac{1}{n^2}$ for $n \neq 2^{k^3}$ for all $k \in \mathbb{N}$. A counterexample is given. There exist freshmen who are able to do it but there not many of them.