INFINITE SEQUENCES

1. On many occasions one considers infinite sequences. If we want to define the area of a circle we may consider for example regular polygons inscribed into it having a growing number of vertices and we may say that the area of the circle is a number approached by the areas of the polygons. The precision grows together with the numbers of the vertices. In fact we consider a sequence of areas of regular polygons inscribed into the circle. This means that to each natural number starting from 3 a real number is assigned. The real numbers are called sequence terms. Usually the third term of a sequence is denoted by a_3, the fourth one a_4 etc.

2. Another example of an infinite sequence was considered by Zeno of Elea (490—425 BC). He said that Achilles known as one of the fastest runners of ancient times cannot catch a turtle. We shall describe his argument using present language and contemporary notation. For simplicity let us assume that the initial distance between Achilles and the turtle is 100 meters and the the velocity of the escaping turtle is 10 times smaller than the velocity of the following man. After some time Achilles will pass 100 meters. At the same time the turtle will pass only 10 meters and is not caught by the man. After ten times smaller amount of time Achilles will pass 10 meters and the turtle is still ahead of him by 1 meter. Achilles will pass the remaining 1 meter but the turtle is still 0.1 of a meter ahead of him. The process may be continued. The subsequent distances passed by Achilles are 100 m, 110 m, 111 m, 111.1 m, 111.11 m, $\ldots$ We are considering a sequence of numbers. Write $a_n = 100 + 10 + 1 + \frac{1}{10} + \cdots + \frac{100}{10^{n-1}} = 111.1\ldots1$, the digit 1 appears in the last number n times. Zeno was unable to say how to determine the sum of infinitely many numbers. A precise definition necessary for the solution of the problem was not ready until the begining of the 19-th century (Gauss, Cauchy, Bolzano). Obviously today one can say that Achilles will catch the turtle after he will pass $\frac{1000}{9} = 111.11\ldots$ meters, there are infinitely many ones in the last expression. In order to make the solution of the problem as clear as possible we shall write down an argument that does not use a notion of an infinite sum explicitely. Let x be a distance passed by the turtle during his escape. At the same time Achilles passed the distance of $10x$. The difference of the distances is $9x = 100$, so $x = \frac{100}{9}$, thus $10x = \frac{1000}{9}$. We did not show yet how to solve an important problem raised by Zeno, we only went around it.* The important problem was to find the limit of the sequence. We shall learn soon how to deal with such problems.

3. We are going to consider another example of great importance for the economists. Let us assume that the deposit of k dollars is in a bank, the interest rate is $100x$ per year. If the deposit remains untouched for e year and the interest is compounded once a year then after a year we shall have $k + xk$ dollars. If the interest is compounded after 6 six months we shall have $k+\frac{x}{2}k$ dollars after 6 months and $k+\frac{x}{2}k+\frac{x}{2}(k+\frac{x}{2}k) = k(1+\frac{x}{2})^2$ dollars after the whole year. This means that under such a policy of the bank we shall $k\frac{x^2}{4}$ dollars more

* There were other paradoxes related to infinity. For example a point has no length, a line segment consists of points but it has a length. This seems to mean that the sum of infinitely many zeros may be positive! A moving object, say an arrow travels no distance at an infinitely short time, so it should not move at all. We shall learn in what way one should talk of such questions and do not end up with a contradictory statements.

than under the previous one. Obviously for small deposit this is not important, if $k = 1000$, $x = \frac{1}{10}$ (10% interest rate is very high!) then $k\frac{x^2}{4} = 2.5$ so it is not much. For big deposits the situation changes. Suppose now that the interest is compounded after each month. Then after the first month the deposit will be $k + \frac{x}{12}k = k(1 + \frac{x}{12})$, after 2 months it will be $k(1 + \frac{x}{12}) \cdot (1 + \frac{x}{12}) = k(1 + \frac{x}{12})^2$, after 3 months $k(1 + \frac{x}{12})^3$, ..., after a year $k(1 + \frac{x}{12})^{12}$. If we divide a year into n equal periods (n may be $1, 2, 3, \ldots$) then after a year the deposit will be equal to $k(1 + \frac{x}{n})^n$. It is more or less clear that when we increase the number of periods after the interest is compounded the amount of money after a year should grow. People responsible for the policy of the bank should know how such changes of the policy influence the income of bank customers because it is tightly related to the income of the bank.

The quantity $k(1 + \frac{x}{n})^n$ with $x \in \mathbb{R}$, not necessarily $x > 0$ appears in different situations. Physicists say that the decay of the mass of an radioactive element is proportional to the given mass and to the time. It is not hard to realize that this statement differs very little from the statement: the amount of money added to the deposit is proportional to the deposit. Therefore the formula obtained after a similar consideration is the same. The only difference is that in this case the $x < 0$. In this case it makes more sense to think that n is very big, in fact the bigger n is the result should be more precise. In the bank it is practically impossible to compound interest continuously but the radiation is not stopped after office hours, it goes on with no breaks. Another example from physics leading to the same formula is the length of steel rail considered as a function of temperature. The length of the rail grows together with the temperature. The increment of the length is proportional to the increment of the temperature and the length of the rail. This physics law is known to everybody who took physics in a high school. In this situation one writes that $\Delta\ell = \lambda \cdot \Delta t$. This leeds to the formula $\ell(t + \Delta t) = \ell(t)(1 + \lambda\Delta t)$, λ plays here the role of x in the story about a deposit in a bank. Very few high school students are able to notice that if this law is applied in a standard way with no comments it leads to contradictory results. If the rail is warmed up say by 20° C at once one gets $\ell(t + 20) = \ell(t)(1 + 20\lambda\Delta t)$. It the rail is warmed up twice by 10° each time then the result is $\ell(t + 20) = \ell(t + 10)(1 + 10\lambda\Delta) = \ell(t)(1 + 10\lambda\Delta)^2 = \ell(t)(1 + 20\lambda\Delta t + 100\lambda^2\Delta t^2)$ so it is bigger. The problem that arises is of mathematical nature. In this particular case the difference $\ell(t) \cdot 100\lambda^2\Delta t^2$ is very small, because λ is very little and it is squared, in fact much smaller than the precision of measurement, so it is practically irrelevant. One should be able to find mathematical formulas that would lead to the same result because in practically the rail is warmed up gradually, so one should be able to compute its length with any of the two methods described above. The is true in the case of radioactive decay. But there is a difference. Frequently people are interested at the time after which the mass of the radioactive element will be twice smaller than the initial one. This means that although the coefficient λ is little also in this case the time is big enough to make the quantity $\ell(t) \cdot 100\lambda^2\Delta t^2$ so big that it is not possible to call it negligible.

In both situations we ended up with the same formula as in the case of a deposit in the bank. This should convince us that the sequence $\left(1 + \frac{x}{n}\right)^n$ is important.

4. A geometric sequence is another important type of a sequence. a_0, q are arbitrary real numbers. We define $a_1 = a_0 q$, $a_2 = a_1 q$, $\ldots$, in general $a_n = a_0 q^n$ for $n = 0, 1, 2 \ldots$. The number q is called a ratio of the geometric sequence. If $q \neq 0$, then $q = \frac{a_1}{a_0} = \frac{a_2}{a_1} = \ldots$. Problems described above lead to different geometric sequences. If the interest rate is fixed then the deposit behaves like a geometric sequence: a_0 is the initial deposit, a_1 is the deposit after the first period after which the interest is compounded, a_2 is the deposit after two periods etc. Another example is a number of people living in some country. If the conditions of life are unchanged (no wars, typhoons, floods, $\ldots$) then the population changes at a certain rate, grows if the population growth rate x is positive, declines if the growth rate x is negative. In this situation $q = 1 + x$.

5. Even simpler than a geometric sequence is an arithmetic one: $a_n = a_0 + nd$, a_0 and d are arbitrary real numbers. The number d is called the difference of a sequence because it is a difference of two subsequent terms. In the 19-th century some people noticed the the amount of grain behaves approximately like an arithmetic sequence (n was a number of the year). Of course from time to time there were floods, droughts and the statistics changes. In the 19-th century people started to use chemical fertilizers, one the first in use was Chile saltpeter. This resulted in crop growth.

6. In the manuscript *Liber abaci (1202)* of Leonardo Pisano, known as Fibonacci, there is a problem: A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive? At first there is one pair of rabbits, a month later there are two pairs of them. The old ones are ready to have one more pair in a month, new born are not, so after 2 months there are 3 pairs of rabbits. 2 pairs of „old" rabbits will give birth to 2 new pairs of rabbits in a month. So after 3 months there will be 5 pairs of rabbits, among them 2 pairs of new born. Therefore after 4 months there are $5 + 3 = 8$ pairs of rabbits, 3 of them new born. After some time we are able to say that after a year there are $377 = 233 + 144$ pairs of rabbits. There is a quite natural question: let $a_0 = 1$, $a_1 = 2$ i $a_n = a_{n-1} + a_{n-2}$ for all $n = 2, 3, 4, \ldots$. Find a formula for a_n. Such a formula was written by J.Binet in $19-$th century. Here it is:

$$a_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}.$$

One can easily verify it using for example mathematical induction. But there is more important question. How it is possible to guess or to derive it? It does not look obvious and it is not surprising that it took so long before mathematicians were able to get it. We are not going to discuss the derivation of the Binet formula now.

7. We shall give a precise definition of a sequence now.

Definition of a sequnce

A sequence is an arbitrary function defined on the set of all integers that are greater than or equal to n_0, n_0 is a given integer. The value of the function at $n \geq n_0$ is called an $n-$th term of the sequence. ■

(a_n) denotes the sequence n–th of which is a_n. When the sequence of regular polygons was considered then $n_0 = 3$, because the smallest number of the vertices was 3. In the next two examples $n_0 = 1$ (the first term of the sequence is a_1). Geometric sequence, arithmetic sequence and Fibonacci sequences started from a_0 so in these cases $n_0 = 0$.

In some cases the initial index n_0 may be negative. The names *arithmetic sequence* or *geometric sequence* will be applied to sequences originating from a_{n_0}, no matter what n_0 is chosen. The only requirement is that $a_{n+1} = a_n + d$ for $n \geq n_0$ in the case of an arithmetic sequence or — $a_{n+1} = a_n \cdot q$ for $n \geq n_0$ in the case of an geometric sequence. Most frequently we shall start with 0 or od 1. Unless something else will be written explicitely the letter n will denote a non–negative integer, i.e. *natural number.**

8. Now we are ready to define a limit of a sequence. We came close to this notion when telling the story about Achilles and the turtle (Zeno paradox).

Definition of the limit of the sequence

a. A number g is a limit if the sequence (a_n) iff, for each real number $\varepsilon > 0$ there exists an integer n_ε such that if $n > n_\varepsilon$ then $|a_n - g| < \varepsilon$.

b. $+\infty$ (plus infinity) is a limit of the sequence (a_n) iff for each real number M there exists an integer n_m such that if $n > n_M$ then $a_n > M$.

c. $-\infty$ (minus infinity) is a limit if the sequence (a_n) iff for each real number M there exists an integer n_m such that if $n > n_M$ then $a_n < M$.

d. It g is a limit of the sequence (a_n), finite or not, we write $g = \lim\limits_{n\to\infty} a_n$ or $a_n \xrightarrow[n\to\infty]{} g$. Some people write $a_n \to g$ as $n \to \infty$ or shortly $a_n \to g$. If the sequence has a finite limit g we say it is convergent to g.■

Suppose $g = \lim\limits_{n\to\infty} a_n$ for some $g \in \mathbb{R}$. The difference $|a_n - g|$ is small ($|a_n - g| < \varepsilon$) if the number n numery is big enough ($n > n_\varepsilon$), so we can say that for n sufficiently big the term a_n is approximately equal to g, moreover the admissible error is $|a_n - g| < \varepsilon$. Let us be more precise. It does not have to be true that later terms necessarily give a better precision. It may happen that for some time an error will grow. What is true is the following: if one wants to approximate g with small error with a_n, he should take n large (sufficiently large). The precise meaning of the statement *sufficiently large* depends on the sequence. One should think of the number ε as of „allowed error", so usually it will be very little. If we are able to prove that for sufficiently big n's the term a_n approximates g with an error smaller then ε then the error is smaller than any $\tilde{\varepsilon} > \varepsilon$. It is one of many obvious statements which are very useful in proofs.

Do not forget that the number $|x - y|$ may be regarded as the distance of points x, y on the real line. Therefore the inequality $|a_n - g| < \varepsilon$ means that that point a_n is inside the open interval of length 2ε centered at g. In particular the sequence all term s of which are equal (or only terms with sufficiently big

* Some mathematicians say that the natural numbers start from 1, so they are 1, 2, ... Others say that they start from 0. At the moment the author supports the later concept. Natural numbers are used primarily for saying how many elements are in a given finite set, since an empty set has 0 elements we regard 0 as a natural number. Obviously an empty set is an abstract notion so it is hard to say whether it is natural to include 0 into natural numbers. Some people do not like this idea. It is obviously possible to discuss such a problem as long as the humanity exists.

indices are equal), is convergent to the common value of all terms.

Instead of *there exists* n_ε *such that for all* $n > n_\varepsilon$ *... holds* we shall say *for sufficiently big* n *... holds* or *for almost all* n *... holds*. So *for almost all* n *'s ...* means *for all except for finitely many* n *'s ...*. On many occasions we shall be interested at g not at n_ε.

One can speak in a very similar way of the equality $+\infty = \lim\limits_{n\to\infty} a_n$, part **b** of the definition of the limit. In this situation the term a_n ($n > n_M$) should be *close* to plus infinity. This should mean that it is a big positive number, so $a_n > M$. We do not assume in the definition that $M > 0$ but it is clear that one may think of positive M only: if $a_n > 0$ then a_n is greater than any negative number.

We leave to the reader restating part (c) of the the definition in the same way.

If $+\infty = \lim\limits_{n\to\infty} a_n$ then some people say that the sequence (a_n) is *convergent* to $+\infty$ while some other people say that the sequence (a_n) is *divergent* to $+\infty$. Usually in this situation we shall say *convergent.*

9. Examples

a. $0 = \lim\limits_{n\to\infty} \frac{1}{n}$. To prove this it is enough to define n_ε as an arbitrary number greater than $\frac{1}{\varepsilon}$. Thus we may set $n_1 = 1$, $n_{1/2} = 3$, $n_{0,41} = 3$ etc. but we may also choose bigger numbers e.g. $n_1 = 10$, $n_{1/2} = 207$, $n_{0,41} = 3$, etc. Always we may replace a chosen n_ε with a bigger number. ■

b. $\frac{1}{2} = \lim\limits_{n\to\infty} \frac{2n+3}{4n-1}$. We are going to show this equality. Obviously the inequality $\left|\frac{1}{2} - \frac{2n+3}{4n-1}\right| = \left|\frac{-7}{2(4n-1)}\right| \leq \frac{7}{6n}$ holds for all $n \geq 1$. Therefore *it suffices* to choose n_ε so that $n_\varepsilon > \frac{7}{6\varepsilon}$. This means that if n_ε is chosen so and $n > n_\varepsilon$ then $\left|\frac{1}{2} - \frac{2n+3}{4n-1}\right| < \varepsilon$. We do **not** claim that the inequality holds for these indices *only.* We say that for them it holds, maybe it holds also for some others! It is not necessary to solve the inequality, we only need to show that it holds *for all sufficiently big* natural numbers n. ■

c. If $d > 0$ then $+\infty = \lim\limits_{n\to\infty} (a_0 + nd)$. Let us show it. If M is an arbitrary number, $n_\varepsilon > \frac{M-a_0}{d}$ and $n > n_\varepsilon$ then $n > \frac{M-a_0}{d}$. Therefore $a_n = a_0 + nd > M$. Thus $+\infty = \lim\limits_{n\to\infty} (a_0 + nd)$. ■

10. Bernoulli Inequality

We are going to prove a useful inequality. Let n be a positive integer, $a > -1$ a real number. Then

$$(1+a)^n \geq 1 + na$$

the equality holds iff $a = 0$ or $n = 1$.

If $n = 1$ then for every a the equality holds: $(1+a)^1 = 1 + 1 \cdot a$. Since $(1+a)^2 = 1 + 2a + a^2 \geq 1 + 2a$ the inequality holds for $n = 2$ and all real numbers a (not for $a > -1$ only). We can multiply the inequality $(1+a)^2 \geq 1 + 2a$ by a positive number $(1+a)$, here we use the hypothesis $a > -1$. The result is $(1+a)^3 \geq (1+2a)(1+a) = 1 + 3a + 2a^2 \geq 1 + 3a$. It is clear that for $a \neq 0$ the strict inequality holds. In the same way we derive the inequality $(1+a)^4 \geq (1+3a)(1+a) \geq 1 + 4a + 3a^2 \geq 1 + 4a$ from the inequality $(1+a)^3 \geq 1 + 3a$. We may repet the procedure as many times as we want to. This way we prove the inequality for $n = 5$ and all $a > -1$, then for $n = 6$ etc. In general, if the inequality is true for $a > -1$ and fixed n then $(1+a)^{n+1} \geq (1+na)(1+a) = 1 + (n+1)a + na^2 \geq 1 + (n+1)a$. It is not hard to show that for $n > 1$ the equality holds for $a = 0$ only. Of course in the proof we used mathematical induction.

We did not start with the name of the procedure in hope that the students would not be terrified by the name. ■

11. The limit of a geometric sequence

Let $a_n = q^n$. This sequence is convergent to 0 iff $|q| < 1$, It is convergent to 1, if $q = 1$, its limit is $+\infty$ if $q > 1$. If $q \le -1$, then the sequence does not have any limit.

The theorem is easy. In the case $q = 0$ or $q = 1$ the claim is obvious, because in these situations the sequence is constant, i.e. its terms are independent of n.

Let $-1 < q < 1$. Then $0 < |q| < 1$. Let $\varepsilon > 0$ be a real number. If $n_\varepsilon > \dfrac{\frac{1}{\varepsilon} - 1}{\frac{1}{|q|} - 1}$ is an integer and $n > n_\varepsilon$, then

$$\frac{1}{|q|^n} = \left(1 + \left(\tfrac{1}{|q|} - 1\right)\right)^n \ge 1 + n(\tfrac{1}{|q|} - 1) > 1 + \tfrac{1}{\varepsilon} - 1 = \tfrac{1}{\varepsilon}.$$

This implies that for $n > n_\varepsilon$ the inequality $\frac{1}{|q|^n} > \frac{1}{\varepsilon}$ holds. Therefore for $n > n_\varepsilon$ we have $|q^n| < \varepsilon$ but this means that $\lim\limits_{n\to\infty} q^n = 0$, by the definition of the limit.

Next case is $q > 1$. One can easily see that $q^n = (1 + (q-1))^n \ge 1 + n(q-1)$. So, if $n > n_M$ and $n_M > \frac{M-1}{q-1}$ then $q^n > 1 + (M-1) = M$. This proves that $\lim\limits_{n\to\infty} q^n = +\infty$.

It is time to assume that $q \le -1$. In this case $q^n \le -1$ for any odd integer n and $q^n \ge 1$ for any even integer n. If the sequence is convergent to a finite limit g then the distance of any term with n sufficiently big from g is less than 1. So the distances from q^n or q^{n+1} from g are less than 1. Therefore the distance from q^n to q^{n+1} is less than $1 + 1 = 2$ i.e. $|q^n - q^{n+1}| < 2$. This is not possible, because one the numbers q^n, q^{n+1} is less than or equal to -1 while the other one is greater than or equal to 1, so the the distance from q^n to q^{n+1} is greater than or equal to $1 - (-1) = 2$*. It is a contradiction, so the sequence has no finite limit. Also $+\infty$ is not the limit of the sequence, because if it had been then for n sufficiently large the following inequality would hold $q^n > 0$ (here $M = 0$) contrary to $q^n < 0$ for all odd n. Analogously $-\infty$ is not a limit of the sequence because for all *even* n, $q^n > 0$.

Therefore the sequence (q^n) nas neither finite nor infinite limit. ■

12. Monotone, strictly monotone and bounded sequences

Definition of monotone sequences

A sequence (a_n) is called non-decreasing (increasing) iff for every number n the inequality $a_n \le a_{n+1}$ ($a_n < a_{n+1}$) is satisfied. Similarly a non-increasing (decreasing) sequence is a sequence such that for every number n the inequality $a_n \ge a_{n+1}$ ($a_n > a_{n+1}$) is satisfied. A monotone sequence is a sequence which is either non-decreasing or non-increasing. A strictly monotone sequence is a sequence which is either increasing or decreasing.■

In some books another terminology is used: instead of non-increasing sequence the authors may say decreasing. In such case the name strictly increasing is applied for increasing sequences. To avoid any misunderstanding one may talk of non-decreasing and of strictly increasing sequences.

* Using the formulas we can write: $2 \le |q^n - q^{n+1}| \le |q^n - g| + |g - q^{n+1}| < 1+1 = 2$ for n sufficiently big.

A geometric sequence that starts from $a_1 = q$ is monotone for $q \ge 0$: for $q = 0$ or $q = 1$ the geometric sequence is constant, in these cases it is non-increasing and non-decreasing at the same time. If $0 < q < 1$ the sequence is decreasing, if $q > 1$ it is increasing. An arithmetic sequence with positive difference d is increasing, if the difference d is negative it is decreasing, if $d = 0$ the sequence is constant.

Definition of bounded sequences

A sequence (a_n) is bounded from above iff there exists a real number M such that for every natural number n the inequality $a_n \le M$ is satisfied. A sequence (a_n) is bounded from below iff there exist a real number m such that for every n the inequality $a_n \ge m$ holds. A sequence which is bounded from above and from below is called bounded. A sequence which is not bounded is called unbounded. ■

A sequence (n) is bounded from below, eg. -13 or 0 are its lower bounds. This sequence is not bounded from above, so we say it is unbounded. The sequence $\big((-1)^n\big)$ is bounded from above, e.g. by 1 or $\sqrt{1000}$. It is also bounded from below, e.g. by -1 or by -13.

It is clear that a sequence (a_n) is bounded iff there exists a non-negative number M such that $|a_n| \le M$ for every n. This is an obvious corollary from the definition of a bounded sequence: a number M must be so big that M is an upper bound and at the same time $-M$ is a lower bound of the sequence (a_n).

13. The sequence $\big((1 + \frac{x}{n})^n\big)$

Let us start with writing down the first ten terms of a sequence.

in the case of $x = 1$:

$\left(1 + \frac{1}{1}\right)^1 = 2$

$\left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2,25$

$\left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} \approx 2,37$

$\left(1 + \frac{1}{4}\right)^4 = \frac{625}{256} \approx 2,44$

$\left(1 + \frac{1}{5}\right)^5 = \frac{7776}{3125} \approx 2,49$

$\left(1 + \frac{1}{6}\right)^6 = \frac{117649}{46656} \approx 2,52$

$\left(1 + \frac{1}{7}\right)^7 = \frac{2097152}{823543} \approx 2,55$

$\left(1 + \frac{1}{8}\right)^8 = \frac{43046721}{16777216} \approx 2,56$

$\left(1 + \frac{1}{9}\right)^9 = \frac{1000000000}{387420489} \approx 2,58$

$\left(1 + \frac{1}{10}\right)^{10} = \frac{25937424601}{10000000000} \approx 2,59$

and in the case of $x = -4$:

$\left(1 + \frac{-4}{1}\right)^1 = -3$

$\left(1 + \frac{-4}{2}\right)^2 = 1$

$\left(1 + \frac{-4}{3}\right)^3 = \frac{-1}{27} \approx -0,37$

$\left(1 + \frac{-4}{4}\right)^4 = 0$

$\left(1 + \frac{-4}{5}\right)^5 = \frac{1}{3125} \approx 0,00032$

$\left(1 + \frac{-4}{6}\right)^6 = \frac{1}{729} \approx 0,0014$

$\left(1 + \frac{-4}{7}\right)^7 = \frac{2187}{823543} \approx 0,0027$

$\left(1 + \frac{-4}{8}\right)^8 = \frac{1}{256} \approx 0,0039$

$\left(1 + \frac{-4}{9}\right)^9 = \frac{1953125}{387420489} \approx 0,0050$

$\left(1 + \frac{-4}{10}\right)^{10} = \frac{59049}{9765625} \approx 0,0060$

It is easy to see that the sequence (a_n) with $a_n = (1 + \frac{x}{n})^n$ is neither arithmetic nor geometric with one exception: $x = 0$. **We are going to show that if $n > -x \ne 0$ then $a_{n+1} > a_n$. This means that the sequence is increasing** *from some place*. If $x > 0$ it is increasing. If $x < 0$ then it may happen that initial terms change their signs. In such situation there is no chance for monotonicity. If all terms of the sequence are positive then it is increasing. This should be proved. The inequality $n > -x$ imply the inequality $n+1 > -x$. The first of the two implies that $1 + \frac{x}{n} > 0$, the second one implies that $1 + \frac{x}{n+1} > 0$. The inequality $a_n < a_{n+1}$ is equivalent to $\left(1 + \frac{x}{n}\right)^n < \left(1 + \frac{x}{n+1}\right)^{n+1}$. Since $1 + \frac{x}{n} > 0$ the last inequality is equivalent to $\left(\frac{1+\frac{x}{n+1}}{1+\frac{x}{n}}\right)^{n+1} > \frac{1}{\left(1+\frac{x}{n}\right)} = \frac{n}{n+x}$. We shall use Bernoulli inequality (see 10.) to prove that the last inequality holds for $n > -x$. One has

$$\left(\frac{1+\frac{x}{n+1}}{1+\frac{x}{n}}\right)^{n+1} = \left(1 - \frac{x}{(n+x)(n+1)}\right)^{n+1} \geq 1-(n+1)\frac{x}{(n+x)(n+1)} = 1 - \frac{x}{n+x} = \frac{n}{n+x}.$$

Let us mention that that the number $\frac{-x}{(n+x)(n+1)}$, it plays the role of a in Bernoulli inequality, is greater than -1 — it is obvious for $x \leq 0$ because in this case the number is positive, if $x > 0$ its absolute value $\frac{x}{(n+x)(n+1)}$ is less than $\frac{1}{n+1} < 1$. We have proved that from the moment at which the quantity $(1+\frac{x}{n})$ becomes positive for all subsequent n's the sequence starts growing (it is constant for $x = 0$). Let us mention that if $x > 0$ then all terms of the sequence are positive, if $x < 0$ then for every even number n the inequality $(1+\frac{x}{n})^n \geq 0$ holds while for odd integers n it holds under the hypothesis $n > -x$.

One interesting question is still open: is the growth of the sequence $\left((1+\frac{x}{n})^n\right)$ unbounded in the case of $x > 0$ or for a given number x one can find a number greater than all terms of the sequence. It turns out that the sequence $\left((1+\frac{x}{n})^n\right)$ is bounded from above for any positive number x. For negative number x this statement is obvious since as it was said before we have $0 < \left((1+\frac{x}{n})^n\right) < 1$ for $n > -x$. If $n > x > 0$ then $\left(1+\frac{x}{n}\right)^n = \frac{\left(1-\frac{x^2}{n^2}\right)^n}{\left(1-\frac{x}{n}\right)^n} < \frac{1}{\left(1-\frac{x}{n}\right)^n}$. The quantity $\frac{1}{\left(1-\frac{x}{n}\right)^n}$ decreases together with n (recall that $n > x$) because the numerator is unchanged while the denominator grows as we proved before. This implies that if $n(x)$ is the smallest integer greater that x then all terms of the sequence are less than $\frac{1}{\left(1-\frac{x}{n(x)}\right)^{n(x)}} = \left(\frac{n(x)}{n(x)-x}\right)^{n(x)}$. For example $n(1) = 2$ so all term of the sequence $\left(1+\frac{1}{n}\right)^n$ are less than $\left(\frac{2}{2-1}\right)^2 = 4$. If $x = -4$ then all terms of the sequence beginning from the fifth one are positive and less than 1, a quick look at the first four of them tells us that the biggest of all terms is $\left(1+\frac{-4}{2}\right)^2 = 1$ and the smallest one is $\left(1+\frac{-4}{1}\right)^1 = -3$.

It easily follows that if $k \geq n(x)$ than the number $\frac{1}{\left(1-\frac{x}{k}\right)^k} = \left(\frac{k}{k-x}\right)^n$ is an upper bound of the sequence $\left(\left(1+\frac{x}{n}\right)^n\right)$ — the reader may prove this easy statement by himself or herself. ■

14. A limit of a monotone sequence

We cannot give any proof of the following theorem

The Monotone Sequence Limit Theorem

Each monotone sequence has a limit. ■

We cannot prove this theorem because we never stated the *Dedekind Continuity Axiom*. In fact this theorem is equivalent to the axiom which is not included into this book. Let us just say that had we used rational numbers only (a number is rational if it can be written as a fraction with integer numerator and integer denominator), the theorem was false. This is due to the fact that there are sequences of rational numbers with irrational limits. This theorem describes in some way a very important property of the set of all real numbers, namely: there are no holes (punctures) in it, geometrically the set of all real numbers is a straight line. The Dedekind axiom describes the same property with other words. The set of all rational numbers is very different. There are punctures in it everywhere. Between any two rational numbers c, d

there is an irrational number, e.g. $c+\frac{d-c}{\sqrt{2}}$ — the number is irrational because $\sqrt{2}>1$ is irrational while c,d are rational. The number lies between c and d because it lies on the same side of c as d and its distance from c is less than that of d: $\frac{|d-c|}{\sqrt{2}}<|d-c|$.

From the monotone sequence limit theorem it follows immediately that the geometric sequence with $q\geq 0$ has a limit. The theorem does not imply that there is a limit in the case of $q<0$ because in this case the sequence is **not** monotone.

The theorem implies also that for every real number x the sequence $\big((1+\frac{x}{n})^n\big)$ has a limit. The sequence is not monotone but it is monotone from some place and it suffices for convergence since a change of finitely many terms of the sequence has no impact on convergence or on the limit. It is so because in the definition of a limit one speaks of a_n with sufficiently big n only, so the change of finitely many terms of the sequence may change only the meaning of the statement *sufficiently big* .

Notation

In the future $\exp(x)$ will denote the limit of the sequence $\big((1+\frac{x}{n})^n\big)$, so

$$\exp(x)=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n .$$

Therefore the symbol exp denotes the function which is defined on the set of all real numbers, the function assigns a positive number $\lim\limits_{n\to\infty}\big(1+\frac{x}{n}\big)^n$ to a real number x.

15. Definition of the operations with $\pm\infty$ symbols

The symbols $+\infty$ and ∞ have been already used. They are not new real numbers, they are new objects. We are going *to define* operations with them.

Definition

$-(+\infty)=-\infty\,,\ \ +(+\infty)=+\infty\,,\ \ -(-\infty)=+\infty\,,\ \ +(-\infty)=-\infty\,.$

$+\infty\pm a=\pm a+(+\infty)=+\infty$ and $-\infty\pm a=\pm a+(-\infty)=-\infty$ for every real number a.

$+\infty+(+\infty)=+\infty\,,\ \ -\infty+(-\infty)=-\infty\,,\ \ +\infty-(-\infty)=+\infty\,,\ \ -\infty-(+\infty)=-\infty\,.$

$+\infty\cdot a=+\infty$ and $-\infty\cdot a=-\infty$ for any real number $a>0$,

$(+\infty)\cdot(+\infty)=(-\infty)\cdot(-\infty)=+\infty\,.$

$+\infty\cdot a=-\infty$ i $-\infty\cdot a=+\infty$ for any real number $a<0$,

$\frac{a}{\pm\infty}=0$ for every real number a,

$\frac{\pm\infty}{a}=\pm\infty\cdot\frac{1}{a}$ for every real number $a\neq 0$.

$a^{+\infty}=+\infty$ and $a^{-\infty}=0$ for every $a>1$.

$a^{+\infty}=0$ and $a^{-\infty}=+\infty$ for every $0<a<1$.

$-\infty<a<+\infty$ for every real number a.

$-\infty<+\infty$ ■

We do not give any definition of $\frac{\pm\infty}{\pm\infty}$, $0\cdot(\pm\infty)$, $1^{\pm\infty}$ and few others. It will become soon clear that it is not possible to do it in a useful way. We call them *indeterminate expressions*. The definitions given above allow to state theorems about existence and evaluating limits in a simpler way, as we shall see in a near future.

Now we are going to state few simple theorems that help a lot when working with limits. After it we shall give few examples showing how the theorems can be applied in solving problems. At the end we shall give the proofs of some them in order to show how this can be done.

Arithmetic Properties of the Limit Theorem

A1. It limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exists and their sum is defined then there exists a limit $\lim\limits_{n\to\infty}(a_n+b_n)$ and the following equality $\lim\limits_{n\to\infty}(a_n+b_n)=\lim\limits_{n\to\infty} a_n+\lim\limits_{n\to\infty} b_n$ holds.

A2. It limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exists and their difference is defined then there exists a limit $\lim\limits_{n\to\infty}(a_n-b_n)$ and the following equality $\lim\limits_{n\to\infty}(a_n-b_n)=\lim\limits_{n\to\infty} a_n-\lim\limits_{n\to\infty} b_n$ holds.

A3. It limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exists and their product is defined then there exists a limit $\lim\limits_{n\to\infty}(a_n\cdot b_n)$ and the following equality $\lim\limits_{n\to\infty}(a_n\cdot b_n)=\lim\limits_{n\to\infty} a_n\cdot\lim\limits_{n\to\infty} b_n$ holds.

A4. It limits $\lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$ exists and their quotient is defined then there exists a limit $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ and the following equality $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ holds. $\boxtimes$

Estimate Theorem

N1. If $C<\lim\limits_{n\to\infty} a_n$ then for sufficiently large n the inequality $C<a_n$ holds.

N2. If $C>\lim\limits_{n\to\infty} a_n$ then for sufficiently large n the inequality $C>a_n$ holds.

N3. If $\lim\limits_{n\to\infty} b_n<\lim\limits_{n\to\infty} a_n$ then for sufficiently large n the inequality $b_n<a_n$ holds.

N4. If the inequality $b_n\le a_n$ holds for all sufficiently large n then $\lim\limits_{n\to\infty} b_n\le\lim\limits_{n\to\infty} a_n$. $\boxtimes$

Uniqueness Proposition

Any sequence has at most one limit.

Proof.

Suppose it has 2 limits, e.g. $g_1<g_2$. Let C be a number that lies between g_1 and g_2: $g_1<C<g_2$. Then for n sufficiently large $a_n<C$ (since g_1 is a limit, see N2) and at the same time $a_n>C$ (since g_2 is a limit, see N1). Therefore $a_n<C<a_n$. It is a contradiction: no number is greater than itself. ■

Boundedness Proposition

Any sequence with a finite limit is bounded.

Proof.

If the limit $\lim\limits_{n\to\infty} a_n$ is finite then there are real numbers $\tilde{C},\tilde{D}$ and a natural number k such that for all $n\ge k$ the inequality $\tilde{C}<a_n<\tilde{D}$ holds. Let $C=\min(\tilde{C},a_1,a_2,\ldots,a_k)$ and $D=\max(\tilde{C},a_1,a_2,\ldots,a_k)$. From this definition it follows at once that $C\le a_n\le D$. This means that the sequence (a_n) is bounded from below by C and from above by D. ■

Remark. The proof is very simple. Nonetheless it worth to mention that among finitely many numbers there always is the smallest and the biggest. It is not so in the case of infinitely many numbers, e.g. there is no smallest number among $1,\frac{1}{2},\frac{1}{3},\ldots$ When we say *for sufficiently big numbers n the inequality $C\le a_n$ holds* it means that there a number k_1 such that $\tilde{C}\le a_n$ for all $n\ge k_1$. In we say that the inequality $a_n\le\tilde{D}$ holds we mean that there is a number k_2 such that $a_n\le D$ for $n\ge k_2$. If we say that $\tilde{C}\le a_n\le\tilde{D}$

holds for sufficiently big n we mean that $n \geq k_1$ **and** $n \geq k_2$, i.e. $n \geq \max(k_1, k_2)$. Obviously it is possible to talk of the biggest of the two numbers. In this proof we imposed only two conditions on n. Sometimes one works with infinitely many conditions. Then it is not possible to pick up the biggest of *infinitely* many numbers k_i. This usually makes the proof more complicated and sometimes a theorem under consideration requires additional hypothesis. ■

The Three Sequence Theorem

If $a_n \leq b_n \leq c_n$ for all sufficiently large n and the sequences (a_n) and (c_n) have *equal* limits then the sequence (b_n) also has a limit and moreover

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n. \quad \boxtimes$$

Definition of a subsequence

If (n_k) is a strictly increasing sequence of natural numbers then the sequence (a_{n_k}) is called a subsequence of (a_n). ■

The sequence $a_2, a_4\, a_6, \ldots,$ i.e. the sequence (a_{2k}) is a subsequence of the sequence (a_n) — in this case $n_k = 2k$. The sequence $a_2, a_3, a_5, a_7, a_{11}, \ldots$ is a subsequence of (a_n) — in this case n_k is a $k-$th prime number. We can give many more examples but hopefully it is sufficient to say that we take out of the given sequence infinitely many of its terms *with no change of their order*.

It follows immediately from the definitions of the limit of the sequence and of the subsequence that all subsequences of a sequence with a limit g have limit g. It is not hard to prove that instead of showing that a sequence has a limit one can look at finitely many properly chosen subsequences of the sequence. It all them have the same limit then it is a limit of the initial sequence.

Junction Theorem*

Suppose it is possible to choose out of the sequence (a_n) two subsequences (a_{k_n}) and (a_{l_n}) that converge to the same limit g and that each term of the sequence (a_n) appears at least in one of them, i.e. for every n there exists an m such that $n = k_m$ or $n = l_m$. In this situation the common limit g of the two sequence is the limit of the sequence (a_n) : $\lim\limits_{n\to\infty} a_n = g$. ⊠

It is time for one of the most important theorems which will intervene in many proofs in the future.

Bolzano – Weierstrassa Theorem

Every sequence of real numbers contains a sequence which has a limit (finite or infinite). ⊠

Corollary 1.

A sequence has a limit iff the limits of its all sequences which have limits are equal. ⊠

The next theorem, which we already partially proved, was shown by A.Cauchy who has been one of the founders of mathematical analysis.

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* The name of the theorem came out of the author's mind, the author hopes the name is not a stupid one.

Cauchy Condition Theorem

A sequence (a_n) has a finite limit wtedy iff the following Cauchy condition is satisfied:

$$\text{dla każdego } \varepsilon > 0 \text{ istnieje liczba naturalna } n_\varepsilon \text{ taka, że jeśli } k,l > n_\varepsilon, \text{ to } |a_k - a_l| < \varepsilon\,. \boxtimes \tag{CC}$$

The Cauchy theorem similarly to monotone sequence limit theorem allows in many cases to prove that a sequence has a finite limit without indicating the limit. This is very useful in many cases. It allows also to prove that some sequences do not have finite limits. In fact we have already used the theorem when proving that a geometric sequence with $q \le -1$ does not have any limit. In that case number 2 play the role of ε.

Essentially we have listed all the theorems about sequences needed in the book. We are going to give one more because usually many students learn de l'Hospital's Rule which can be found in one the later chapters while at a high school. The theorem is useful and a student can understand de l'Hospital's Rule better when he will see its discrete version. The theorem helps to deal with the indeterminancies of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

Stolz Theorem

Let all terms of the strictly increasing sequence (b_n) be different from 0 and let there exists a limit $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$. If one of the conditions is satisfied:

(i) $\lim\limits_{n\to\infty} b_n = \pm\infty$,

(ii) $\lim\limits_{n\to\infty} a_n = 0 = \lim\limits_{n\to\infty} b_n$

the the sequence $\left(\frac{a_n}{b_n}\right)$ has also a limit and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}\,. \boxtimes$$

17. Examples and comments

We are going to show how the machinery can be applied in practice. The examples **d,e,f,g** are very important and they will be used in the later chapters.

a. We shall start with an example we described already. Namely we shall deal with the sequence $\left(\frac{2n+3}{4n-1}\right)$ (cf. 9b.). We proved that the sequence converges to $\frac{1}{2}$ and we never explained how it is possible to guess the value of the limit. Notice that both the numerator and the denominator tend to ∞ as $n \longrightarrow \infty$. This not the best situation: $\frac{+\infty}{+\infty}$ and this quotient remains undefined. It is not hard to get rid of this difficulty: $\frac{2n+3}{4n-1} = \frac{2+\frac{3}{n}}{4-\frac{1}{n}}$. It is possible to apply now the theorem about the limit of the sum of two sequences (A1), then we use the theorem about the limit of the difference of two sequences (A2) in order to see that $\lim\limits_{n\to\infty}(2+\frac{3}{n}) = 2 + \lim\limits_{n\to\infty}\frac{3}{n} = 2+0 = 2$ and $\lim\limits_{n\to\infty}(4-\frac{1}{n}) = 4 - \lim\limits_{n\to\infty}\frac{1}{n} = 4-0 = 4$ — we know already that $\lim\limits_{n\to\infty}\frac{1}{n} = 0$ (cf. 9a), so $\lim\limits_{n\to\infty}\frac{3}{n} = 3\cdot\lim\limits_{n\to\infty}\frac{1}{n} = 3\cdot 0 = 0$. We are now dealing with the quotient. Its numerator tends to 2 while the denominator to $4 \ne 0$. Therefore we may use the theorem about the limit of the quotients (A4). From this theorem it follows immediately that the limit is equal to $\frac{2}{4} = \frac{1}{2}$. That is all since the arithmetic properties of the limit theorem guarantees the existence of all necessary limits and the equalities.

We shall show one more way of finding the limit although it is like hunting a mosquito with an A–bomb. Stolz theorem can be applied since $\lim_{n\to\infty}(4n-1)=+\infty$ and the sequence $(4n-1)$ is strictly increasing. It is enough to find the limit $\lim_{n\to\infty}\frac{2(n+1)+3-(2n+3)}{4(n+1)-1-(4n-1)}$, if it exists. We have $\frac{2(n+1)+3-(2n+3)}{4(n+1)-1-(4n-1)}=$ $=\frac{2}{4}=\frac{1}{2}$. The obtained sequence is constant, i.e. the value of all terms is $\frac{1}{2}$, independently of n, so its limit equals $\frac{1}{2}$, too. Lets make it very clear: there is an easy solution without Stolz theorem, we only wanted to show on a very simple examples how the theorem can be applied. ■

b. We shall show that $\lim_{n\to\infty}(n^5-100n^4-333978)=+\infty$. The reader will notice that the first 100 terms (possibly many more) are negative, $n^5-100n^4-333978=n^4(n-100)-333978\le 0$. We do not care how many of sequence terms are negative. Let's write

$$n^5-100n^4-333978=n^5(1-\tfrac{100}{n}-\tfrac{333978}{n^5}).$$

Obviously $\lim_{n\to\infty}n^5=(\lim_{n\to\infty}n)\cdot(\lim_{n\to\infty}n)\cdot(\lim_{n\to\infty}n)\cdot(\lim_{n\to\infty}n)\cdot(\lim_{n\to\infty}n)=(+\infty)\cdot(+\infty)\cdot(+\infty)\cdot(+\infty)\cdot(+\infty)=$ $=+\infty$, the limit of the product equals to the product of the limits (A3). Then $\lim_{n\to\infty}\frac{100}{n}=0$ and $\lim_{n\to\infty}\frac{333978}{n^5}=0$, because the limit of the quotient is equal to the quotient of the limits (A4). Then we apply (A2) twice to see that $\lim_{n\to\infty}\left(1-\frac{100}{n}-\frac{333978}{n^5}\right)=1-0-0=1$. The sequence is now a product of the two sequences the first of then tends to $+\infty$ while the second one has a positive limit, namely 1. It follows from the definition of the product of $+\infty$ and a positive number that the sequence $(n^5-100n^4-333978)$ tends to $+\infty$.

Also this limit can be found in a different way. Write $n^5-100n^4-333978\ge n^5-334078n^4=n^4(n-334078)$. The last sequence is a product of two sequences: $(n-334078)$ and (n^4). They both tend to $+\infty$ so their product tends to $+\infty\cdot+\infty=+\infty$. So does the sequence with a greater term. ■

c. We showed that the geometric sequence (q^n) with $q\in(-1,1)$ is convergent to 0. We shall show now how one can prove it avoiding any estimates. Instead of the estimates we are going to use theorems that guarantee existence of certain limits. At the beginning let us assume that $0\le q<1$. Clearly $q^{n+1}\le q^n$ so the sequence as a non-increasing one has a limit. Let $\lim_{n\to\infty}q^n=g$. All terms of the sequence (q^n) are in the interval $(0,1)$. Therefore $g\in[0,1]$. All subsequences of (q^n) converge to g. Therefore $g=\lim_{n\to\infty}q^{n+1}=\lim_{n\to\infty}(q\cdot q^n)=q\cdot\lim_{n\to\infty}q^n=q\cdot g$, so $g=qg$. Since $q\ne 1$ it immediately implies $g=0$. Assume now that $-1<q<0$. Then $-|q|^n\le q^n\le|q|^n$. It follows from what is already proved and the three sequence theorem that $0=\lim_{n\to\infty}(-|q|^n)=\lim_{n\to\infty}q^n=\lim_{n\to\infty}|q|^n=0$.

One may deal in the same way with $q>1$. The sequence (q^n) is strictly increasing, therefore it has some limit g. As before we prove that $g=qg$. This may happen only when $g=0$ or $g=\pm\infty$. Obviously $g>0$, -the limit of an *increasing* sequence of positive numbers is necessarily greater than 0, so $g=+\infty$. In the case of $q\le -1$ the sequence has no limit because there exist subsequences with different limits, e.g. $g_1=\lim_{n\to\infty}q^{2n+1}=\lim_{n\to\infty}\left[q\cdot(q^2)^n\right]=q\cdot\lim_{n\to\infty}(q^2)^n$ and $g_2=\lim_{n\to\infty}q^{2n}=\lim_{n\to\infty}(q^2)^n$, $g_1\le -1<1\le g_2$. ■

d. Let $a>0$ be an arbitrary real number. We shall show that $\lim_{n\to\infty}\sqrt[n]{a}=1$. As in the previous example two methods will be shown. This time we start with more theoretical solution, so less computational solution.

Let $a > 1$. The sequence $(\sqrt[n]{a})$ is strictly decreasing, its terms are less than 1. Therefore it has a limit g. g is *finite* and it is greater than or equal to 1. All subsequences of (q^n) converge to g. Therefore $g = \lim\limits_{n\to\infty} \sqrt[2n]{a}$. The limit of a product equals to the product of the limits so: $g^2 = g \cdot g = = \lim\limits_{n\to\infty} \sqrt[2n]{a} \cdot \lim\limits_{n\to\infty} \sqrt[2n]{a} = \lim\limits_{n\to\infty} (\sqrt[2n]{a})^2 = \lim\limits_{n\to\infty} \sqrt[n]{a} = g$, hence $g^2 = g$. Therefore $g = 0 < 1$ or $g = 1$ ($\pm\infty$ have been already excluded). We already know that $g \geq 1$, so $g = 1$.

If $a = 1$ the $\sqrt[n]{a} = 1$ and there is nothing to prove. Let $0 < a < 1$. Then $\lim\limits_{n\to\infty} \sqrt[n]{a} = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{1/a}} = = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{1/a}} = \frac{1}{1} = 1$, because the limit of the quotient equals to the quotient of the limits and $\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1$ by already proved part of the theorem.

Now we shall prove that $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 1$ using the definition of the limit and some estimates. Let $\varepsilon > 0$ be an arbitrary real number. We wnat to show that for n sufficiently big $|\sqrt[n]{a} - 1| < \varepsilon$, i.e. $1 - \varepsilon < \sqrt[n]{a} < 1 + \varepsilon$. Since $a > 1$, the inequality $1 - \varepsilon < \sqrt[n]{a}$ holds for all $\varepsilon > 0$. It suffices to show that $\sqrt[n]{a} < 1 + \varepsilon$, i.e. $a < (1+\varepsilon)^n$ for n big enough. By Bernoulli inequality $1 + n\varepsilon < (1+\varepsilon)^n$, so it is enough to prove that $a < 1 + n\varepsilon$ for sufficiently big n. Let $n_\varepsilon > \frac{a-1}{\varepsilon}$. Then $n > \frac{a-1}{\varepsilon}$ for $n > n_\varepsilon$, so $1 + n\varepsilon > a$.

Remark: we have not solved the inequality $\sqrt[n]{a} < 1 + \varepsilon$ *because it requires logarithms* $n > \frac{\log a}{\log(1+\varepsilon)}$, *we have only shown that* $\sqrt[n]{a} < 1 + \varepsilon$ *for* $n > \frac{a-1}{\varepsilon}$ and we did not bother of any $n \leq \frac{a-1}{\varepsilon}$, for some of them the inequality may hold. ■

e. The next interesting sequence is $(\sqrt[n]{n})$. Its limit is 1.

Let us start with writing down the first few terms: $\sqrt[1]{1} = 1$, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{4} = \sqrt{2}$, ... It is clear that $\sqrt[3]{3} > \sqrt{2}$ — to prove it we can raise both sides to 6-th power. Therefore $\sqrt{2} < \sqrt[3]{3} > \sqrt[4]{4}$. The sequence is neither increasing nor decreasing. It may be monotone *from some place on.* We shall prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ using the definition of the limit, another proof will be given later.

Let ε be an arbitrary positive number. All terms of the sequence are greater than or equal to 1. Therefore it suffices to show that for n big enough the inequality $\sqrt[n]{n} < 1 + \varepsilon$ holds. An equivalent inequality is $n < (1+\varepsilon)^n$. In this case Bernoulli inequality is insufficient. Let $\varepsilon > 0$ and $n \geq 2$. In such case $(1+\varepsilon)^n \geq 1 + \binom{n}{1}\varepsilon + \binom{n}{2}\varepsilon^2 > \binom{n}{2}\varepsilon^2$. It is enough to know that $n < \binom{n}{2}\varepsilon^2 = \frac{n(n-1)}{2}\varepsilon^2$ for all n big enough. It is equivalent to $\frac{2}{\varepsilon^2} + 1 < n$. This ends the proof.

Now we are going to show the proof without any estimates. Let us raise the inequality $\sqrt[n+1]{n+1} < \sqrt[n]{n}$ to $n(n+1)$-th power. The result is $(n+1)^n < n^{n+1}$. Deciding by n^n we get $n > \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n$. We showed already (c.f. 13.) that the sequence $\left(\left(1 + \frac{1}{n}\right)^n\right)$ is bounded. From this it follows that $n > \left(1 + \frac{1}{n}\right)^n$ for all n *sufficiently large.* There is no need to find out what *sufficiently large* means in this case. Therefore the sequence $(\sqrt[n]{n})$ decreases from some term, it is bounded by 1 from below, therefore it is convergent to a finite limit g. All its subsequences, e.g. $\sqrt[2n]{2n}$ converge to the same limit g. Therefore

$$g^2 = g \cdot g = \lim_{n\to\infty} \sqrt[2n]{2n} \cdot \lim_{n\to\infty} \sqrt[2n]{2n} = \lim_{n\to\infty} \left(\sqrt[2n]{2n}\right)^2 = \lim_{n\to\infty} \left(\sqrt[n]{2} \cdot \sqrt[n]{n}\right) = \lim_{n\to\infty} \sqrt[n]{2} \cdot \sqrt[n]{n} = 1 \cdot g = g\,.$$

Therefore $g^2 = g$. From this equality and $1 \le g < +\infty$ it follows that $g = 1$. The proof is over. It turns out that also in this case it is possible to find the limit using the theoretical theorems instead of estimates. It required slightly more work because the sequence was not monotone. ■

f. Let k be an arbitrary positive integer and $q > 1$ an arbitrary real number. We shall show that $\lim\limits_{n\to\infty} \frac{n^k}{q^n} = 0$. The sequence (q^n) increases, its limit is $= \infty$. We may try to apply Stolz theorem. Let us start with $k = 1$.

$$\lim_{n\to\infty} \frac{n+1-n}{q^{n+1}-q^n} = \lim_{n\to\infty} \frac{1}{q^n(q-1)} = \frac{1}{q-1} \cdot \lim_{n\to\infty} \left(\frac{1}{q}\right)^n = 0.$$

From Stolz theorem it follows that

$$\lim_{n\to\infty} \frac{n}{q^n} = \lim_{n\to\infty} \frac{n+1-n}{q^{n+1}-q^n} = 0\,.^*$$

We are to study the sequence $\left(\frac{n^2}{q^n}\right)$. As before we shall use Stolz theorem. Let us look at the quotient of differences of two successive numerators and two successive denominators.

$$\lim_{n\to\infty} \frac{(n+1)^2 - n^2}{q^{n+1}-q^n} = \lim_{n\to\infty} \frac{2n+1}{q^n(q-1)} = \frac{2}{q-1} \lim_{n\to\infty} \frac{n}{q^n} + \frac{1}{q-1} \lim_{n\to\infty} \left(\frac{1}{q}\right)^n = 0$$

— the last equality follows from the theorem for $k = 1$ which has been proved already. This suggests a possibility of using mathematical induction. The details (easy) are left to the readers. In the prof the most important statement is that $(n+1)^k - n^k$ is a polynomial in n of degree $(k-1)$ — it follows immediately from the Newton binomial formula. Therefore the quotient of the difference of two successive numerators and the difference of two successive denominators can be written as a sum of at most k expressions of type $c \cdot \frac{n^j}{q^n(q-1)}$, c is a real number and $j < k$ — a natural number. By the induction hypothesis such a sum converges to 0.

Let us sketch another proof. Let $q = 1 + r$. $r > 0$ because $q > 1$. If $n > k$ then

$$q^n = (1+r)^n = \sum_{j=0}^{n} \tbinom{n}{j} r^j > \tbinom{n}{k+1} r^{k+1}\,.$$

The expression $\binom{n}{k+1} r^{k+1}$ is $k+1$–th degree polynomial in n, so

$$\lim_{n\to\infty} \frac{n^k}{\binom{n}{k+1} r^{k+1}} = 0 \ \text{ and } \ 0 < \frac{n^k}{q^n} < \frac{n^k}{\binom{n}{k+1} r^{k+1}}\,.$$

From three sequence theorem it follows now that $\lim\limits_{n\to\infty} \frac{n^k}{q^n} = 0$. ■

g. Let $a_n = \frac{q^n}{n!}$, q be an arbitrary real number. In this situation $\lim\limits_{n\to\infty} a_n = 0$.

From th definition of the sequence (a_n) it follows that $a_n = \frac{q \cdot q \cdot q \cdot \ldots \cdot q}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$. The quotient $\frac{|q|}{n}$ decreases when

* In 1798 an English economist Th.R.Malthus published an essay at which he predicted that the population would grow as a geometric sequence whereas the a food supplies would grow as as an arithmetic sequence. This in view of just shown equality would imply that food supplies per person in a long run would decline below acceptable level, in fact they would tend to 0 even for $q \approx 1$. Although a catastrophe would happen after a long time, n should be big, it did not look good. Fortunately Malthus did not take into account technological advances in agriculture, e.g. the result of applying chemical fertilizers and we can safely discuss his theories instead of starving.

n grows. Moreover $\lim\limits_{n\to\infty}\frac{|q|}{n}=0$. This implies that if n is big then a_{n+1} very small when compared with a_n. This should imply convergence of the sequence (a_n) to 0. Let $m\geq 2|q|$ be a natural number and let $n>m$. Clearly

$$0<\left|\frac{q^n}{n!}\right|=\frac{|q^m|}{m!}\cdot\frac{|q|}{m+1}\cdot\frac{|q|}{m+2}\cdot\ldots\cdot\frac{|q|}{n}<\frac{|q^m|}{m!}\cdot\left(\frac{1}{2}\right)^{n-m}.$$

The last expression approaches 0 because it is a term of a geometric sequence with the quotient $\frac{1}{2}$. Now we apply three sequence theorem. It implies $\lim\limits_{n\to\infty}\left|\frac{q^n}{n!}\right|=0$. QED ■

h. $\lim\limits_{n\to\infty}\frac{n!}{n^n}=0$. It easily follows from $0<\frac{n!}{n^n}=\frac{1}{n}\cdot\frac{2}{n}\cdot\ldots\cdot\frac{n}{n}\leq\frac{1}{n}$ and $\lim\limits_{n\to\infty}\frac{1}{n}=0$. QED ■.

i. Let $k>1$ be a natural number, let $x_1,\ x_2,\ \ldots$ be non-negative numbers such that $\lim\limits_{n\to\infty}x_n=g$. Then $\lim\limits_{n\to\infty}\sqrt[k]{x_n}=\sqrt[k]{g}$. Let $\left(\sqrt[k]{x_{l_n}}\right)$ be a subsequence of the sequence $\left(\sqrt[k]{x_n}\right)$ convergent to x. Since the limit of the product equals to the product of the limits we can write $x^k=\left(\lim\limits_{n\to\infty}\sqrt[k]{x_{l_n}}\right)^k=\lim\limits_{n\to\infty}x_{l_n}=g$. $x\geq 0$ because it is the limit of the nonnegative sequence. Therefore $x=\sqrt[k]{g}$. We have just shown that all convergent sequences of the sequence $\left(\sqrt[k]{x_n}\right)$ converge to $\sqrt[k]{g}$. From the corollary 1 (cf. Bolzano–Weierstrass theorem) it follows that the sequence $\left(\sqrt[k]{x_n}\right)$ converges to $\sqrt[k]{g}$. It is clear that the theorem is true also when $g<0$ and k is odd.

One can prove this theorem using the inequality $\left|\sqrt[k]{x}-\sqrt[k]{y}\right|\leq\sqrt[k]{|x-y|}$ instead od Bolzano–Weierstrass theorem. The only problem is to realize that such an equality holds. ■

x. We shall explain briefly why some operations have been not defined. Let us write few easy equalities:
$\lim\limits_{n\to\infty}\left(n-(n-\frac{1}{n})\right)=\lim\limits_{n\to\infty}\frac{1}{n}=0$, this suggests the following definition $+\infty-(+\infty)=0$;
$\lim\limits_{n\to\infty}(n-(n-1))=\lim\limits_{n\to\infty}1=1$, this suggests the following definition $+\infty-(+\infty)=1$;
$\lim\limits_{n\to\infty}\left(n-(n-\frac{n}{2})\right)=\lim\limits_{n\to\infty}\frac{n}{2}=+\infty$, this suggests the following definiton $+\infty-(+\infty)=+\infty$;
$\lim\limits_{n\to\infty}(n-(2n))=\lim\limits_{n\to\infty}(-n)=-\infty$, this suggests the following definiton $+\infty-(+\infty)=-\infty$.
We have just seen that if two sequences converge to $+\infty$ then nothing can be said of the limit of their difference. Let $a_n=n$ and $b_n=n+(-1)^n$. Obviously $\lim\limits_{n\to\infty}a_n=+\infty$, $\lim\limits_{n\to\infty}b_n=+\infty$ whereas the difference (a_n-b_n) of sequences (a_n) i (b_n) has no limit at all because it is a geometric sequence with the quotient equal to -1. All this proves that if two sequences converge to $+\infty$ then we can say nothing of existence of the limit of the difference of the two sequences, if the limit exists we can say nothing of its value. Therefore any definition of $+\infty-(+\infty)$ would be misleading in many situations. One can say the same about other indeterminancies, e.g. $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $1^{\pm\infty}$, 0^0... The reader should make up examples showing that the definitions of the mentioned indeterminancies does not make any sense.

Remark.

Many students in the past had serious problems with the indeterminancies — the future is not known to the author. The author is sure that students who have made up examples by themselves will have almost no problems with understanding the indeterminancies.

Before we start proving the theorems we say a little bit about limits and inequalities. One could think that if for all natural numbers n the strict inequality $a_n < b_n$ holds and both sequences (a_n), (b_n) have limits then $\lim_{n\to\infty} b_n < \lim_{n\to\infty} a_n$. Unfortunately it is not so. Let $a_n = \frac{1}{2n}$, $b_n = \frac{1}{n}$. It is clear that $a_n = \frac{1}{2n} < \frac{1}{n} = b_n$ for all natural numbers n. At the same time $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$. ■

18. The Proofs

We are going to prove the theorems that where stated but not proved yet. The first in the row is estimate theorem.

Proof of the estimate theorem

The first part of the theorem is N1. A number C is less than the limit of the sequence (a_n). We have to show that for n big enough the inequality $C < a_n$ is satisfied. Assume that the limit $\lim_{n\to\infty} a_n$ is infinite. Since the limit is greater than the number C the equality $\lim_{n\to\infty} a_n = +\infty$ must be satisfied (note that $-\infty < C$). Right from the the definition it follows then for any real number M the inequality $a_n > M$ holds for all n sufficiently big. Therefore it holds for $M = C$, too.
Now we assume that the limit $\lim_{n\to\infty} a_n$ is finite. Let $\varepsilon = \lim_{n\to\infty} a_n - C$. For sufficiently big n the inequality $|a_n - \lim_{n\to\infty} a_n| < \varepsilon$ holds so $a_n > \lim_{n\to\infty} a_n - \varepsilon = C$.
The same proof works for N2. One has only to reverse few inequalities and replace $+\infty$ with $-\infty$.
Let $\lim_{n\to\infty} b_n < \lim_{n\to\infty} a_n$. There exists a number C such that $\lim_{n\to\infty} b_n < C < \lim_{n\to\infty} a_n$. From the already proved part of the theorem it follows that for all sufficiently big n's the inequalities $b_n < C$ and $C < a_n$ are satisfied. From the it follows that $b_n < a_n$. Part N3 is proved.
Let the inequality $b_n \leq a_n$ be satisfied for all sufficiently big numbers n. We want to show that $\lim_{n\to\infty} b_n \leq$ $\leq \lim_{n\to\infty} a_n$. Suppose this is no so, i.e. $\lim_{n\to\infty} b_n > \lim_{n\to\infty} a_n$. This implies that for all sufficiently big numbers n the inequality $b_n > a_n$ is satisfied, a contradiction. The proof of the estimate theorem is done. ■

From the estimate theorem we already deduced that a sequence cannot have two limits, any sequence has at most one limit and that the sequence with a finite limit is bounded.

Remark on convergence of the opposite sequence

The sequence (c_n) has a limit iff the sequence $(-c_n)$ has a limit, no matter if the limit is finite or not. If the limits exist then $\lim_{n\to\infty} (-c_n) = -\lim_{n\to\infty} c_n$. ■

It is a very simple remark. We stated it because in many cases the use of it shortens proofs, number of cases to be considered is reduced.

It is time for arithmetic properties of the limit theorem. Let $g_a = \lim_{n\to\infty} a_n$ and $g_b = \lim_{n\to\infty} b_n$. Three cases will be considered: g_a, g_b are real numbers, g_a is a real number whereas g_b is $\pm\infty$, both g_a, g_b are infinities of the same sign.

The limits g_a, g_b are finite.

Let $\varepsilon > 0$ be an arbitrary real number. Let n'_ε be na natural number such that if $n > n'_\varepsilon$ then $|a_n - g_a| < \frac{\varepsilon}{2}$. Let n''_ε be na natural number such that if $n > n''_\varepsilon$ then $|b_n - g_b| < \frac{\varepsilon}{2}$. Let n_ε be the biggest of the two numbers n'_ε, n''_ε. If $n > n_\varepsilon$ then both inequalities are satisfied, so

$$|a_n + b_n - (g_a + g_b)| \leq |a_n - g_a| + |b_n - g_b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have showed that for sufficiently big n ($n > n_\varepsilon$) the absolute value of the difference $(a_n + b_n) - (g_a + g_b)$ is less than ε, so $\lim\limits_{n\to\infty}(a_n + b_n) = g_a + g_b$.

The limit $g = \lim\limits_{n\to\infty} a_n$ *is finite and* $\lim\limits_{n\to\infty} b_n = +\infty$

We shall prove that $\lim\limits_{n\to\infty}(a_n + b_n) = +\infty$. Let M be an arbitrary real number. There exists a natural number n''_{M-g+1} such that if $n > n''_{M-g+1}$ then $b_n > M - g + 1$. There exists a natural number n'_1 such that if $n > n'_1$ then $|a_n - g| < 1$. Let n_M be the biggest of the two numbers n''_{M-g+1} and n'_1. For $n > n_M$ both inequalities are satisfied, so

$$a_n + b_n = b_n + g + (a_n - g) \geq b_n + g - |a_n - g| > (M - g + 1) + g - 1 = M\,.$$

We have showed that if n is big enough then $a_n + b_n > M$, więc $\lim\limits_{n\to\infty}(a_n + b_n) = +\infty$. The proof is done.

The limit $g = \lim\limits_{n\to\infty} a_n$ *is finite and* $\lim\limits_{n\to\infty} b_n = -\infty$

It the limit $\lim\limits_{n\to\infty} a_n$ is finite and $\lim\limits_{n\to\infty} b_n = -\infty$ then from what is proved above it follows that the sequence $\big(-a_n + (-b_n)\big)$ has a limit and $\lim\limits_{n\to\infty}(-a_n - b_n) = -\lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty}(-b_n) = +\infty$. From this and from the remark preceding the proof one can deduce that the limit $\lim\limits_{n\to\infty}(a_n + b_n)$ exists and equals $-\infty$.

Both limits are infinite: $g = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = -\infty$ *or* $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = +\infty$

From the remark preceding the proof it follows that it is enough to consider one of the two cases. Let $\lim\limits_{n\to\infty} a_n = +\infty = \lim\limits_{n\to\infty} b_n$. If M is an arbitrary real number then there exist natural numbers $n'_{M/2}$ and $n''_{M/2}$ such that if $n > n'_{M/2}$ then $a_n > \frac{M}{2}$ and if $n > n''_{M/2}$ then $b_n > \frac{M}{2}$. Let n_M be the bigger of $n > n'_{M/2}$ and $n > n''_{M/2}$. Then both inequalities hold, so $a_n + b_n > \frac{M}{2} + \frac{M}{2} = M$, so for sufficiently big n th e inequality $a_n + b_n > M$ holds, so $\lim\limits_{n\to\infty}(a_n + b_n) = +\infty$

From the remark on convergence of the opposite sequence and the part (A1) of the theorem part (A2) of the theorem follows right away, it is enough to use the formula $x - y = x + (-y)$.

Now we shall prove (A3), i.e. the limit of the product equals the product of the limits. As in the cases of sum and difference the number of cases may be reduced to the following three: both limits are finite, both limits equal $+\infty$ and one limit is $+\infty$ and the second one is a real *positive* number

Both limits are finite

Since the limits are finite, both sequences are bounded. This implies that there is a number $K' > 0$ such that $|a_n| \leq K'$ and the number K'' such that $|b_n| < K''$ for every natural number n. Let K be greater of the two numbers K', K''. Clearly $|a_n|, |b_n| \leq K$. Let $g_a = \lim\limits_{n\to\infty} a_n$, $g_b = \lim\limits_{n\to\infty} b_n$. From estimate theorem it follows that $|g_a|$, $|g_b| \leq K$. Let $\varepsilon > 0$ be an arbitrary real number. There exists n_ε such that if $n > n_\varepsilon$ then $|a_n - g_a| < \frac{\varepsilon}{2K}$ and at the same time $|b_n - g_b| < \frac{\varepsilon}{2K}$. Therefore

$$|a_n b_n - g_a g_b| = |(a_n - g_a)b_n + g_a(b_n - g_b)| \leq |a_n - g_a| \cdot |b_n| + |g_a| \cdot |b_n - g_b| < \tfrac{\varepsilon}{2K} \cdot K + K \cdot \tfrac{\varepsilon}{2K} = \varepsilon.$$

We have proved that for sufficiently large n the distance $a_n b_n$ from $g_a g_b$ is less than ε, this means that $g_a g_b = \lim\limits_{n\to\infty}\big(a_n \cdot b_n\big)$ as we wanted to show.

One limit is finite, the second one is infinite

Let $g_a = \lim\limits_{n\to\infty} a_n$ be a positive real number and let $+\infty = \lim\limits_{n\to\infty} b_n$. Let M be an arbitrary real number. It follows from the estimate theorem that there is a natural number n_M such that if $n > n_M$ then $a_n > \frac{1}{2}g_a > 0$ and $b_n > \frac{2|M|}{g_a} > 0$. So we have $a_n b_n > \frac{1}{2}g_a \frac{2|M|}{g_a} = |M| \ge M$, thus $\lim\limits_{n\to\infty}(a_n b_n) = +\infty$. This ends the proof in this case.

Both limits are infinite

Let $\lim\limits_{n\to\infty} a_n = +\infty = \lim\limits_{n\to\infty} b_n$. If M is a real number then there exists an natural number n_M such that for all $n > n_M$ the inequalities $a_n > 1 + |M|$ and $b_M > 1 + |M|$ hold. Then for $n > n_M$ the inequality $a_n b_n > (1 + |M|)^2 > 2 \cdot |M| \ge |M| \ge M$ is satisfied.This proves that $+\infty = \lim\limits_{n\to\infty} a_n b_n$.
The theorem about the limit of the product of two sequences has been proved.
The last part of the theorem is related to the limit of the quotient of two sequences. As before we start with finite limits. Let $g_a = \lim\limits_{n\to\infty} a_n$ and $g_b = \lim\limits_{n\to\infty} b_n \ne 0$. We shall show that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{g_a}{g_b}$. Let $\varepsilon > 0$ be a real number. There exist a number n_ε such that if $n > n_\varepsilon$ then $|b_n| > \frac{|g_b|}{2}$, $|a_n - g_a| < \frac{\varepsilon\cdot|g_b|}{4}$, $|b_n - g_b| < \frac{\varepsilon\cdot|g_b|^2}{4(|g_a|+1)}$.* If $n > n_\varepsilon$ then

$$\left|\frac{a_n}{b_n} - \frac{g_a}{g_b}\right| = \frac{|a_n g_b - g_a b_n|}{|g_b b_n|} \le \frac{|a_n g_b - g_a g_b| + |g_a g_b - g_a b_n|}{|g_b|^2/2} = \frac{2}{|g_b|}|a_n - g_a| + \frac{2|g_a|}{|g_b|^2}|g_b - b_n| < \varepsilon.$$

The theorem is rpoved in the case of finite limits. If $\lim\limits_{n\to\infty} a_n = +\infty$ whereas $\lim\limits_{n\to\infty} b_n$ is finite and different from 0 then the sequence $\left(\frac{1}{b_n}\right)$ is convergent to a limit which finite and different from 0 — this follows from the theorem for finite limits which has been proved already. Now we can apply the theorem about the limit of the product of the two sequences: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty}\left(a_n \cdot \frac{1}{b_n}\right) = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} \frac{1}{b_n} = +\infty \cdot \lim\limits_{n\to\infty} \frac{1}{b_n}$. The last product is well defined because $\lim\limits_{n\to\infty} \frac{1}{b_n} \ne 0$.
One more case is left: the limit of (a_n) is finite and the limit of (b_n) is infinite. In this case the sequence (a_n) is bounded, i.e. there exists $K > 0$ such that for every n the inequality $|a_n| < K$ holds. If $\varepsilon > 0$ then there exists a number n_ε such that if $n > n_\varepsilon$ then $|b_n| > \frac{K}{\varepsilon}$. Therefore $\left|\frac{a_n}{b_n}\right| < K \cdot \frac{\varepsilon}{K} = \varepsilon$. We have proved that for n sufficiently large the quotient $\frac{a_n}{b_n}$ is less in absolute value than ε. This means that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$. The proof has been completed. ■

Remark

From the **proof** of the theorem it follows that if a sequence (a_n) is bounded and $\lim\limits_{n\to\infty} |b_n| = +\infty$ then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ — we do not assume that the sequence (b_n) has a limit, we assume less, namely that the sequence $\big(|b_n|\big)$ tends to $+\infty$, the convergence of the sequence (a_n) has been used in the proof only to prove that the sequence (a_n) is bounded. ■

* We have not assumed that $g_a \ne 0$. Therefore $|g_a|+1 > |g_a|$ occurs in the denominator.

Next theorem to be proved is the three sequence theorem. For sufficiently large n the double inequality $a_n \le b_n \le c_n$ is satisfied. The sequences (a_n) and (c_n) converge to the same limit g. We want to show that the sequence (b_n) also tends to g. Assume at first that the limit g is finite. Let $\varepsilon > 0$ an arbitrary number. There exists a natural number n_ε such that if $n > n_\varepsilon$ $|a_n - g| < \varepsilon$ and $|c_n - g| < \varepsilon$. Therefore $g - \varepsilon < a_n \le b_n \le c_n < g + \varepsilon$ so $|b_n - g| < \varepsilon$. We have proved that $g = \lim\limits_{n\to\infty} b_n$. It is time for the infinite limits. It suffices to prove the theorem for one type of infinity only, e.g. $g = -\infty$. Let M be a real number. Since $\lim\limits_{n\to\infty} c_n = -\infty$, there exists a natural number n_M such that dla $n > n_M$ the inequality $b_n \le c_n < M$ holds, in particular $b_n < M$. This ends the proof. ■

Remark

From the **proof** of the theorem it follows that in the case of an infinite limit, e.g. $\lim\limits_{n\to\infty} a_n = \pm\infty = \lim\limits_{n\to\infty} c_n$ we need to use one of them only. In the situation discussed above we have not used the sequence (a_n) at all. This implies the following theorem *if for n big enough the inequality $b_n \le c_n$ holds and $\lim\limits_{n\to\infty} c_n = -\infty$ then the sequence (b_n) tends to $-\infty$; if for n big enough the inequality $a_n \le b_n$ holds and $\lim\limits_{n\to\infty} a_n = \infty$ then $\lim\limits_{n\to\infty} b_n = +\infty$*..■

The proof of the junction theorem.

This proof is very simple. Let the limit g be finite and let $\varepsilon > 0$. There exist numbers n'_ε and n''_ε such that if $n > n'_\varepsilon$ then $|a_{k_n} - g| < \varepsilon$, if $n > n''_\varepsilon$ then $|a_{l_n} - g| < \varepsilon$. Since $k_n \to \infty$ and $l_n \to \infty$, there exists n_ε such that if $n > n_\varepsilon$ and m is chosen so that $a_n = a_{k_m}$ or $a_n = a_{l_m}$ then $m > n'_\varepsilon$ oraz $m > n''_\varepsilon$, therefore $|a_n - g| < \varepsilon$. This means that $g = \lim\limits_{n\to\infty} a_n$. Small modifications of the proof are required in the case of an inifite limit. ■

The proof of Bolzano – Weierstrassa theorem.

If the sequence (a_n) is not bounded from above then there it contains a strictly increasing subsequence: let $n_1 = 1$; since the sequence (a_n) is not bounded from above it contains terms greater than a_{n_1} with $n > n_1$; let n_2 be such a number that $n_2 > n_1$ and $a_{n_2} > a_{n_1}$; since the sequence (a_n) is not bounded from above it contains terms a_n such that $a_n > a_{n_2}$ with $n > n_2$, let n_3 be the index of one of them, so $n_3 > n_2$ oraz $a_{n_3} > a_{n_2}$; we can go on with this construction. When the sequence is unbounded from below one can construct a decreasing subsequence making very little changes in the proof in the previous case. Most important case is a bounded sequence.

Let c, d be such real numbers that the inequality $c \le a_n \le d$ holds for every n; in this situation c is a lower bound of the sequence (a_n) while d is an upper bound. If the sequence (a_n) contains a constant sequence then this sequence is convergent. Let us assume that (a_n) contains no constant subsequence. This means that every real number may show up in the sequence at most finitely many times. This observation is not very significant but makes the presentation of the proof shorter. Let $n_1 = 1$, $c_1 = c$, $d_1 = d$. One of the two halves of the interval $[c, d]$ (or both) contains infinitely many terms of the sequence (a_n), let $[c_2, d_2]$ be this half (if the interval $\left[c, \frac{c+d}{2}\right]$ contains infinitely many terms of the sequence (a_n) then we set $c_2 = c_1 = c$ and $d_2 = \frac{c+d}{2}$, if the interval $\left[c, \frac{c+d}{2}\right]$ contains only finitely many terms of the sequence (a_n) then there are infinitely many of them in $\left[\frac{c+d}{2}, d\right]$, in such case we set $c_2 = \frac{c+d}{2}$ and $d_2 = d_1 = d$), let $n_2 > n_1$ be such a number that $a_{n_2} \in [c_2, d_2]$. Now we repeat the same procedure taking into account the interval $[c_2, d_2]$ instead of $[c_1, d_1]$ and the sequence terms that follow a_{n_2} rather than a_{n_1}. At the end we get a natural number $n_3 > n_2$ and an interval $[c_3, d_3] \subseteq [c_2, d_2]$ that contains infinitely many terms of the sequence (a_n), among them a_{n_3}. For $j = 1, 2, 3$ the inequalities $c_j \le a_{n_j} \le d_j$, $c_1 \le c_2 \le c_3$ and $d_1 \ge d_2 \ge d_3$ and the equality $d_j - c_j = \frac{d-c}{2^j}$ are satisfied. If we shall continue the procedure a non-decreasing sequence (c_j) and non-increasing sequence (d_j) such that $d_j - c_j = \frac{d-c}{2^j}$ will be obtained. Both sequences have limits because they are monotone. These limits are equal because $\lim\limits_{n\to\infty}(d_j - c_j) = \lim\limits_{n\to\infty} \frac{1}{2^j} \cdot (d-c) = 0$. Since $c_j \le a_{n_j} \le d_j$ for all indices j, the sequence (a_{n_j}) converges to the same limit. This ends the proof.■

Proof of the corollary 1

In view of Bolzano–Weierstrass theorem it is enough to prove that the sequence (a_n) which does not have any limit contains two subsequences with distinct limits (finite or not). Let us assume that the sequence (a_n) contains a sequence tending to $+\infty$. Since $+\infty$ is not a limit of (a_n) there exists a real number B such that for infinitely many n the inequality $a_n < B$ is satisfied. Let a_{k_n} be a susequence of the sequence (a_n) consisting of those terms of a_n which are less than B. It follows from Bolzano–Weierstrass theorem that the sequence (a_{k_n}) contains a sequence with a limit g. Obviously $g \le B$. Therefore the sequence (a_n) contains two subsequences: one tends to $+\infty$, the other one to $g \le B < +\infty$. If the sequence (a_n) contains a subsequence tending to $-\infty$ then we replace it with the sequence $(-a_n)$ and apply to it the already proved part ot the theorem to see that the sequence $(-a_n)$ contains two subsequences with distinct limits. Now we shall consider a sequence (a_n) which does not contain any subsequence with an infinite limit. Since (a_n) contains no subsequence tending to $+\infty$, it must be bounded from above. In the same way we may notice that it is bounded from below. Therefore the sequence (a_n) is bounded. It contains a subsequence (a_n) convergent to some number g. Since g is **not** a limit of (a_n), there exists a number $\varepsilon > 0$ such that outside of the interval $(g-\varepsilon, g+\varepsilon)$ there are infinitely many terms of the sequence (a_n). Bolzano–Weierstrass theorem guarantees that it is possible to choose a convergent subsequence out of them. The chosen subsequence cannot converge to g, if fact the distance from its limit to g cannot be less than ε. The proof has been completed. ■

Remark.

It is pretty clear that infinite limits look different from the finite ones. On the other hand the theorems on them are not very different from the theorems on finite limits, also the ideas of proofs are quite close. Some differences could vanish if we decided to define a distance from $\pm\infty$. This could be done so that the new way of measuring the distances would change neither the set of convergent sequences nor their limits (including the infinite limits).* There are many ways of defining such a „distance". We shall give one example.

Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ for every real number x, $f(+\infty) = 1$ and $f(-\infty) = -1$. Clearly $\lim\limits_{n\to\infty} x_n = g$ iff

* A sequence (a_n) converges to a limit g iff the distance from a_n to g tends to 0 as $n\to\infty$.

$\lim_{n\to\infty} f(x_n) = f(g)$. The last equality is equivalent to $\lim_{n\to\infty} |f(x_n) - f(g)| = 0$. Therefore it makes sense to say that the distance from x to y is equal to $|f(x) - f(y)|$. In this definition x or y mah be infinite. The idea is to identify real numbers with the points of $(-1,1)$, $+\infty$ with 1, $-\infty$ with -1. There some disadvantages. One of them is that $f(x+y) \neq f(x) + f(y)$, so the proposed identification is not related to addition or multiplication. ■

We are going to prove that the Cauchy condition is equivalent to existence a finite limit for the sequence. If a sequence (a_n) is convergent to g and $\varepsilon > 0$, then for all sufficiently big natural numbers n the inequality $|a_n - g| < \frac{\varepsilon}{2}$ holds. Therefore if natural numbers k and l are sufficiently large then $|a_k - a_l| = |a_k - g + g - a_l| \leq |a_k - g| + |g - a_l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This proves that for a sequence convergent to a finite limit the Cauchy condition is satisfied.

Let a sequence (a_n) satisfies the Cauchy condition. There exists a number n_1 such that if $k, l > n_1$ then $|a_k - a_l| < 1$. Let $l = n_1 + 1$. Then $|a_k| - |a_l| \leq |a_k - a_l| < 1$ so $|a_k| \leq 1 + |a_l|$ for all k big enough. This implies that the sequence (a_n) is bounded, therefore it contains a convergent subsequence (a_{n_m}). Let g be its limit. We shall prove that g is a limit of the whole sequence. If $\varepsilon > 0$ then for $k, \tilde{k}, m$ big enough the inequalities $|a_k - a_{\tilde{k}}| < \frac{\varepsilon}{2}$ and $|a_{n_m} - g| < \frac{\varepsilon}{2}$ hold. Numbers $m, \tilde{k}$ are chosen arbitrarily, the only condition imposed of the choice is that both numbers must be sufficiently big. Since $n_m \geq m$, we may choose $\tilde{k}$ so that $\tilde{k} = n_m$. For k sufficiently large we have $|a_k - g| \leq |a_k - a_{\tilde{k}}| + |a_{n_m} - g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This implies that $g = \lim_{n\to\infty} a_n$. ■

The last and the most difficult theorem in this chapter is Stolz theorem. The students who really want to understand the theory should study this proof carefully. Everyone who will understand the proof will be able to understand many later theorems faster than without it. At the same time one must say that one can study economy or even mathematics without this theorem.

Proof of the Stolz theorem

There is no loss of generality if one assumes that the sequence (b_n) is strictly increasing — if necessary we can replace the sequence (b_n) with the sequence $(-b_n)$. Let m, M be such real numbers that $m < g < M$, in case of $g = -\infty$ only M is considered, if $g = +\infty$ only m is considered. Choose m', M' so that $m < m' < g < M' < M$. Since the limit of the sequence $\left(\frac{a_{n+1}-a_n}{b_{n+1}-b_n}\right)$ equals g there exists a number n_0 such that if $n > n_0$ then $m' < \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < M'$. Multiplying this inequality by $b_{n+1} - b_n > 0$ we obtain:

$$m'(b_{n+1} - b_n) < a_{n+1} - a_n < M'(b_{n+1} - b_n)\,. \qquad (N_{n,n+1})$$

Add the inequalities $(N_{n,n+1})$, $(N_{n+1,n+2})$, ..., $(N_{n+k-1,n+k})$ to get

$$m'(b_{n+k} - b_n) < a_{n+k} - a_n < M'(b_{n+k} - b_n) \qquad (N_{n,n+k})$$

We use the assumption (ii). The sequence (b_n) increases (strictly) and converges to 0, so its terms are less than 0. Therefore

$$-mb_n < -m'b_n = \lim_{k\to\infty} m'(b_{n+k} - b_n) \leq -a_n = \lim_{k\to\infty}(a_{n+k} - a_n) \leq \lim_{k\to\infty} M'(b_{n+k} - b_n) = -M'b_n < -Mb_n.$$

Let us divide the inequality by $-b_n > 0$. The result is $m < \frac{a_n}{b_n} < M$. The numbers m, M have been chosen arbitrarily, so $g = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$. Case (ii) is finished.

Now we use hypothesis (i). The increasing sequence (b_n) tends to $+\infty$, so its terms are positive except for finitely many of them. In the sequel we assume that $a_n > 0$ for $n > n_0$, this may require some increase of n_0 already chosen. Divide the inequality $(N_{n,n+k})$ by b_{n+k}. The result is

$$m'\left(1 - \frac{b_n}{b_{n+k}}\right) < \frac{a_{n+k}}{b_{n+k}} - \frac{a_n}{b_{n+k}} < M'\left(1 - \frac{b_n}{b_{n+k}}\right),$$

so

$$m'\left(1 - \frac{b_n}{b_{n+k}}\right) + \frac{a_n}{b_{n+k}} < \frac{a_{n+k}}{b_{n+k}} < M'\left(1 - \frac{b_n}{b_{n+k}}\right) + \frac{a_n}{b_{n+k}}.$$

Since

$$\lim_{k\to\infty}\left[m'\left(1 - \frac{b_n}{b_{n+k}}\right) + \frac{a_n}{b_{n+k}}\right] = m' > m, \text{ oraz } \lim_{k\to\infty}\left[M'\left(1 - \frac{b_n}{b_{n+k}}\right) + \frac{a_n}{b_{n+k}}\right] = M' < M,$$

there exists k_n such that if $k > k_n$ then the inequalities

$$m'\left(1 - \tfrac{b_n}{b_{n+k}}\right) + \tfrac{a_n}{b_{n+k}} > m \text{ and } M'\left(1 - \tfrac{b_n}{b_{n+k}}\right) + \tfrac{a_n}{b_{n+k}} < M$$

hold, so $m < \frac{a_{n+k}}{b_{n+k}} < M$ for $n > n_0$ and $k > k_n$. This implies that $\lim\limits_{m\to\infty} \frac{a_m}{b_m} = g$. The case (i) is fully discussed. ■

19. Exponential function $\exp(x)$, number e.

We have shown already (cf. 14.) that for every real number x there exists a finite limit $\lim\limits_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. The limit has been denoted by $\exp(x)$. This means that a function has been defined on the whole real line. We are going to study the most important properties of this function.

a. $\exp(x) > 0$ for real numbers x.

It is so because from some place ($n > -x$) the sequence $\left(\left(1 + \frac{x}{n}\right)^n\right)$ is non-decreasing and its terms are posiotive. ■

b. For every real number x the inequality $\exp(x) \geq 1 + x$ is satisfied.

For $n > -x$ we have $\frac{x}{n} > -1$ so — Bernoulli inequality implies that $\left(1 + \frac{x}{n}\right)^n \geq 1 + n \cdot \frac{x}{n} = 1 + x$. Since from some place all terms of the sequence are greater than or equal to $1 + x$, so is the limit of this sequence.

Later on we shall see that for $x \neq 0$ the inequality is sharp. ■

c. Lemma on limits of n-th powers of the sequences „converging quickly” to 1

If $\lim\limits_{n\to\infty} n \cdot a_n = 0$ then $\lim\limits_{n\to\infty} (1 + a_n)^n = 1$.

Proof.

Since $\lim\limits_{n\to\infty} n \cdot a_n = 0$, there exists n_0 such that if $n > n_0$ then $|n \cdot a_n| < \frac{1}{2}$. For such n we have $|a_n| = \frac{1}{n} \cdot (|n \cdot a_n|) < \frac{1}{n} \cdot \frac{1}{2} \leq \frac{1}{2}$. Therefore for every $n > n_0$ three inequlities $n \cdot a_n > -\frac{1}{2} > -1$, $\frac{a_n}{1+a_n} > -1$ and $\frac{n \cdot a_n}{1+a_n} < 1$ are satisfied. Therefore below we can apply twice Beronulli inequality

$$1 + n \cdot a_n \leq (1 + a_n)^n = \frac{1}{\left(1 - \frac{a_n}{1+a_n}\right)^n} \leq \frac{1}{1 - \frac{n a_n}{1+a_n}}$$

Note please that n_0 has been chosen so that the denominators are positive, so we can switch to the reciprocal expressions after the Bernoulli inequality has been applied. The assertion follows from the three sequence theorem because $\lim\limits_{n\to\infty}(1+n\cdot a_n)=1=\lim\limits_{n\to\infty}\frac{1}{1-\frac{na_n}{1+a_n}}$. The Lemma is proved. ■

d. Basic equation

For any real numbers x,y the equation $\exp(x+y)=\exp(x)\cdot\exp(y)$ is satisfied.

We shall use the definition of the number $\exp(x)$ and its previously proved properties, in particular the inequality $\exp(x)>0$. The equation to be proved is equivalent to $\frac{\exp(x)\cdot\exp(y)}{\exp(x+y)}=1$. We have

$$\frac{\exp(x)\cdot\exp(y)}{\exp(x+y)}=\lim_{n\to\infty}\left(\frac{(1+\frac{x}{n})(1+\frac{y}{n})}{1+\frac{x+y}{n}}\right)^n=\lim_{n\to\infty}\left(1+\frac{\frac{xy}{n^2}}{1+\frac{x+y}{n}}\right)^n=1$$

The last equation follows from the lemma on powers of sequences converging quickly to 1 and the equation $\lim\limits_{n\to\infty}\left(n\cdot\frac{\frac{xy}{n^2}}{1+\frac{x+y}{n}}\right)=0$. ■

e. For every real number x the equation $\exp(-x)=\frac{1}{\exp(x)}$ holds.

Notice that $\exp(0)=\exp(0+0)=\exp(0)\cdot\exp(0)$, $+\infty>\exp(0)>0$ so $\exp(0)=1$.* Therefore $1=\exp(0)=\exp(-x+x)=\exp(-x)\cdot\exp(x)$ so $\exp(-x)=\frac{1}{\exp(x)}$. ■

f. For every real number x, any integer p and any positive integer q the equation: $\exp(\frac{p}{q}x)=(\exp(x))^{p/q}$ holds.

If m is a natural number, y — a real number then $\exp(my)=\exp(y+y+\ldots+y)=$ $=\exp(y)\cdot\exp(y)\cdot\ldots\cdot\exp(y)=(\exp(y))^m$. This implies that $\exp(\frac{x}{q})=\sqrt[q]{\exp(x)}=(\exp(x))^{1/q}$, the previous equation is applied with $y=\frac{x}{m}$ and $m=q$. Thus for $p>0$ the equality

$$\exp(\tfrac{p}{q}x)=\left(\exp(\tfrac{x}{q})\right)^p=\left((\exp(x))^{1/q}\right)^p=(\exp(x))^{p/q}$$

is satisfied. If $p<0$ then $\exp(\frac{p}{q}x)=\frac{1}{\exp(\frac{-p}{q}x)}=\frac{1}{(\exp(x))^{-p/q}}=(\exp(x))^{p/q}$. QED. ■

g. Definition of e

Number e is equal to $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, i.e. $e=\exp(1)$.

The study of this number and functions related to it, e.g. e^x was initiated by L.Euler, Swiss mathematician who worked at St Petersburg Academy of Sciences (1727-1744,1766-1783) and Berlin Academy of Science (1744-1766). The number e is one of the most important in mathematics. In this course it is the most important base of powers and logarithms. From the already proved theorems it follows that $\exp(w)=e^w$ for all rational numbers w — it suffices to set $\frac{p}{q}=w$ and $x=1$ in **f.** We know that $e=\exp(1)\geq 1+1=2$. ■

h. If $x<1$ then the double inequality $1+x\leq\exp(x)\leq\frac{1}{1-x}$ is satisfied.

The left hand side inequality has been proved already (cf. b) for all real x. Let us switch to the right hand side. Substitute $-x$ for x in $\exp(x)\geq 1+x$ to get $\exp(-x)\geq 1-x$. Thus $\exp(x)=\frac{1}{\exp(-x)}\leq\frac{1}{1-x}$. ■

* another proof: $\exp(0)=\lim_{n\to\infty}\left(1+\frac{0}{n}\right)^n=\lim_{n\to\infty}1=1$.

i. Continuity of the function $\exp$

If $\lim\limits_{n\to\infty} x_n = x$ then $\lim\limits_{n\to\infty} \exp(x_n) = \lim\limits_{n\to\infty} \exp(x)$, too.

This property of the function $\exp$ is called continuity of it. Properties of continous functions will be studied later in this course. Now we shall prove that the function $\exp$ is continuous. Let $|h| < \frac{1}{2}$. Then $h \le \exp(h) - 1 \le \frac{1}{1-h} - 1 = \frac{h}{1-h}$. Therefore if $|h| < \frac{1}{2}$ then $|\exp(h) - 1| \le 2|h|$. If $\lim\limits_{n\to\infty} x_n = x$ then for all n sufficiently large $|x_n - x| < \frac{1}{2}$ so

$$0 \le |\exp(x_n) - \exp(x)| = |\exp(x)\,(\exp(x_n - x) - 1)| \le \exp(x) \cdot 2 \cdot |x_n - x|.$$

Now it is enough to apply the three sequence theorem. ■

j. The characterization of the exponential function

Suppose that a function f defined on the set of all real numbers has the properties:

(i) if $\lim\limits_{n\to\infty} x_n = x$ then $\lim\limits_{n\to\infty} f(x_n) = f(x)$, i.e the function f is continuous;

(ii) for arbitrary real numbers x, y the equation $f(x+y) = f(x)f(y)$ is satisfied;

(iii) $f(1) = e = \exp(1)$.

In this situation for every real number x the equality $f(x) = \exp(x)$ holds.

According to this theorem the properties (i) and (ii) define an exponential function. The property (iii) fixes the base of the exponential function. Had we omitted in the formulation the property (iii), the assertion was $f(x) = (f(1))^x$. Now we are going to prove the theorem.

$f(x) = f(\frac{x}{2} + \frac{x}{2}) = f(\frac{x}{2}) \cdot f(\frac{x}{2}) = f(\frac{x}{2})^2 \ge 0$ for all $x \in \mathbb{R}$. If for some real number x_1 the equality $f(x_1) = 0$ is satisfied then $f(x) = f(x_1)f(x - x_1) = 0$ for all real numbers x. Therefore the function f is either positive at all points or is constant and equal to 0. In our case $f(1) \ne 0$ so our function is positive everywhere. We can repeat everything said in **f** about the function $\exp$, replacing $\exp$ with f. In this way we can see that for every real number x, every integer p and every positive integer q the equation $f(\frac{p}{q}x) = (f(x))^{p/q}$ holds. In particular for $x = 1$. This means that $f(\frac{p}{q}) = (f(1))^{p/q} = e^{p/q} = \exp(\frac{p}{q})$. We have just shown that f coincides with $\exp$ on the set of all rational numbers. For every real number x there exists a sequence of rational numbers (w_n) convergent to x. Therefore using the continuity of both functions f and $\exp$ we can write $f(x) = \lim\limits_{n\to\infty} f(w_n) = \lim\limits_{n\to\infty} \exp(w_n) = \exp(x)$. * This ends the proof. ■

k. The values of the exponential function $\exp$**.**

For every real number $y > 0$ there exists a real number x such that $y = e^x = \exp(x)$.

Let us give a proof of this statement. It follows right away from the properties of the geometric series that $\lim\limits_{n\to\infty} e^n = +\infty$ and $\lim\limits_{n\to\infty} e^{-n} = 0$. From this it follows that there exists a natural number n such that $e^{-n} < y < e^n$. Let $c = e^{-n}$, $d = e^n$. There are two possibilities $\exp\left(\frac{c+d}{2}\right) \le y$ or $\exp\left(\frac{c+d}{2}\right) > y$. In the first case we define: $c_1 = \frac{c+d}{2}$, $d_1 = d$, in the second one: $c_1 = c$, $d_1 = \frac{c+d}{2}$. In both cases the

* The author has no idea how a power a^x with irrational x is defined at high schools, this may depend on a teacher, on the book or other factors and suspects that majority of high school students can give no definition of a power with an irrational exponent. In fact all possible definitions of it must somehow refer to continuity and the definition of the function for the rational exponents, this may be done implicitly or explicitly, instead of continuity one may talk of monotonicity. One way of avoiding this long path is to set $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

interval $[c_1, d_1]$ contained in $[c, d]$ is twice shorter than $[c, d]$ and $\exp(c_1) \le y \le \exp(d_1)$. In the same way one can replace the interval $[c_1, d_1]$ with an interval $[c_2, d_2]$ twice shorter than $[c_1, d_1]$ contained in $[c_1, d_1]$ with $\exp(c_2) \le y \le \exp(d_2)$. This process can be continued. At the end we obtain two sequences (c_n) and (d_n) such that $d_n - c_n = \frac{d-c}{2^n}$, $c_n \le c_{n+1}$, $d_n \ge d_{n+1}$ and $\exp(c_n) \le y \le \exp(d_n)$ for every natural number n. Obviously the sequences (c_n) and (d_n) converge to the same finite limit that we denote by g. Therefore $\exp(g) = \lim\limits_{n\to\infty} \exp(c_n) \le y$ and $\exp(g) = \lim\limits_{n\to\infty} \exp(d_n) \ge y$. It follows from the last two inequalities that $\exp(g) = y$. QED. ■

l. Monotonicity of the exponential function

The function $\exp$ is strictly increasing, i.e. if $x < y$ then $\exp(x) < \exp(y)$.

Proof: $\exp(y) = \exp(y-x) \cdot \exp(x) > (1 + y - x) \cdot \exp(x) > 1 \cdot \exp(x) = \exp(x)$. ■

m. Important limit *

If $h_n \ne 0$ for every n and $\lim\limits_{n\to\infty} h_n = 0$ then $\lim\limits_{n\to\infty} \frac{\exp(x+h_n)-\exp(x)}{h_n} = \exp(x)$.

It is enough to prove that $\lim\limits_{n\to\infty} \frac{\exp(h_n)-1}{h_n} = 1$ because $\frac{\exp(x+h_n)-\exp(x)}{h_n} = \exp(x) \cdot \frac{\exp(h_n)-1}{h_n}$. Let $0 \ne h < \frac{1}{2}$. This implies that $0 < \frac{1}{1-h} < 2$. The equality $\frac{\exp(h)-1}{h} - 1 = \frac{\exp(h)-1-h}{h}$ holds. From the inequality $\frac{1}{1-h} \ge \exp(h) \ge 1 + h$ it follows immediately that

$$0 \le \exp(h) - 1 - h \le \tfrac{1}{1-h} - 1 - h = \tfrac{h^2}{1-h} = \tfrac{|h|}{1-h} \cdot |h|.$$

Divide the inequality by $|h| > 0$ to obtain

$$0 \le \left| \frac{\exp(h)-1}{h} - 1 \right| = \left| \frac{\exp(h)-1-h}{h} \right| \le \frac{1}{1-h} \cdot |h| < 2|h|. \tag{19.m}$$

The theorem follows from this inequality and the three sequence theorem. ■

n. Estimates, finding decimal estimates of the number e

We know already quite a lot about the number e and the exponential function with base e. It is time for some explanations. We know already that the sequence $\left(1 + \frac{x}{n}\right)^n$ converges to $e^x = \exp(x)$. A natural question arises: how big the number n should be to know that the distance from the sequence term to the limit is small. It is not quite clear what „*small*" means. If we want to find the dimensions of the desk on which the device used for writing these sentences stands then an error of 1 m is incredibly large because the desk is 182 cm long. If we talk of the distance from Warsaw to Kraków then it is impossible or at least it is very hard to define this distance with such a precision. Usually we interested in a relative errors. The inequality (19.m) may be read this way. An approximate formula $e^h = \exp(h) \approx 1 + h$ is used. We are interested if the error is small when compared to h. If $|h| < \frac{1}{200}$ then the relative error, i.e. the quotient $\frac{e^h-1-h}{h}$ is less than $\frac{1}{100}$, so it is less than 1%. If we know only that $|h| < 1$ then the inequality (19.m) allows to say only that the relative error is less than 200%, this is of no help in any problem, moreover we do not know what is the real size of the error, we have only an estimate from above. We shall prove later that for $h \approx 1$ the precision of an estimate is unsatisfactory. The situation

* In this paragraph the derivative of *exp* will be found, the definition of the derivative will be given in the later chapters!

is very different if small h are considered. In the case of a small number h the precision is high. This means that in particular in the case of a bank account with a low interest considered in a relatively short period of time it is not important how one interprets the rules of compounding interest. The situation becomes different if long periods of time are under consideration or the interest rate is high. We mentioned before a problem of another type at which an exponential function shows up, namely the length of a steel rail considered as a function of temperature. In this situation h is very small because it depends on a coefficient which is very small and on the change of the temperature which is moderate. In this situation it makes no sense to look at the precise formula with the exponential function, because its linear approximation does the job with sufficient precision and its easier in use. In case of the radioactive decay also mentioned before the error is much too big, in this case one has to use an exponential! How one can estimate the error we shall show when Taylor's formula will appear. In general precise estimates are hard to get, although theoretically it is possible to find them.

In the next part of this chapter we shall show some estimates. They will be obtained with very elementary techniques and therefore sometimes students with smaller experience in mathematics may find them lengthy or hard. The author's suggestion is: omit the calculations and go straight to the results. If one finds the results interesting he/she may try to go back to the estimates but have in mind that in the next chapters some theorems will be shown which will allow many simplifications in these proofs. Next important thing to realize is that there is no need to memorize the details though it is worth to go through the harder parts of the book to understand better how mathematicians work. Another reason is that after some study of elementary proofs it will be much easier to realize the advantages of the calculus presented later.

At this point we should say that although $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ the initial terms of the sequence do not give reasonable approximation of $e \approx 2,718281828459\ldots$, in the paragraph **13** the decimal approximations of the first 10 terms of the sequence are given and even in the tenth one we do not see the digit 7. This means that although the sequence converges to e, the better precision, the smaller difference from the sequence term will be seem for much bigger n. There is another sequence that converges to e^x. We are going to show that for every real number x the equality

$$e^x = \lim_{n\to\infty}\sum_{j=0}^{n}\frac{x^j}{j!} = \lim_{n\to\infty}\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\right)$$

holds. Usually we use the symbol $\sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ to denote $\lim\limits_{k\to\infty}\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^k}{k!}\right)$. The new notation allows to write $e^x = \sum\limits_{k=0}^{\infty}\frac{x^k}{k!}$ for $x\in\mathbb{R}$.

We start with $x>0$ because in this case the estimates are somewhat easier than for negative x. Let k be a fixed natural number and let n be a number *not less* than k. Then

$\left(1+\frac{x}{n}\right)^n = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{x}{n}\right)^j \geq \sum_{j=0}^{k} \binom{n}{j} \left(\frac{x}{n}\right)^j =$
$= 1 + \frac{n}{n}\frac{x}{1!} + \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{x^2}{2!} + \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\frac{x^3}{3!} + \ldots + \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\ldots\cdot\frac{n-(k-1)}{n}\frac{x^k}{k!} =$
$= 1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \ldots + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!}$. It is clear that $\lim\limits_{n\to\infty}\left\{1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \ldots + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!}\right\} = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!}$.
Therefore for every natural number k and every real number $x > 0$ the inequality

$$e^x = \exp(x) \geq 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!}$$

is satisfied. The previous inequalities (set $n = k$) imply that $1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} \geq \left(1+\frac{x}{k}\right)^k$. Thus

$$e^x \geq 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} \geq \left(1+\frac{x}{k}\right)^k$$

This and the three sequence theorem imply that $e^x = \exp(x) = \lim\limits_{k\to\infty}\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!}\right)$.

Now let $x < 0$. We must change somehow the estimates because the expression $\binom{n}{j}\left(\frac{x}{n}\right)^j$ is positive for an even j whereas for an odd j this expression is negative. If $n > j > |x|$ then

$$0 > \frac{\binom{n}{j+1}\left(\frac{x}{n}\right)^{j+1}}{\binom{n}{j}\left(\frac{x}{n}\right)^j} = \frac{n-j}{j+1}\cdot\frac{x}{n} = \frac{n-j}{n}\cdot\frac{x}{j+1} > -1\,.$$

This means that from some place on the absolute values of the summands in $\sum_{j=0}^{n}\binom{n}{j}\left(\frac{x}{n}\right)^j$ decrease. This together with the fact that these summands are positive for even j and negative for odd j implies the inequality

$$\left(1+\frac{x}{n}\right)^n \geq 1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \ldots + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!}$$

is satisfied for an odd k and $n > k > |x| = -x$; if k is even then the reverse inequality is satisfied. Let k be an *even* number. We can write:

$$1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \ldots + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!} \geq$$

$$\geq \left(1+\frac{x}{n}\right)^n \geq$$

$$\geq 1 + x + \left(1-\frac{1}{n}\right)\frac{x^2}{2!} + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\frac{x^3}{3!} + \ldots + \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{k}{n}\right)\frac{x^{k+1}}{(k+1)!}\,.$$

Look at the limits as $n\to\infty$:

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} \geq e^x \geq 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}\,.$$

This implies that

$$0 \geq e^x - \left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!}\right) \geq \frac{x^{k+1}}{(k+1)!}$$

Taking into account that $\lim\limits_{k\to\infty}\frac{x^{k+1}}{(k+1)!} = 0$ one can write

$$e^x = \exp(x) = \lim_{k\to\infty}\left(1 + x + \frac{x^2}{2!} + \ldots + \frac{x^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\,.$$

Look at the decimal approximations of the first ten terms of the sequence $\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)$:

$2,000000000$, $2,500000000$, $2,666666667$, $2,708333333$, $2,716666667$,

$2,718055556$, $2,718253968$, $2,718278770$, $2,718281526$, $2,718281801$.

In this case the digit 7 can be seen already in the fourth term of the sequence, the first eight digits of the tenth term are exact — the ninth is 0 instead of 2. This is quite good precision at a little cost. We shall show later that this is not a result of an unexplainable luck but the result of the very fast convergence of the sequence to its limit e.

It is clear that

$$e-\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{k!}\right)=\lim_{n\to\infty}\left(\frac{1}{(k+1)!}+\frac{1}{(k+2)!}+\frac{1}{(k+3)!}+\ldots+\frac{1}{(k+n)!}\right).$$

For $n\geq 2$ the inequality

$$\frac{1}{(k+1)!}+\frac{1}{(k+2)!}+\frac{1}{(k+3)!}+\ldots+\frac{1}{(k+n)!}\leq\frac{1}{(k+1)!}+\frac{1}{(k+2)(k+1)!}+\frac{1}{(k+2)^2(k+1)!}+\ldots+\frac{1}{(k+2)^{n-1}(k+1)!}=$$

$$=\frac{1}{(k+1)!}\cdot\frac{1-\frac{1}{(k+2)^n}}{1-\frac{1}{k+2}}<\frac{1}{(k+1)!}\cdot\frac{1}{1-\frac{1}{k+2}}=\frac{k+2}{(k+1)!\cdot(k+1)}=\frac{k+2}{k!(k+1)^2}=\frac{k+2}{k![k(k+2)+1]}<\frac{1}{k\cdot k!}$$

— holds, obvious inequalities: $k+3\geq k+2$, $k+4\geq k+2$, $\ldots$, $k+n\geq k+2$ have been used. We have shown that the k-th term of the sequence $\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)$ differs from the number e less than $\frac{1}{k\cdot k!}$, clearly this number is little even if k is not very big .

Before we suggested that the difference between $\left(1+\frac{1}{n}\right)^n$ and e is not so small — this was the result of the numerical experiment with the first ten terms of the sequence. We are going to show that it was not random. Stolz theorem will be applied to show that the limit $\lim\limits_{n\to\infty}\frac{e-\left(1+\frac{1}{n}\right)^n}{1/n}$ equals to $\frac{e}{2}$. This means that the difference $e-\left(1+\frac{1}{n}\right)^n$ is approximately equal to the quotient of this limit and the number n, i.e. $\frac{e}{2n}>\frac{1}{n}$, of course for n sufficiently large. The result tells us that this approximation $e\approx\left(1+\frac{1}{n}\right)^n$ is not very useful because for small error one needs a very big n, this requires lengthy calculations. Even if a calculator or a computer is used we have a problem due to the truncation error that may accumulate and we cannot control this error — the electronic devices use approximations of the numbers!

Both the numerator and the denominator of the fraction $\frac{e-\left(1+\frac{1}{n}\right)^n}{1/n}$ tend to 0, the denominator decreases as n increases. Therefore we can look at the quotient of the differences of the two subsequent numerators and the two subsequent denominators, i.e. we can try to find the limit of the expression

$$\frac{\left(e-\left(1+\frac{1}{n+1}\right)^{n+1}\right)-\left(e-\left(1+\frac{1}{n}\right)^n\right)}{\frac{1}{n+1}-\frac{1}{n}}=\frac{\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n}{\frac{1}{n}-\frac{1}{n+1}}.$$

To simplify the notation we set $x=\frac{1}{n+1}$. Then $n=\frac{1}{x}-1=\frac{1-x}{x}$ so $1+\frac{1}{n}=1+\frac{x}{1-x}=\frac{1}{1-x}$. The expression which we want to study looks like this (we do not use x in the exponents):

$$\frac{(1+x)^{n+1}-\left(\frac{1}{1-x}\right)^n}{\frac{x}{1-x}-x}=\frac{(1+x)^{n+1}-\frac{1}{(1-x)^n}}{\frac{x-x(1-x)}{1-x}}=\frac{\left(1-x^2\right)^{n+1}-(1-x)}{x^2(1-x)^n}.$$

Clearly $\lim\limits_{n\to\infty}(1-x)^n = \lim\limits_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n = \lim\limits_{n\to\infty}\frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{1-\frac{1}{n+1}} = \frac{e^{-1}}{1} = \frac{1}{e}$. We need to find the limit $\lim\limits_{n\to\infty}\frac{\left(1-x^2\right)^{n+1}-(1-x)}{x^2}$. Newton's binomial formula will be used for the numerator :
$(1-x^2)^{n+1}-1+x = 1-\binom{n+1}{1}x^2+\binom{n+1}{2}x^4-\binom{n+1}{3}x^6+\cdots+(-x^2)^n-1+x =$

$$= x-(n+1)x^2+\binom{n+1}{2}x^4-\binom{n+1}{3}x^6+\cdots+(-x^2)^n = \binom{n+1}{2}x^4-\binom{n+1}{3}x^6+\cdots+(-x^2)^n$$

— the last equality follows from the equation $x=\frac{1}{n+1}$ which is equivalent to the equation $x(n+1)=1$. Obviously $\lim\limits_{n\to\infty}\frac{\binom{n+1}{2}x^4}{x^2} = \lim\limits_{n\to\infty}\frac{(n+1)\cdot n}{2(n+1)^2} = \frac{1}{2}$. If $n\geq k-1$ and $k\geq 3$ then

$$\binom{n+1}{k}x^{2k} = \frac{(n+1)n\dots(n+2-k)}{k!}x^{2k} \leq \frac{(n+1)^k}{k!}x^{2k} < \frac{1}{(n+1)^k}.$$

Therefore $\frac{\binom{n+1}{3}x^6}{x^2} \leq \frac{1}{n+1}$ oraz $\left|\binom{n+1}{4}x^8-\binom{n+1}{5}x^{10}+\cdots+(-x^2)^n\right| \leq (n-3)\frac{1}{x^2(n+1)^4} < \frac{1}{n+1}$. So we can write

$$\frac{\left|-\binom{n+1}{3}x^6+\binom{n+1}{4}x^8-\binom{n+1}{5}x^{10}+\cdots+(-x^2)^n\right|}{x^2} \leq \frac{2}{n+1}\xrightarrow[n\to\infty]{}0.$$

This together with $\lim\limits_{n\to\infty}\frac{\binom{n+1}{2}x^4}{x^2}=\frac{1}{2}$ implies that $\lim\limits_{n\to\infty}\frac{(1-x^2)^{n+1}-1+x}{x^2}=\frac{1}{2}$. From this formula and from the equality $\lim\limits_{n\to\infty}(1-x)^n=\frac{1}{e}$ the equation follows

$$\lim_{n\to\infty}\frac{\left(e-\left(1+\frac{1}{n+1}\right)^{n+1}\right)-\left(e-\left(1+\frac{1}{n}\right)^n\right)}{\frac{1}{n+1}-\frac{1}{n}} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n}{\frac{1}{n}-\frac{1}{n+1}} = \frac{e}{2}.$$

Now we apply Stolz theorem

$$\lim_{n\to\infty}n\left(e-\left(1+\frac{1}{n}\right)^n\right)=\frac{e}{2}.$$

The result tells us something of an error made when we use the approximate formula $\left(1+\frac{1}{n}\right)^n\approx e$ but only for sufficiently big n the error is $\frac{e}{2n}$. We do not know what sufficiently big means. Now we shall show a better result although majority of economists never saw them and they do not need them. The reader who is not interested in these considerations can simply go on without reading, nothing in the later parts of the course depends on these results. We shall prove that if $n\geq 1$ then

$$e-\left(1+\frac{1}{n}\right)^n \geq \frac{1}{n+2}.$$

Let us start with an estimate from below of the difference of two subsequent terms if the sequence:

$$\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n\left\{\left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^n-1\right\}.$$

For $n>3$ we have $\left(1-\frac{1}{(n+1)^2}\right)^n > 1-\frac{n}{(n+1)^2}+\frac{n(n-1)}{2(n+1)^4}-\frac{n(n-1)(n-2)}{6(n+1)^6}$

— it is so because for $j<n$ the inequality $\binom{n}{j}\left(\frac{n}{(n+1)^2}\right)^j > \binom{n}{j+1}\left(\frac{n}{(n+1)^2}\right)^{j+1}$ holds, therefore after Newton's binomial formula is applied we get the sum of $n+1$ terms the absolute values of which *decrease* with j, any two terms in a row differ in the sign, so if we stop adding at a negative term we get a sum smaller than the sum of all terms, if we stop adding at a positive term the obtained sum is greater than the sum of all terms. We have

$1-\frac{n}{(n+1)^2}+\frac{n(n-1)}{2(n+1)^4}-\frac{n(n-1)(n-2)}{6(n+1)^6}=$

$=1-\frac{1}{n+1}\left(1-\frac{1}{n+1}\right)+\frac{1}{2(n+1)^2}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)-\frac{1}{6(n+1)^3}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\left(1-\frac{3}{n+1}\right).$

We shall show that

$$\left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{(n+1)^2}+\frac{n(n-1)}{2(n+1)^4}-\frac{n(n-1)(n-2)}{6(n+1)^6}\right)-1>\frac{1}{2(n+1)^2}-\frac{1}{2(n+1)^3}\,.$$

Let $y=\frac{1}{n+1}$. Then $\frac{n}{n+1}=1-y$, $\frac{n-1}{n+1}=1-2y$, $\frac{n-2}{n+1}=1-3y$. We should prove that

$$(1+y)\left(1-y(1-y)+\frac{1}{2}y^2(1-y)(1-2y)-\frac{1}{6}y^3(1-y)(1-2y)(1-3y)\right)-1>\frac{1}{2}(y^2-y^3)\,.$$

This is equivalent to (multiply and reorder to see it):

$$\frac{1}{2}(y^2-y^3)<\frac{1}{2}y^2-\frac{1}{6}y^3+\frac{1}{3}y^4+\frac{1}{6}y^5-\frac{5}{6}y^6+y^7=\frac{1}{2}y^2-\frac{1}{2}y^3+\frac{1}{3}y^3+\frac{1}{3}y^4+\frac{1}{6}y^5-\frac{5}{6}y^6+y^7.$$

Recall that $0<y<1$, so $y^3>y^6$, $y^4>y^6$, $y^5>y^6$, therefore

$$\frac{1}{3}y^3+\frac{1}{3}y^4+\frac{1}{6}y^5-\frac{5}{6}y^6>y^6\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{6}-\frac{5}{6}\right)=0\,,$$

so the above inequality is proved. We have transformed many times different expressions but are coming to an end:

$$\left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{(n+1)^2}\right)^n-1>\frac{1}{2(n+1)^2}-\frac{1}{2(n+1)^3}=\frac{n}{2(n+1)^3}>\frac{1}{2(n+2)(n+3)}\,.$$

Notice that $\frac{1}{2(n+2)(n+3)}=\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$. If n and $k>n$ are natural numbers then to

$\left(1+\frac{1}{k}\right)^k-\left(1+\frac{1}{n}\right)^n=\left(1+\frac{1}{k}\right)^k-\left(1+\frac{1}{k-1}\right)^{k-1}+\left(1+\frac{1}{k-1}\right)^{k-1}-\left(1+\frac{1}{k-2}\right)^{k-2}+$

$+\cdots+\left(1+\frac{1}{n+2}\right)^{n+2}-\left(1+\frac{1}{n+1}\right)^{n+1}+\left(1+\frac{1}{n+1}\right)^{n+1}-\left(1+\frac{1}{n}\right)^n>$

$>\frac{1}{2}\left(1+\frac{1}{k-1}\right)^{k-1}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)+\frac{1}{2}\left(1+\frac{1}{k-2}\right)^{k-2}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\cdots+$

$+\frac{1}{2}\left(1+\frac{1}{n+1}\right)^{n+1}\left(\frac{1}{n+3}-\frac{1}{n+4}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)^n\left(\frac{1}{n+2}-\frac{1}{n+3}\right)>$

$>\frac{1}{2}\left(1+\frac{1}{n}\right)^n\left\{\frac{1}{k+1}-\frac{1}{k+2}+\frac{1}{k}-\frac{1}{k+1}+\cdots+\frac{1}{n+3}-\frac{1}{n+4}+\frac{1}{n+2}-\frac{1}{n+3}\right\}=\frac{1}{2}\left(1+\frac{1}{n}\right)^n\left(\frac{1}{n+2}-\frac{1}{k+2}\right)$

— the second inequality in the row follows from the monotonicity of the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)$ which is increasing. We find the limits of the left hand side and of the right hand side as $k\longrightarrow\infty$, we may forget of all middle terms. The conclusion is

$$e-\left(1+\frac{1}{n}\right)^n\geq\frac{1}{2(n+2)}\left(1+\frac{1}{n}\right)^n\,.$$

From this inequality it follows that for natural numbers n the inequality

$$e-\left(1+\tfrac{1}{n}\right)^n \geq \tfrac{1}{n+2}$$

holds, recall that $\left(1+\frac{1}{n}\right)^n \geq 2$. The obtained result tells us that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)$ converges very slowly to the number e. For example

$e-\left(1+\frac{1}{998}\right)^{998} \geq \frac{1}{1000}$, the situation is even worse because if $n \geq 6$ then $\left(1+\frac{1}{n}\right)^n \geq \frac{5}{2}$, so

$e-\left(1+\frac{1}{n}\right)^n \geq \frac{2{,}5}{2(n+2)} = \frac{5}{4(n+2)}$, surely $e-\left(1+\frac{1}{1248}\right)^{1248} \geq \frac{1}{1000}$, so the fourth digit of the number $\left(1+\frac{1}{1248}\right)^{1248}$ differs from that of the number e. It is clear that finding the decimal approximations of the number e with the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)$ no sense at all. ■

At the end we should say once again that we shall learn in the near future how to derive such estimates and many other theorems in a much shorter way. This requires some knowledge. We shall develop the theory which is used in many branches of science. The readers will have an opportunity to compare the amount of work necessary to get some results (as above) with bare hands with the amount of work when using differential calculus.

20. Natural logarithms

We proved that for every real number $y > 0$ there exists a real number x such that $y = e^x$.

Definition of the natural logarithm

The natural logarithm (logarithm to the base e) of the positive number y is a real number x such that $y = e^x$. We write $x = \ln y$. ■

Since the inequality $x_1 < x_2$ implies that $e^{x_1} < e^{x_2}$, the function $\ln$ is well defined: assigned to the number y is precisely one number x, moreover $\ln y$ grows with y.

Since $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$, the equality $\ln(y_1 \cdot y_2) = \ln y_1 + \ln y_2$ holds for all $y_1, y_2 > 0$.

If $a > 0$ them a power of a can be defined with a formula $a^x = e^{x\ln a} = \exp(x\ln a)$. This implies immediately that $\ln(a^x) = x\ln a$. Using the definition of the exponential function and the properties of e^x one can see that że

$$a^{x_1+x_2} = a^{x_1} \cdot a^{x_2} \text{ and } (a^{x_1})^{x_2} = \exp(x_2 \ln a^{x_1}) = \exp(x_2 x_1 \ln a) = a^{\mathbf{x_1 x_2}}.$$

Some inequalities will be useful. We know that $1+x \geq e^x$ for every real number x. This implies that $\ln(1+x) \leq x$ for $x > -1$, because the number $1+x$ must positive if we want to talk about its logarithm. If $x < 1$ then $e^x \leq \frac{1}{1-x}$, cf. 19.g. Taking logarithms of both sides one gets $x \leq \ln\left(\frac{1}{1-x}\right) = \ln\left(1+\frac{x}{1-x}\right)$. Let $y = \frac{x}{1-x}$. Then $x = \frac{y}{1+y}$, notice that $x < 1$ iff $y > -1$. If $y > -1$ then

$$\frac{y}{1+y} \leq \ln(1+y) \leq y. \tag{20.1}$$

This implies that if $y > 0$ then $\frac{1}{1+y} \leq \frac{\ln(1+y)}{y} \leq 1$, if $-1 < y < 0$ then: $\frac{1}{1+y} \geq \frac{\ln(1+y)}{y} \geq 1$. From these inequalities and the three sequence theorem it follows that if a sequence (y_n) converges to 0 and $y_n \neq 0$ for all n then

$$\lim_{n\to\infty} \frac{\ln(1+y_n)}{y_n} = 1. \tag{20.2}$$

Let $\lim\limits_{n\to\infty} x_n = x$ for some positive real numbers $x, x_1, x_2, \ldots$. Let $y_n = \frac{x_n - x}{x}$. Then

$$\ln x_n - \ln x = \ln\left(1 + \tfrac{x_n - x}{x}\right) = \ln(1+y_n) = \tfrac{\ln(1+y_n)}{y_n} \cdot y_n \xrightarrow[n\to\infty]{} 1 \cdot 0 = 0\,.$$

Therefore if $\lim\limits_{n\to\infty} x_n = x$ then $\lim\limits_{n\to\infty} \ln x_n = \ln x$. It is worth to show another proof of this statement. The reader may recall how we proved that if $\lim\limits_{n\to\infty} x_n = g$ then $\lim\limits_{n\to\infty} \sqrt[k]{x_n} = \sqrt[k]{g}$. This will not be done now but in the chapter at which we shall give basic properties of continuous functions, among them a theorem that guarantees continuity of an inverse function. In fact the proof of continuity of a root given in **17.** will be rewritten in the general case.

The equality (20.2) gives some information about the magnitude of the natural logarithm of numbers close to 1. It tells nothing of logarithms of other numbers.

From the formulas derived in **19** it follows that if $x > 0$ then for every natural number k the inequality $e^x > \frac{x^{k+1}}{(k+1)!}$ holds. This implies that if $\lim\limits_{n\to\infty} x_n = +\infty$ then $\lim\limits_{n\to\infty} \frac{x_n^k}{\exp(x_n)} = 0$.

If (x_n) is a sequence of positive numbers then $\lim\limits_{n\to\infty} x_n = +\infty$ iff $\lim\limits_{n\to\infty} \ln x_n = +\infty$. To prove it note that if M is a real number and $\lim\limits_{n\to\infty} x_n = +\infty$ then for n sufficiently large $x_n > e^M$, so $\ln x_n > M$, therefore $\lim\limits_{n\to\infty} \ln x_n = +\infty$. If M is a real number and $\lim\limits_{n\to\infty} \ln x_n = +\infty$ then for all n sufficiently large the inequality $\lim\limits_{n\to\infty} x_n > M$ holds, so $x_n > e^M \geq 1 + M > M$, therefore $\lim\limits_{n\to\infty} x_n = +\infty$.

Let $\lim\limits_{n\to\infty} x_n = +\infty$ and $x_n > 0$ for all n. Let $y_n = \ln x_n$. In this situation $\lim\limits_{n\to\infty} y_n = +\infty$. Therefore $0 = \lim\limits_{n\to\infty} \frac{y_n}{\exp(y_n)} = \lim\limits_{n\to\infty} \frac{\ln x_n}{x_n}$. We have shown that for n sufficiently large the number $\lim\limits_{n\to\infty} x_n$ incomparably smaller than the number x_n.

It is easy to find an estimate that shows that if x is a big positive number then the quotient $\frac{\ln x}{x}$ is very small. We know that $\lim\limits_{n\to\infty} x \leq x - 1$ for all positive x. Therefore $\frac{1}{2}\ln x = \ln(\sqrt{x}) \leq \sqrt{x} - 1 < \sqrt{x}$, hence $\frac{\ln x}{x} < \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$.

Let us give an example with specific numbers. $e^{10} > 2,7^{10} = 7,29^5 > 7^5 = 16807$, $\ln(e^{10}) = 10$. Therefore

$$\frac{\ln 16807}{16807} < \frac{\ln\left(e^{10}\right)}{16807} = \frac{10}{16807} < 0,0006$$

Also $e^{100} = \left(e^{10}\right)^{10} > 16807^{10} > 16000^{10} = 2^{40} \cdot 10^{30} = 1024^4 \cdot 10^{40} > 10^{32}$. It implies that

$$\frac{\ln\left(10^{42}\right)}{10^{42}} < \frac{\ln(e^{100})}{10^{42}} = 10^{-40} = 0,0000000000\,0000000000\,0000000000\,0000000001$$

It was proved that

the natural logarithm of a big positive number is very little compared with this number

and for those readers who like to see concrete numbers we provided two examples and electronic devices were not necessary for the calculations!

This property of logarithms is the reason for which they are used in situations where people have to deal with numbers very big and also with numbers very close to 0 they may vary a lot. Usually instead of natural logarithms logarithms to base 10 are used. Chemists use pH factor. It is minus logarithm of of the molar concentration of hydronium ions. If a plain water is considered than the concentration of hydronium ions in it is around $0,0000001 = 10^{-7}$, so in this case pH equals 7. Another example is Richter scale for the magnitude of earthquakes. It also logarithmic. Instead of having many zeros at the end on the number or in front one deals with logarithms of these numbers and they turn out to be of moderate size. It is also worth to mention that it is easier to sketch graphs of functions after changing the scale, sometimes without swtitching to the logarithms it is impossible to sketch a graph.

Logarithms were invented in the XVII century by J.Napier. People who performed complicated calculations wanted to replace multiplication with addition (try to multiply 5 digit numbers without calculator or computer). It was possible to do it using logarithms and the formula $\ln(xy) = \ln x + \ln y$. Tables of logarithms were made. People multiplied numbers finding their logarithms in the table, then adding them, then finding the result in the table and the number corresponding to it. It was possible to find quite fast roots of different degrees, powers. Today these methods are not used because computers and calculators are everywhere around. But it was not so until 1979 or 1980.

Logarithms of infinities

In the future we shall use the following notation $\ln(+\infty) = \infty$ and $\ln 0 = -\infty$.

It is consistent with the formulas $e^{+\infty} = +\infty$ i $e^{-\infty} = 0$ adopted earlier. Even more important reason for these definition is that if $x_n \to +\infty$ then $\ln x_n \to +\infty$, if $0 < x_n \to 0$ then $\ln x_n \to -\infty$. ■

21. Trigonometric functions

We are going to recall the high school definitions of the sine and cosine. At the beginning one should say that most often used unit for measuring angles is rather artificial. Why to divide a right angle into 90 equal parts? Maybe 100 would be better (France after the Revolution) or maybe 8 is better (sailors liked consequitive divisions by 2). In the theoretical considerations something less artificial is useful. These are radians.

We assume that the angles are oriented, this means that we have decided which arm is first an which is second, if we change the names of the arms of the angle then the angle changes to minus the initial one. Angles measured clockwise are called negative, angles measured counterclockwise are positive. Let us place an angle so that its first arm coincide with the positive x–axis, i.e. with the set of all points of the form $(x, 0)$ with $x \geq 0$. The value of a positive angle is t if the second arm of the angle meets the unit circle centered at the origin at a point P such that the length of the arc that starts at the point $(1, 0)$ and ends at C equals t. So the value of the positive right angle is $\frac{\pi}{2}$ instead of $90°$. The value of the negative right angle is of course $-\frac{\pi}{2}$, the second arm of it contains the point $(0, -1)$. We are allowed to talk of angles greater than 2π. If the value of the angle is $\frac{9\pi}{4}$ the it means that we go around the unit circle counterclockwise and then further by $\frac{1}{8}$ of the circle; the angle of value $\frac{-5\pi}{2}$ is the angle measured clockwise: we go around the circle once and then further by a quarter of the circle, all the time clockwise.

Let P be the end point of an arc of length t that starts at $(1, 0)$. We say that the first coordinate of P is $\cos t$, the second one is $\sin t$.* This is the *definition* of cosine and sine e.g. $\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=1$, $\cos\frac{3\pi}{2}=0$, $\sin\frac{3\pi}{2}=-1$, $\cos\frac{-\pi}{4}=\frac{\sqrt{2}}{2}$, $\sin\frac{-\pi}{4}=-\frac{\sqrt{2}}{2}$. Let us remind that $\tan t=\frac{\sin t}{\cos t}$, $\cot t=\frac{\cos t}{\sin t}$, $\sec t=\frac{1}{\cos t}$, $\csc t=\frac{1}{\sin t}$. In this book we shall use cosine, sine and tangent.

We give some basic properties of cosine and sine.

T1. For every real number t the formula $\sin^2 t+\cos^2 t=1$ is satisfied.

T2. For all real numbers t,s the formula $\sin(s+t)=\sin s\cos t+\sin t\cos s$ is satisfied.

T3. For all real numbers t,s the formula $\cos(s+t)=\cos s\cos t-\sin s\sin t$ is satisfied.

T4. For every real number t the equations $\cos(-t)=\cos t$ and $\sin(-t)=-\sin t$ is satisfied because the points $(\cos t,\sin t)$ and $(\cos(-t),\sin(-t))$ are symmetric relative to the x–axis.

T5. For every real number t the equations $\cos(t+\frac{\pi}{2})=-\sin t$ and $\sin(t+\frac{\pi}{2})=\cos t$ is satisfied — these equations follow from the fact that counterclockwise rotation by $\frac{\pi}{2}$ radians around $(0,0)$ brings a point (x,y) onto the point $(-y,x)$, one may derive them also from T2 and T3 and obvious statements $\cos\frac{\pi}{2}=0$, $\sin\frac{\pi}{2}=1$.

T6. For every real number t the equation $\cos(t+2\pi)=\cos t$ and $\sin(t+2\pi)=\sin t$ is satisfied. It is so because rotation by 2π moves no point. One can also use four times T5.**

T7. For all real numbers s,t the equations: $\sin s\pm\sin t=2\sin\frac{s\pm t}{2}\cos\frac{s\mp t}{2}$, $\cos s+\cos t=2\cos\frac{s+t}{2}\cos\frac{s-t}{2}$ oraz $\cos s-\cos t=-2\sin\frac{s-t}{2}\sin\frac{s+t}{2}$ are satisfied — this is an easy consequence of T2, T3 i T4.

T8. If $0<t<\frac{\pi}{2}$ then $0<\sin t<t<\operatorname{tg} t$.

The proof of the inequality will follow. Let $O=(0,0)$, $A=(1,0)$, $P=(\cos t,\sin t)$, $Q=(1,\operatorname{tg} t)$. The isosceles triangle POA is contained in the sector $\widehat{POA}$, and this sector is contained in the right triangle QOA. Therefore the area of the triangle POA is less than the area of the sector $\widehat{POA}$ the area of which is less than that of the sector QOA. Computing these areas with very well known formulas we obtain the inequality $\frac{1}{2}\cdot 1\cdot\sin t<\frac{t}{2\pi}\cdot\pi\cdot 1^2<\frac{1}{2}\cdot 1\cdot\operatorname{tg} t$ obviously equivalent to the one we are proving. ■

The inequality $\sin t<t$ is satisfied for *positive* numbers t because if $t\geq\frac{\pi}{2}$ then $t>1\geq\sin t$. Due to the equality $\sin(-t)=-\sin t$, for $t\neq 0$ the inequality $|\sin t|<|t|$ holds. Therefore $|\sin s-\sin t|=\left|2\sin\frac{s-t}{2}\cos\frac{s+t}{2}\right|\leq 2\cdot\left|\frac{s-t}{2}\right|\cdot 1=|s-t|$ for all numbers s,t. In the same way one proves that $|\cos s-\cos t|\leq|s-t|$. So we proved that

T9. For all numbers s,t the inequalities $|\sin s-\sin t|\leq|s-t|$ and $|\cos s-\cos t|\leq|s-t|$ are satisfied. ■

T10. If $\lim\limits_{n\to\infty}t_n=t$ then $\lim\limits_{n\to\infty}\sin t_n=\sin t$ and $\lim\limits_{n\to\infty}\cos t_n=\cos t$, so sine and cosine are continuous functions — the proof follows from the three sequence theorem and the property T9. ■

T11. If $\lim\limits_{n\to\infty}t_n=0$ and $t_n\neq 0$ for all n then $\lim\limits_{n\to\infty}\frac{\sin t_n}{t_n}=1$.

We shall prove this theorem. $\frac{\sin(-t)}{-t}=\frac{\sin t}{t}$, so we may assume that $t_n>0$ for all n. Since $\lim\limits_{n\to\infty}t_n=0$,

* These names come in an alphabetic order.

** There many more formulas of that sort. All these formulas are very easy, in fact obvious for people who draw a figure instead of trying to recall the formula seen some time ago or try to find somewhere in the net or in a book.

for n big enough the inequality $|t_n| < 1$ is satisfied, so $0 < t_n < 1$ because $t_n > 0$. For such numbers t, due to T8, the inequality $t(1-t^2) < t(1-\sin^2 t) = t\cos^2 t < t\cos t < \sin t < t$ holds. Therefore $t - t^3 < \sin t < t$ and $1 - t^2 < \frac{\sin t}{t} < 1$. Now we can apply the three sequence theorem. ■

The above proof may shortened. It follows from **T8.** then $\cos t_n < \frac{\sin t_n}{t_n} < 1$ and since $\lim\limits_{n\to\infty} \cos t_n = \cos 0 = 1$, the theorem follows from the three sequence theorem. A longer proof has been provided to obtain a concrete estimate of the error resulting from replacing $\sin t$ with $t \approx 0$. This estimate is not the best known. Later on we shall be able to show easily that $t - \frac{t^3}{6} < \sin t$ for $t > 0$. This will not change much.

So if $0 < t < 0.1$ then $0 < t - \sin t < t^3 < 0.01 \cdot t$. This means that the relative error due to replacement of $\sin t$ with t is less than 1% of the number t (in fact less than $< \frac{1}{6}\%$). It is quite reasonable precision, do not forget that we express all angles in radians (0.1 rad is more than $5°$). Such angles appear if one considers problems in geometric optics, mathematical pendulum or gun shooting to far away located goals.

At high school textbooks one can easily find a theorem: the period of the mathematical pendulum is independent of the amplitude. Not too many people some attention to the hypothesis: *amplitude has to be sufficiently small.* It must be so in order to guarantee that the error in the approximate formula $\sin t \approx t$ would not interfere. It is easy to notice that if the amplitude is quite big, say greater than $\frac{\pi}{2}$ then the period of the pendulum grows significantly. In the case of very little amplitudes the differences in periods are so little that it is very hard to measure the periods so precisely that they appeared noticable.

P R O B L E M S

0. Show that for any natural number n and arbitrary real numbers a, b the following equalities hold:

(a) $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n$, where $\binom{c}{k} = \frac{c(c-1)(c-2)\ldots\big(c-(k-1)\big)}{k!}$ for every real number c and $k \in \{1,2,3,\ldots\}$,

(b) $a^n - b^n = (a-b)\big(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}\big)$.

(c) $a^{2n+1} + b^{2n+1} = (a+b)\left(a^{2n} - a^{2n-1}b + a^{2n-2}b^2 - a^{2n-3}b^3 + \cdots + b^{2n}\right)$.

1. Let $a_{n+1} = qa_n$ for $n = 1, 2, \ldots,$ $a_1 \in \mathbb{R}$. Prove that $a_n = a_1q^{n-1}$ for every n. Prove that for $q \neq 1$ the equality $a_1 + a_2 + \ldots + a_n = a_1\frac{1-q^n}{1-q}$ holds.

2. Write the number $0.12345123451234512345\ldots = 0.(12345)$ as a quotient of two integers.

3. What is greater the number 1 or the number $0.99999\ldots = 0{,}(9)$? — give the detailed explanation.

4. Show that if for every natural number n the equality $a_{n+1} = qa_n + p$ is satisfied, p and q are arbitrary fixed real numbers, then for every n the equality $a_n = \frac{p}{1-q} + (a_1 - \frac{p}{1-q})q^{n-1}$ holds. In particular if $a_1 = \frac{p}{1-q}$, then the term of the sequence (a_n) does not depend on n (this sequence is called by economists an equilibrium of the equation $a_{n+1} = qa_n + p$).

5. The price of the fuel went up 10 times, each time by 3%. Give an estimate of the price increase after all 10 changes. The allowed error is at most 0,1 .

6. (cobweb model). Let P_t be the price of a good at the time t, D_t be the demand for the good at the time t, S_t be the supply of the good at the time t. Let for every non-negative integer t and some positive numbers t $\alpha, \beta, \gamma, \delta$ that do not depend on t the following equations be fulfilled $S_t = D_t$, $D_t = \alpha - \beta P_t$, $S_{t+1} = \gamma + \delta P_t$. The first of these equations means that the manufacturer makes as much good as he can sell at the price P_t, the second equation implies that the demand decays if the price grows, from the third one one can deduce that the manufacturer supplies more good when the price grows. Find explicit formulas for P_t and S_t in terms of t and constants $\alpha, \beta, \gamma, \delta$. Explain what happens to the demand and the supply as t grows. The description of the model can be found in many books, in particular in A.Ostaszewski, Mathematics in Economics. Models and methods", or A.C.Chianga, Foundations of Mathematical Economy.

7. There is a fund of 1000000, kept at a bank, interest rate is 8% per year. Each year 300000 is used for scholarships paid at the end of the year after the interes was compounded starting from the year after the money was put into the bank. How long under these circumstances the fund will last? What would happen if the scholarships had been ten times smaller, i.e. 30000 yearly?

8. In some country the rate of inflation is 5% monthly. What will be the rate of inflation after a year. Give the answer so that the error will not exceed 1%. Instead of a calculator or a computer try to use Newton's binomial formula.

9. Assuming that the population growth rate on the Earth is $1{,}3\%$ per year find out after what time the population will be doubled.

10. Show that a sequence convergent to a finite limit must be bounded.

11. Show that a sequence (a_n) converges to 0 iff the sequence $(|a_n|)$ converges to 0.

12. Prove that the product $(a_n b_n)$ of the sequence (a_n) that converges to 0 and of a bounded sequence (b_n) converges to 0.

13. Prove that if $n > 1000000$ then $\sqrt[n]{2} < 1.000001$.

14. Prove that for sufficiently big numbers n the inequalities: $n! > 1000000^n$ oraz $1.000001^n > n^{1000000}$ are satisfied. In both cases prove that they are not satisfied for $n = 2, 3, 4, 5$ and find a number n_0 (not necessarilly the smallest one!) such that if $n > n_0$ then the first set of inequalities is satisfied.

15. Find the limit of the sequence (a_n) if it exists, $a_n = \sqrt{n + \sqrt[3]{n}} - \sqrt{n}$.

16. Find the limit of the sequence (a_n) if it exists, $a_n = \sqrt{n + \sqrt{n}} - \sqrt{n}$.

17. Find the limit of the sequence (a_n) if it exists, $a_n = \sqrt[n]{3^n - 2^n}$.

18. Find the limit of the sequence (a_n) if it exists, $a_n = \sqrt[n]{3^n + \sin n}$.

19. Find the limit of the sequence (a_n) if it exists, $a_n = 1 + \frac{n}{n+1} \cos \frac{n\pi}{2}$.

20. Find the limit of the sequence (a_n) if it exists, $a_n = \sin n$.

21. Find the limit of the sequence (a_n) if it exists, $a_n = \frac{\ln\left(n^2+n+1000\right)}{\ln(n^{1000}+999n-1)}$.

22. Find the limit of the sequence (a_n) if it exists, $a_n = \frac{n^2+n+1000}{n^{1000}+999n-1}$.

23. Find the limit of the sequence (a_n) if it exists, $a_n = \left(1+\sin\frac{1}{n}\right)^n$.

24. Find the limit of the sequence (a_n) if it exists, $a_n = \sqrt[n]{n!}$.

25. Find the limit of the sequence (a_n) if it exists, $a_n = \frac{10\cdot 11\cdot 12\cdot\ldots\cdot(n+9)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}$.

26. Find the limit of the sequence (a_n) if it exists, $a_n = \left(1+\frac{\sin n}{n^2}\right)^n$.

27. Find the limit of the sequence (a_n) if it exists, $a_n = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n}\right)$.

28. Find the limit of the sequence (a_n) if it exists, $a_n = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$.

29. Find the limit of the sequence (a_n) if it exists, $a_n = \frac{3^n-2^n}{3^n+n^2\cdot 2^n}$.

30. Find the limit of the sequence (a_n) if it exists, $a_n = \frac{3^n+2^n\cdot\sin n}{3^{n+1}+n^{2002}}$.

31. Does the inequality

a. $\lim\limits_{n\to\infty} a_n \le \lim\limits_{n\to\infty} b_n$ imply that for sufficiently large natural numbers n the inequality $a_n \le b_n$ holds?

b. $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$ imply that for sufficiently large natural numbers n the inequality $a_n < b_n$ holds?

32. Let $\lim\limits_{n\to\infty} a_n = g$. Show that $\lim\limits_{n\to\infty}\left(1+\frac{a_n}{n}\right)^n = e^g$. In the proof one can use the theorem about the limits of n-th powers of sequences converging quickly to 1.

33. Find sequences (a_n), (b_n) such that $\lim\limits_{n\to\infty} a_n = 0 = \lim\limits_{n\to\infty} b_n$ and:

a. $\lim\limits_{n\to\infty} a_n^{b_n} = 1$, b. $\lim\limits_{n\to\infty} a_n^{b_n} = 0$, c. $\lim_{n\to\infty} a_n^{b_n} = \frac{2}{3}$.

34. Evaluate $\lim\limits_{n\to\infty}\left(\frac{n+\sqrt[5]{n^5+160n^4}}{2n+2}\right)^n$. *Hint:* $\lim\limits_{x\to 0}\dfrac{(1+x)^a-1}{x} = a$.

35. Let $a_n = \left(1+\dfrac{1}{n}\sin\dfrac{n\pi}{200}\right)^n$. Find $\lim\limits_{n\to\infty} a_n$ if the limit exists. If not find two subsequences with different limits.

36. Lest $a_n = \left(1+\dfrac{1}{n^3}\cos\dfrac{n\pi}{200}\right)^{n^2}$. Find $\lim\limits_{n\to\infty} a_n$ if it exists. If it does not find two subsequences with distinct limits.

37. Let $a_n = \left(\dfrac{n+n\cos\frac{1}{n}}{n+\sqrt{n}}\right)^n$. Find $\lim\limits_{n\to\infty} a_n$ if it exists. If it does not find two subsequences with distinct limits.

38. Let $a_n = \left(\dfrac{n+n\sin\frac{1}{n}}{n-1}\right)^n$. Find $\lim\limits_{n\to\infty} a_n$ if it exists. If it does not find two subsequences with distinct limits.

39. Find $\lim_{n\to\infty} a_n$, if it exists, $a_n = \left(1+\frac{2}{n+2}\right)^n$.

40. Find $\lim_{n\to\infty} a_n$, if it exists, $a_n = \left(\frac{n^2+5n-3}{n^2+13}\right)^n$.

41. Find $\lim_{n\to\infty} a_n$, if it exists, $a_n = \left(0.999999+\frac{1}{n}\right)^n$.

42. Find $\lim_{n\to\infty} a_n$, if it exists, $a_n = \left(1.000001+\frac{1}{n}\right)^n$.

43. Find $\lim_{n\to\infty} a_n$, if it exists, $a_n = \frac{1000000^n}{n!}$.

44. Find $\lim_{n\to\infty} a_n$ ciągu (a_n) if it exists, $a_n = \frac{n^{100000}}{1.000001^n}$.

45. Let $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$. Prove that the sequence (a_n) has a limit. Is the limit finite or not? Is the limit equal 0?

46. Let $a_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$. Prove that the sequence (a_n) has a limit. Is the limit finite or not?

47. Let $a_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$. Prove that the sequence (a_n) has a limit. Is the limit finite or not?

48. Let $a_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that the sequence (a_n) has a limit. Is the limit finite or not?

49. Let a_0 be an arbitrary non-negative number. Let $a_n = \sqrt{6 + a_{n-1}}$ for $n = 1, 2, \ldots$. Prove that the sequence (a_n) has a limit. Is the limit finite or not? Find the limit.

50. Let a_0 be an arbitrary positive number. Let $a_n = \frac{1}{2}\left(a_{n-1} + \frac{5}{a_{n-1}}\right)$ dla $n = 1, 2, 3, \ldots$. Find $\lim_{n\to\infty} a_n$, if it exists. *Hint:* Prove that if $n = 0, 1, 2\ldots$ then $a_{n+1} \geq \sqrt{5}$ and $a_{n+1} \geq a_{n+2}$.

51. Let $a_n = \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right)\ldots\left(1 + \frac{1}{n^2}\right)$. Prove that the sequence (a_n) has a limit. Is the limit finite or not? *Hint:* Use the inequality $1 + x \leq e^x$.

52. Let $a_n = \left(1 - \frac{1}{2^1}\right)\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^3}\right)\ldots\left(1 - \frac{1}{2^n}\right)$. Prove that the sequence (a_n) has a finite limit. Is it true that $\lim_{n\to\infty} a_n = 0$?

Hint: show that for all $x < 1$ the inequality $1 - x = \frac{1}{1+\frac{x}{1-x}} \geq e^{-x/(1-x)}$ holds.

53. Find the limit $\lim\limits_{n\to\infty}\left(1 + \frac{\sqrt[n]{2}}{n}\right)^n$.

54. Find the limit $\lim_{n\to\infty}\left(1 + \sqrt[n]{2}\right)^n$.

55. Find the limit $\lim\limits_{n\to\infty} \frac{\ln n}{n}$;

56. Find the limit $\lim_{n\to\infty} \frac{e^{b_n}-1}{b_n}$, b_n is an n-th term of a sequence that converges to 0, $b_n \neq 0$ for $n = 1, 2, \ldots$ *Hint:* $1 + x \leq e^x \leq \frac{1}{1-x}$ for $x < 1$.

57. Find the limit $\lim\limits_{n\to\infty} \frac{\sqrt[n]{2}-1}{1/n}$;

58. Find the limit $\lim\limits_{n\to\infty}\left(\frac{1+\sqrt[n]{2}}{2}\right)^n$;

59. Find the limit $\lim\limits_{n\to\infty}(n - \ln n)$;

60. Find the limit $\lim\limits_{n\to\infty} \frac{1^5+2^5+3^5+\cdots+n^5-\frac{1}{6}n^6}{n^5}$;

61. Find the limit $\lim\limits_{n\to\infty} \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}$.

62. Find the limit $\lim\limits_{n\to\infty} n^2\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right)$.

63. Find the limit $\lim\limits_{n\to\infty} n\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(2n)^2}\right)$;

64. Find the limit $\lim\limits_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$;

65. Find the limit $\lim\limits_{n\to\infty} \frac{1}{\ln n}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.

66. Find the limit $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$.

67. Let $a_0 = \frac{1}{10}$ and $a_{n+1} = \sin a_n$ for $n = 0, 1, 2, \ldots$. Find the limit $\lim\limits_{n\to\infty} a_n$ or prove that if does not exist.

68. Let $a_0 = \frac{1}{10}$ oraz $a_{n+1} = 2^{a_n} - 1$. Find the limit $\lim\limits_{n\to\infty} a_n$ or prove that if does not exist.

69. Let $a_0 = \frac{1}{10}$ oraz $a_{n+1} = -a_n + \frac{1}{2}a_n^3$. Find the limit $\lim\limits_{n\to\infty} a_n$ or prove that if does not exist.

70. Let $a_n = \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{n^n}$. Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?

71. Let $a_n = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[4]{4}} + \cdots + \frac{1}{\sqrt[n]{n}}$. Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?

72. Let $a_n = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 3^3} + \cdots + \frac{1}{n\cdot n^n}$. Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?

73. Let $a_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n}$. Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?

74. Let $a_n = \frac{\ln 2}{2^2} + \frac{\ln 3}{3^3} + \frac{\ln 4}{4^4} + \cdots + \frac{\ln n}{n^n}$ Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?

75. Let $a_n = 1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-2} - \frac{1}{4n-1} - \frac{1}{4n}$. Does the sequence (a_n) have a limit? If the limit exists, is it finite or not?