28 October 2021, 8:00 a.m.

LINEAR ALGEBRA: 45 – MINUTES TEST

Solution of each problem should be written on a separate page. If you write with a pen on a sheet of paper please keep a distance at least one inch from each border.

Sign each paper with your first name, your last name and your students number.

At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $\mathbf{V}$ be a linear subspace of $\mathbb{R}^5$ defined as a set of solutions of the system of linear equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0 \\ 6x_1 + 5x_2 + 4x_3 + 3x_4 + 2x_5 = 0 \\ 9x_1 + 4x_2 - x_3 - 6x_4 - 11x_5 = 0 \\ 11x_1 + 8x_2 + 5x_3 + 2x_4 - x_5 = 0 \end{cases}$$

a. Find a basis and the dimension of $\mathbf{V}$.

Solution. Let us look at the matrix of the system (I shall not right the last column because it consists of zeros only and this will never change).

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 5 & 4 & 3 & 2 \\ 9 & 4 & -1 & -6 & -11 \\ 11 & 8 & 5 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -7 & -14 & -21 & -28 \\ 0 & -14 & -28 & -42 & -56 \\ 0 & -14 & -28 & -42 & -56 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The row 1 multiplied by 6, then by 9 and at the end by 11 and it was subtracted from the following rows. Then row was multiplied by 2 and subtracted from the next rows, the this row was divided by -7. Then the row was multiplied by 2 and subtracted from the row 1. We arrived with the matrix in the reduced echelon form. This means that $x_1 = x_3 + 2x_4 + 3x_5$ and $x_2 = -2x_3 - 3x_4 - 4x_5$. This means that we can choose the values of x_3, x_4, x_5 arbitrarily (they are free variables) and after this is done both x_1, x_2 are uniquely defined.

This proves that the space of solutions (which is a vector space) is three dimensional, three numbers define each solution which consists of five numbers.

Let us set $x_3 = 1, x_4 = 0, x_5 = 0$. This implies that $x_1 = 1$ and $x_2 = -2$ so the first solution is $\mathbf{v}_1 = (1, -2, 1, 0, 0)$. Our next choice is $x_3 = 0 = x_5$ and $x_4 = 1$. This implies that $x_1 = 2$ and $x_2 = -3$ so the next solution is $\mathbf{v}_2 = (2, -3, 0, 1, 0)$. The last vector we are going to consider is defined by $x_3 = x_4 = 0$ and $x_5 = 1$ so $\mathbf{v}_3 = (3, -4, 0, 0, 1)$. The chosen solutions are linearly independent vectors. To see this it suffices to look at the last three coordinates and apply the definition of the linear independence: if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (0, 0, 0, 0, 0)$ then $c_1 = c_2 = c_3 = 0$.

b. Let $\mathbf{v} = (6, -9, 1, 1, 1)$. Does $\mathbf{v}$ belong to $\mathbf{V}$? If $\mathbf{v} \in \mathbf{V}$, find a basis B of $\mathbf{V}$ such that all coordinates of $\mathbf{v}$ relative to B are equal to 1.

Solution. It is easy to see that $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. This proves that $\mathbf{v} \in \mathbf{V}$ and at the same time it shows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is the basis we are looking for. Let me mention that this one of infeniteli many possibilities. Another basis could be $\left(\frac{5}{6}, -\frac{17}{12}, \frac{1}{3}, \frac{1}{4}, 0\right)$, $\left(\frac{11}{6}, -\frac{35}{12}, \frac{1}{3}, \frac{3}{4}, 0\right)$, $\left(\frac{10}{3}, -\frac{14}{3}, \frac{1}{3}, 0, 1\right)$. The problem is completely solved. □

2. Let $W = \mathrm{lin}((2, 3, -13), (5, 2, -16), (8, 1, -19)) \subseteq \mathbb{R}^3$.

a. Find a system of linear homogeneous equations such that the set of the solutions of the system equals W. How many equations do you need?

Solution. We need few equations of the form $c_1x_1 + c_2x_2 + c_3x_3 = 0$ satisfied by the three triples given in the statement of the problem. This means that the following equalities

$$\begin{aligned} 2c_1 + 3c_2 - 13c_3 &= 0 \\ 5c_1 + 2c_2 - 16c_3 &= 0 \\ 8c_1 + c_2 - 19c_3 &= 0 \end{aligned}$$

must hold. This leads to the matrix

$$\begin{pmatrix} 2 & 3 & -13 \\ 5 & 2 & -16 \\ 8 & 1 & -19 \end{pmatrix} \to \begin{pmatrix} -22 & 0 & 44 \\ -11 & 0 & 22 \\ 8 & 1 & -19 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 8 & 1 & -19 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix}$$

so the initial system of equations is equivalent to $c_1 = 2c_3$ i $c_2 = 3c_3$. The equations satisfied by the three vectors are of the form $2c_3x_1 + 3c_3x_2 - c_3x_3 = 0$. If $c_3 \neq 0$ then this equation is equivalent to $2x_1 + 3x_2 - x_3 = 0$ ($c_3 = 0$ gives us uninteresting equation $0 = 0$ satisfied for all points of $\mathbb{R}^3$). The space W is two-dimensional. In fact it one could notice that $(2, 3, -13) + (8, 1, -19) = 2(5, 2, -16)$ and prove this way the linear dependence. On the other hand the vectors $(2, 3, -13)$, $(8, 1, -19)$ are linearly independent so the dimension is not less than 2. It is 2 because the three vectors are linearly dependent so the dimension is 2. Therefore one equation $2x_1 + 3x_2 + x_3 = 0$ describes the space W (it is a plane through the origin). Therefore one equation is necessary and suffcient. □

b. For what $t \in \mathbb{R}$ does the vector $(t^2, 2t, 1)$ belong to the space W.

Solution. The vector $(t^2, 2t, 1)$ belongs to W iff $0 = 2t^2 + 6t + 1 = 2(t + \frac{3}{2})^2 - \frac{7}{2}$, so $t = \frac{-3 \pm \sqrt{7}}{2}$. So there are these two possibilities only. We are done.