25 November 2021, 8:00 a.m.

LINEAR ALGEBRA: 45 – MINUTES TEST

Solution of each problem should be written on a separate page. If you write with a pen on a sheet of paper please keep a distance at least one inch from each border.

Sign each paper with your first name, your last name and your students number.

At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $A = \left(\begin{pmatrix} -5 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right)$ be the sequence of two vectors. Is A a basis of the plane $\mathbb{R}^2$? Do the terms of the sequence $B = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$ constitute a basis of the space $\mathbb{R}^3$?

Let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map with $M(\psi)_A^B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}$ so $M(\varphi)_{st}^{st} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

a. Find the matrix $M(\psi)_{st}^{st}$.

b. Find the matrix $M(\psi \circ \varphi)_A^B$.

Solution. Suppose $c_1 \begin{pmatrix} -5 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $-5c_1 + 3c_2 = 0$ and $-3c_1 + 2c_2 = 0$. Therefore $0 = 2(-5c_1+3c_2) - 3(-3c_1+2c_2) = -c_1$. Once $c_1 = 0$ then also $c_2 = 0$. This proves that the two vectors are linearly independent. Another possibility of proving the linear independence is to say that they are linearly independent iff $\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} \neq 0$, but this determinant equals -1. In the same way one can check the three terms of the sequence B are linearly independent: $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -1 \neq 0$ – we have expanded the 3×3 determinant along the second row because there the two zeros appeared. We have proved that the elements of B are linearly independent so B is a basis of $\mathbb{R}^3$. Since $M(\psi)_A^B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$ we may write

$$\psi\begin{pmatrix}-5\\-3\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}3\\4\\4\end{pmatrix} \text{ and}$$

$$\psi\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}+2\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}4\\5\\4\end{pmatrix}.$$

One can easily see that

$(M_A^{st})^{-1}=\begin{pmatrix}-5&3\\-3&2\end{pmatrix}^{-1}=\begin{pmatrix}-2&3\\-3&5\end{pmatrix}=M_{st}^A$. This in particular shows that

$\begin{pmatrix}1\\0\end{pmatrix}=-2\begin{pmatrix}-5\\-3\end{pmatrix}-3\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}=3\begin{pmatrix}-5\\-3\end{pmatrix}+5\begin{pmatrix}3\\2\end{pmatrix}$. Therefore

$$\psi\begin{pmatrix}1\\0\end{pmatrix}=-2\psi\begin{pmatrix}-5\\-3\end{pmatrix}-3\psi\begin{pmatrix}3\\2\end{pmatrix}=-2\begin{pmatrix}3\\4\\4\end{pmatrix}-3\begin{pmatrix}4\\5\\4\end{pmatrix}=\begin{pmatrix}-18\\-23\\-20\end{pmatrix} \text{ and}$$

$$\psi\begin{pmatrix}0\\1\end{pmatrix}=3\psi\begin{pmatrix}-5\\-3\end{pmatrix}+5\psi\begin{pmatrix}3\\2\end{pmatrix}=3\begin{pmatrix}3\\4\\4\end{pmatrix}+5\begin{pmatrix}4\\5\\4\end{pmatrix}=\begin{pmatrix}29\\37\\32\end{pmatrix}.$$

This way we obtained the formula $M(\psi)_{st}^{st}=\begin{pmatrix}-18&29\\-23&37\\-20&32\end{pmatrix}$. Part a is done.

We know that $M(\psi\circ\phi)_{st}^{st}=M(\psi)_{st}^{st}\cdot M(\phi)_{st}^{st}=\begin{pmatrix}-18&29\\-23&37\\-20&32\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}=\begin{pmatrix}11&29\\14&37\\12&32\end{pmatrix}$. To end the solution it suffices to multiply the three natrices $M(\psi\circ\phi)_A^B=M_{st}^B\cdot M(\psi\circ\phi)_{st}^{st}\cdot M_A^{st}$. We need $M_{st}^B=(M_B^{st})^{-1}=\begin{pmatrix}1&1&1\\2&1&1\\3&0&1\end{pmatrix}^{-1}=\begin{pmatrix}-1&1&0\\-1&2&-1\\3&-3&1\end{pmatrix}$. So $M(\psi\circ\phi)_A^B=M_{st}^B\cdot M(\psi\circ\phi)_{st}^{st}\cdot M_A^{st}=$

$$=\begin{pmatrix}-1&1&0\\-1&2&-1\\3&-3&1\end{pmatrix}\begin{pmatrix}11&29\\14&37\\12&32\end{pmatrix}\begin{pmatrix}-5&3\\-3&2\end{pmatrix}=\begin{pmatrix}3&8\\5&13\\3&8\end{pmatrix}\begin{pmatrix}-5&3\\-3&2\end{pmatrix}=\begin{pmatrix}-39&25\\-64&41\\-39&25\end{pmatrix}\ \square$$

2. Let $A = \begin{pmatrix} 5 & 10 & -4 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ -1 & 1 & -1 \end{pmatrix}$.

a. Find A^{-1}.

b. Compute the determinants of A, B, B^{-1}, $(A^T)^{-1} \cdot B^2$.

Solution. Denote $\begin{pmatrix} 5 & 10 & -4 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. This means that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$

$$= \begin{pmatrix} 5 & 10 & -4 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 5a + 10d - 4g & 5b + 10e - 4h & 5c + 10f - 4i \\ a + 2d - g & b + 2e - h & c + 2f - i \\ 2a + 5d - g & 2b + 5e - h & 2c + 5f - i \end{pmatrix}.$$

This implies that the following nine equations hold

$$5a + 10d - 4g = 1, \quad a + 2d - g = 0, \quad 2a + 5d - g = 0,$$
$$5b + 10e - 4h = 0, \quad b + 2e - h = 1, \quad 2b + 5e - h = 0,$$
$$5c + 10f - 4i = 0, \quad c + 2f - i = 0, \quad 2c + 5f - i = 1.$$

The next corollary is $1 = 5a + 10d - 4g - 5(a + 2d - g) = g$, $-5 = 5b + 10e - 4h - 5(b + 2e - h) = h$, $0 = 5c + 10f - 4i - 5(c + 2f - i) = i$. This in turn shows that

$$a + 2d = g = 1, \quad 2a + 5d = g = 1,$$
$$b + 2e = h + 1 = -4, \quad 2b + 5e = h = -5,$$
$$c + 2f = i = 0, \quad 2c + 5f = i + 1 = 1.$$

Therefore $-1 = 2a + 5d - 2(a + 2d) = d$ so $a = 3$; $\quad 3 = 2b + 5e - 2(b + 2e) = e$ so $b = -10$ and $1 = 2c + 5f - 2(c + 2f) = f$, so $c = -2$. Our result is

$$A^{-1} = \begin{pmatrix} 5 & 10 & -4 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -10 & -2 \\ -1 & 3 & 1 \\ 1 & -5 & 0 \end{pmatrix}.$$

$$|A| = \begin{vmatrix} 5 & 10 & -4 \\ 1 & 2 & -1 \\ 2 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ – expansion along the first row,}$$

$$|B| = \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 3 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 5 \\ 0 & -4 \end{vmatrix} = -12 \begin{vmatrix} 1 & 5 \\ 0 & 1 \end{vmatrix} = -12.$$

Here we expanded the 3×3 determinant along the first column. We are almost done. Now it is time to apply Cauchy's theorem about the determinant of the product of matrices: the determinant of the product of two matrices equals to the product of their determinants. The theorem implies that $1 = |I| = |M \cdot M^{-1}| = |M| \cdot |M^{-1}|$, so $|M^{-1}| = \frac{1}{|M|}$. We also know that $|M^T| = |M|$. Therefore $|B^{-1}| = \frac{1}{-12} = -\frac{1}{12}$, $|(A^T)^{-1}| = \frac{1}{|A^T|} = \frac{1}{|A|} = 1$ and $|B^2| = |B|^2 = (-12)^2 = 144$. Hence $|(A^T)^{-1} \cdot B^2| = 144$. $\square$

Note that unlike in the case of matrices the rows and the columns of the determinant play exactly the same role. In particular one may operate on columns in the same way as on rows.

25 November 2021, 9:45 a.m.

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At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $A = \left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$ be the sequence of two vectors. Is A a basis of the plane $\mathbb{R}^2$? Do the terms of the sequence $B = \left(\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$ constitute a basis of the space $\mathbb{R}^3$?

Let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map with $M(\psi)_A^B = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 2 & 2 \end{pmatrix}$ and $\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ x+y \end{pmatrix}$ so $M(\varphi)_{st}^{st} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

a. Find the matrix $M(\psi)_{st}^{st}$.

b. Find the matrix $M(\psi \circ \varphi)_A^B$.

Solution. Suppose $c_1 \begin{pmatrix} 4 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $4c_1 + 3c_2 = 0$ and $3c_1 + 2c_2 = 0$. Therefore $0 = 2(-5c_1 + 3c_2) - 3(-3c_1 + 2c_2) = -c_1$. Once $c_1 = 0$ then also $c_2 = 0$. This proves that the two vectors are linearly independent. Another possibility of proving the linear independence is to say that they are linearly independent iff $\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} \neq 0$, but this determinant equals -1. In the same way one can check the three terms of the sequence B are linearly independent: $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 5 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$. Here the 3×3 determinant has been expanded along the third column. We have proved that the elements of B are linearly independent so B is a basis of $\mathbb{R}^3$. Since $M(\psi)_A^B = \begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 2 & 2 \end{pmatrix}$ we may write

$$\psi\begin{pmatrix}4\\3\end{pmatrix}=2\begin{pmatrix}1\\3\\2\end{pmatrix}+2\begin{pmatrix}2\\5\\3\end{pmatrix}+2\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}8\\18\\12\end{pmatrix}\text{ and}$$

$$\psi\begin{pmatrix}3\\2\end{pmatrix}=3\begin{pmatrix}1\\3\\2\end{pmatrix}+\begin{pmatrix}2\\5\\3\end{pmatrix}+2\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}7\\16\\11\end{pmatrix}.$$

One can easily see that $(M_A^{st})^{-1}=\begin{pmatrix}4&3\\3&2\end{pmatrix}^{-1}=\begin{pmatrix}-2&3\\3&-4\end{pmatrix}=M_{st}^A$.
This in particular shows that
$\begin{pmatrix}1\\0\end{pmatrix}=-2\begin{pmatrix}4\\3\end{pmatrix}+3\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}=3\begin{pmatrix}4\\3\end{pmatrix}-4\begin{pmatrix}3\\2\end{pmatrix}$. Therefore

$$\psi\begin{pmatrix}1\\0\end{pmatrix}=-2\psi\begin{pmatrix}4\\3\end{pmatrix}+3\psi\begin{pmatrix}3\\2\end{pmatrix}=-2\begin{pmatrix}8\\18\\12\end{pmatrix}+3\begin{pmatrix}7\\16\\11\end{pmatrix}=\begin{pmatrix}5\\12\\9\end{pmatrix}\text{ and}$$

$$\psi\begin{pmatrix}0\\1\end{pmatrix}=3\psi\begin{pmatrix}4\\3\end{pmatrix}-4\psi\begin{pmatrix}3\\2\end{pmatrix}=3\begin{pmatrix}8\\18\\12\end{pmatrix}-4\begin{pmatrix}7\\16\\11\end{pmatrix}=\begin{pmatrix}-4\\-10\\-8\end{pmatrix}.$$ This way we

obtained the formula $M(\psi)_{st}^{st}=\begin{pmatrix}5&-4\\12&-10\\9&-8\end{pmatrix}$. Part a is done.

We know that $M(\psi\circ\phi)_{st}^{st}=M(\psi)_{st}^{st}\cdot M(\phi)_{st}^{st}=\begin{pmatrix}5&-4\\12&-10\\9&-8\end{pmatrix}\begin{pmatrix}2&1\\1&1\end{pmatrix}=\begin{pmatrix}6&1\\14&2\\10&1\end{pmatrix}$. To end the solution it suffices to multiply the three natrices $M(\psi\circ\phi)_A^B=M_{st}^B\cdot M(\psi\circ\phi)_{st}^{st}\cdot M_A^{st}$. We need $M_{st}^B=(M_B^{st})^{-1}=\begin{pmatrix}1&2&1\\3&5&1\\2&3&1\end{pmatrix}^{-1}=\begin{pmatrix}-2&-1&3\\1&1&-2\\1&-1&1\end{pmatrix}$. So $M(\psi\circ\phi)_A^B=M_{st}^B\cdot M(\psi\circ\phi)_{st}^{st}\cdot M_A^{st}=$

$$=\begin{pmatrix}-2&-1&3\\1&1&-2\\1&-1&1\end{pmatrix}\begin{pmatrix}6&1\\14&2\\10&1\end{pmatrix}\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}4&-1\\0&1\\2&0\end{pmatrix}\begin{pmatrix}4&3\\3&2\end{pmatrix}=\begin{pmatrix}13&10\\3&2\\8&6\end{pmatrix}\ \square$$

2. Let $A = \begin{pmatrix} 5 & 1 & 2 \\ 10 & 2 & 5 \\ -4 & -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$.

a. Find A^{-1}.

b. Compute the determinants of A, B, B^{-1}, $(A^T)^{-3} \cdot B^2$.

Solution. Denote $\begin{pmatrix} 5 & 1 & 2 \\ 10 & 2 & 5 \\ -4 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. This means that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$

$$= \begin{pmatrix} 5 & 1 & 2 \\ 10 & 2 & 5 \\ -4 & -1 & -1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 5a+d+2g & 5b+e+2h & 5c+f+2i \\ 10a+2d+5g & 10b+2e+5h & 10c+2f+5i \\ -4a-d-g & -4b-e-h & -4c-f-i \end{pmatrix}.$$

This implies that the following nine equations hold

$$5a+d+2g=1, \quad 10a+2d+5g=0, \quad -4a-d-g=0,$$
$$5b+e+2h=0, \quad 10b+2e+h=1, \quad -4b-e-h=0,$$
$$5c+f+2i=0, \quad 10c+2f+5i=0, \quad -4c-f-i=1.$$

The next coclusion is $2 = 2(5a+d+2g) - (10a+2d+5g) = -g$,

$-1 = 2(5b+e+2h) - (10b+2e+5h) = -h, \quad 0 = 2(5c+f+2i) - (10c+2f+5i) = -i$.

This in turn shows that

$$5a+d = 1-2g = 5, \quad 4a+d = -g = 2,$$
$$5b+e = -2h = -2, \quad 4b+e = -h = -1,$$
$$5c+f = -2i = 0, \quad 4c+f = -i-1 = -1.$$

Therefore $3 = 5a+d-(4a+d) = a$ so $d=-10$; $\quad -1 = 5b+e-(4b+e) = b$ so $e=3$ and $1 = 5c+f-(4c+f) = c$, so $f=-5$. Our result is

$$A^{-1} = \begin{pmatrix} 5 & 1 & 2 \\ 10 & 2 & 5 \\ -4 & -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -10 & 3 & -5 \\ -2 & 1 & 0 \end{pmatrix}.$$

$|A| = \begin{vmatrix} 5 & 1 & 2 \\ 10 & 2 & 5 \\ -4 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ -4 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ – expansion of the 3×3 determinant along the second column,

$$|B| = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 4 & 8 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 4 & 8 \end{vmatrix} = 4\begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} = 4\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} = -20$$ – expansion of the 3×3 determinant along the firstcolumn.

We are almost done. Now it is time to apply Cauchy's theorem about the determinant of the product of matrices: the determinant of the product of two matrices equals to the product of their determinants. The theorem implies that $1 = |I| = |M \cdot M^{-1}| = |M| \cdot |M^{-1}|$, so $|M^{-1}| = \frac{1}{|M|}$. We also know that $|M^T| = |M|$. Therefore $|B^{-1}| = \frac{1}{-20} = -\frac{1}{20}$, $|(A^T)^{-1}| = \frac{1}{|A^T|} = \frac{1}{|A|} = 1$ and $|B^2| = |B|^2 = (-20)^2 = 400$. Hence $|(A^T)^{-3} \cdot B^2| = 400$. $\square$