December 10-th, 2020, 8 a.m.

45 – MINUTES TEST FOR PEOPLE WHOSE STUDENT'S NUMBER IS EVEN

Solution of each problem should be written on a separate page. If you write with a pen on a sheet of paper please keep a distance at least one inch (2.5 cm) from each border.

Sign **each** paper with your first name, your last name and your students number.

The name of file is: your last name dash your first name dash problem number. For example Vladimir-Putin-3. The files should be turned into .pdf

At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 5 & 2 \\ 1 & -2 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & 2 & 1 \end{pmatrix}$.

Compute the determinants of all five matrices $A \cdot B$, $B \cdot A$, C, C^{-2} and $C^3 \cdot D^2$.

Solution. Let us start with saying that the determinant of the matrix AB is 0 because the range of a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by the matrix B is at most 2. Therefore the dimension of the range of AB is also at most 2 because the map defined by the matrix A starts from the two dimensional space so the range of it cannot exceed 2.

If one does not notice it then she or he computes the product $AB = \begin{pmatrix} 2 & 7 & 1 \\ 5 & 17 & 3 \\ 1 & 3 & 1 \end{pmatrix}$ evaluating all 18 products and adding them as required. Then using the properties of determinants we can compute $\begin{vmatrix} 2 & 7 & 1 \\ 5 & 17 & 3 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 3 & 1 \end{vmatrix} = 0$. Now $BA = \begin{pmatrix} 11 & 7 \\ 13 & 9 \end{pmatrix}$ and $\begin{vmatrix} 11 & 7 \\ 13 & 9 \end{vmatrix} = \begin{vmatrix} 11 & 7 \\ 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 11 & 7 \\ 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} = 8$. We proceed in the same way:

$\det(C) = \begin{vmatrix} 3 & -1 & -1 \\ -1 & 5 & 2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = -1$ and $\det(D) = \begin{vmatrix} 3 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} =$

$= \begin{vmatrix} 0 & 5 & 2 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 5 & 2 \\ -1 & 2 & 1 \end{vmatrix} = 0$. This implies that $\det(C^{-2}) = (\det(C))^{-2} = (-1)^{-2} = 1$ and $\det(C^3D^2) = \det(C^3D) \cdot \det(D) = \det(C^3D) \cdot 0 = 0$.

2. Let φ be the symmetry with respect to the line $2x-3y=0$

Find $\varphi\begin{pmatrix} 9 \\ 6 \end{pmatrix}$, $\varphi\begin{pmatrix} -3 \\ -2 \end{pmatrix}$, $\varphi\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Let $A=\left\{\begin{pmatrix} -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$, $B=\left\{\begin{pmatrix} -6 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}\right\}$ and $st=\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

Find $M(\varphi)_A^A$, M_A^{st}, M_{st}^A, M_B^{st}, M_{st}^B and $M(\varphi)_{st}^{st}$.

Solution. All points from the line $2x-3y=0$ are mapped to themselves (these are fixed points of the symmetry) so $\varphi\begin{pmatrix} 9 \\ 6 \end{pmatrix}=\begin{pmatrix} 9 \\ 6 \end{pmatrix}$ and $\varphi\begin{pmatrix} -3 \\ -2 \end{pmatrix}=\begin{pmatrix} -3 \\ -2 \end{pmatrix}$. The vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is perpendicular to the line $2x-3y=0$ so by the definition of the symmetry we have $\varphi\begin{pmatrix} 2 \\ -3 \end{pmatrix}=-\begin{pmatrix} 2 \\ -3 \end{pmatrix}$. The last two equations imply that $M(\varphi)_A^A=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$M_A^{st}=\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}$ which means that $\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} -3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 2 \\ -3 \end{pmatrix}$. This implies that $x\begin{pmatrix} 1 \\ 0 \end{pmatrix}+y\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=$
$=\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}=u\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}+v\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=$
$=u\begin{pmatrix} -3 \\ -2 \end{pmatrix}+v\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, so the vector written as a linear combination of the elements of the standard basis of $\mathbb{R}^2$ was written as a linear combination of the elements if the basis A. The formulas are: $x=-3u+2v$ and $y=-2u-3v$. To express the numbers u,v with x,y we can solve the system of equations (the last two equations) for u and v: $3x+2y=$ $=3(-3u+2v)+2(-2u-3v)=-13u$ so $u=\frac{-1}{13}(3x+2y)$ and in the same way $v=\frac{1}{13}(2x-3y)$. This is equivalent to the formula $\begin{pmatrix} u \\ v \end{pmatrix}=\frac{1}{13}\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. This proves the equlity $M_{st}^A=\frac{1}{13}\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}$. We could have also written $M_{st}^A=(M_A^{st})^{-1}=\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}^{-1}$.

Now shortly $M_B^{st}=\begin{pmatrix} -6 & 6 \\ 6 & 6 \end{pmatrix}$, $M_{st}^B=(M_B^{st})^{-1}=\frac{-1}{72}\begin{pmatrix} 6 & -6 \\ -6 & -6 \end{pmatrix}$. The last step in this solution is $M(\varphi)_{st}^{st}=M_A^{st}\cdot M(\varphi)_A^A\cdot M_{st}^A=\begin{pmatrix} -3 & 2 \\ -2 & -3 \end{pmatrix}\cdot\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cdot\frac{1}{13}\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}=$
$=\frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}$. We can write $\varphi\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{13}\begin{pmatrix} 5 & 12 \\ 12 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Obviously there was no need to solve the system if the linear equations. We could find the inverse of the matrix. I included the solution above just to show that everyhing could be done using „school mathematics” only. I hope that some students will understand better what is going on and take an adentage of looking in a better way at some problems. It may help to understand the way used to answer questions.

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At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 3 & -1 \\ 3 & 5 & -2 \\ -1 & -2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 3 & -1 \\ 1 & -1 & 1 \\ -1 & -2 & 1 \end{pmatrix}$.

Compute the determinants of all five matrices $A \cdot B$, $B \cdot A$, C^2, C^{-1} and $C^2 \cdot D^3$.

$$AB = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{vmatrix} 2 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 0, \quad BA = \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}, \quad \begin{vmatrix} 5 & 1 \\ 1 & 3 \end{vmatrix} = 14,$$

$\det(C) = 1$, $\det(D) = 0$, $\det(C^{-2}) = 1$, $\det(D) = 0$, $\det(C^3 D^2) = 0$.

2. Let φ be the symmetry with respect to the line $2x + 3y = 0$

Find $\varphi\begin{pmatrix} 9 \\ -6 \end{pmatrix}$, $\varphi\begin{pmatrix} -6 \\ -4 \end{pmatrix}$, $\varphi\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Let $A = \left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} -6 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \end{pmatrix}\right\}$ and $st = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

Find $M(\varphi)_A^A$, M_A^{st}, M_{st}^A, M_B^{st}, M_{st}^B and $M(\varphi)_{st}^{st}$.

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

$$\varphi\begin{pmatrix} 9 \\ -6 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}\begin{pmatrix} 9 \\ -6 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix},$$

$$\varphi\begin{pmatrix} -6 \\ -4 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}\begin{pmatrix} -6 \\ -4 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 18 \\ 92 \end{pmatrix},$$

$$\varphi\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix},$$

$$M(\varphi)_A^A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_A^{st} = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}, \quad M_{st}^A = \tfrac{1}{13}\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix},$$

$$M_B^{st} = \begin{pmatrix} -6 & 6 \\ 6 & 6 \end{pmatrix}, \quad M_{st}^B = \tfrac{-1}{72}\begin{pmatrix} 6 & -6 \\ -6 & -6 \end{pmatrix}, \quad M(\varphi)_{st}^{st} = \tfrac{1}{13}\begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}.$$

December 10-th, 2020, 9:45 a.m.

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At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 10 & 3 \\ 0 & 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & -2 & 0 \\ 5 & 4 & 2 \\ 0 & 3 & 1 \end{pmatrix}$.

Compute the determinants of all five matrices $A \cdot B$, $B \cdot A$, C, C^{-2} and $C^3 \cdot D^2$.

$$AB = \begin{pmatrix} 13 & 5 & 3 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \quad \begin{vmatrix} 13 & 5 & 3 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 0, \quad BA = \begin{pmatrix} 5 & 7 \\ 7 & 11 \end{pmatrix}, \quad \begin{vmatrix} 5 & 7 \\ 7 & 11 \end{vmatrix} = 6,$$

$\det(C) = 1$, $\det(D) = 0$, $\det(C^{-2}) = 1$, $\det(D) = 0$, $\det(C^3D^2) = 0$.

2. Let φ be the symmetry with respect to the line $4x - 3y = 0$

Find $\varphi\begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $\varphi\begin{pmatrix} -3 \\ -4 \end{pmatrix}$, $\varphi\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

Let $A = \left\{\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -8 \\ 6 \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} -12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \end{pmatrix}\right\}$ and $st = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

Find $M(\varphi)_A^A$, M_A^{st}, M_{st}^A, M_B^{st}, M_{st}^B and $M(\varphi)_{st}^{st}$.

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

$$\varphi\begin{pmatrix} 6 \\ 8 \end{pmatrix} = \tfrac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix},$$

$$\varphi\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \tfrac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix},$$

$$\varphi\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \tfrac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix},$$

$$M(\varphi)_A^A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_A^{st} = \begin{pmatrix} 3 & -8 \\ 4 & 6 \end{pmatrix}, \quad M_{st}^A = \tfrac{1}{50}\begin{pmatrix} 6 & 8 \\ -4 & 3 \end{pmatrix},$$

$$M_B^{st} = \begin{pmatrix} -12 & 12 \\ 12 & 12 \end{pmatrix}, \quad M_{st}^B = \tfrac{-1}{288}\begin{pmatrix} 12 & -12 \\ -12 & -12 \end{pmatrix}, \quad M(\varphi)_{st}^{st} = \tfrac{1}{25}\begin{pmatrix} -7 & 24 \\ 24 & 7 \end{pmatrix}.$$

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1. Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 6 & 4 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 6 & 4 & 1 \\ 8 & 8 & 3 \\ 1 & 2 & 1 \end{pmatrix}$.

Compute the determinants of all five matrices $A \cdot B$, $B \cdot A$, C^2, C^{-1} and $C^2 \cdot D^3$.

$$AB = \begin{pmatrix} 5 & 8 & 1 \\ 8 & 13 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad \begin{vmatrix} 5 & 8 & 1 \\ 8 & 13 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0, \quad BA = \begin{pmatrix} 14 & 7 \\ 7 & 6 \end{pmatrix}, \quad \begin{vmatrix} 14 & 7 \\ 7 & 6 \end{vmatrix} = 35,$$

$\det(C) = 1$, $\det(D) = 0$, $\det(C^{-2}) = 1$, $\det(D) = 0$, $\det(C^3 D^2) = 0$.

2. Let φ be the symmetry with respect to the line $4x + 3y = 0$

Find $\varphi\begin{pmatrix} -6 \\ 8 \end{pmatrix}$, $\varphi\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $\varphi\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

Let $A = \left\{\begin{pmatrix} 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 8 \\ 6 \end{pmatrix}\right\}$, $B = \left\{\begin{pmatrix} -12 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 12 \end{pmatrix}\right\}$ and $st = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.

Find $M(\varphi)_A^A$, M_A^{st}, M_{st}^A, M_B^{st}, M_{st}^B and $M(\varphi)_{st}^{st}$.

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

$$\varphi\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}\begin{pmatrix} -6 \\ 8 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix},$$

$$\varphi\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix},$$

$$\varphi\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix},$$

$$M(\varphi)_A^A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M_A^{st} = \begin{pmatrix} 3 & 8 \\ -4 & 6 \end{pmatrix}, \quad M_{st}^A = \frac{1}{50}\begin{pmatrix} 6 & -8 \\ 4 & 3 \end{pmatrix},$$

$$M_B^{st} = \begin{pmatrix} -12 & 12 \\ 12 & 12 \end{pmatrix}, \quad M_{st}^B = \frac{-1}{288}\begin{pmatrix} 12 & -12 \\ -12 & -12 \end{pmatrix}, \quad M(\varphi)_{st}^{st} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ -24 & 7 \end{pmatrix}.$$