28 October 2021, 9:45 a.m.

LINEAR ALGEBRA: 45 – MINUTES TEST

Solution of each problem should be written on a separate page. If you write with a pen on a sheet of paper please keep a distance at least one inch from each border.

Sign each paper with your first name, your last name and your students number.

At the end of your test please add a statement (if it is true as it should be): *I solved the problems by myself without any assistance.*

1. Let $\mathbf{V}$ be a linear subspace of $\mathbb{R}^5$ defined as a set of solutions of the system of linear equations:

$$\begin{cases} x_1 + 16x_2 + 4x_3 - 8x_4 - 2x_5 = 0 \\ x_1 + 81x_2 + 9x_3 + 27x_4 + 3x_5 = 0 \\ 5x_1 + 275x_2 + 35x_3 + 65x_4 + 5x_5 = 0 \\ 5x_1 + 210x_2 + 30x_3 + 30x_4 = 0 \end{cases}$$

a. Find a basis and the dimension of $\mathbf{V}$.

Solution. Let us look at the matrix of the system (I shall not right the last column because it consists of zeros only and this will never change).

$$\begin{pmatrix} 1 & 16 & 4 & -8 & -2 \\ 1 & 81 & 9 & 27 & 3 \\ 5 & 275 & 35 & 65 & 5 \\ 5 & 210 & 30 & 30 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 16 & 4 & -8 & -2 \\ 0 & 65 & 5 & 35 & 5 \\ 0 & 195 & 15 & 105 & 15 \\ 0 & 130 & 10 & 70 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 16 & 4 & -8 & -2 \\ 0 & 13 & 1 & 7 & 1 \\ 0 & 13 & 1 & 7 & 1 \\ 0 & 13 & 1 & 7 & 1 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 16 & 4 & -8 & -2 \\ 0 & 13 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 42 & 6 & 6 & 0 \\ 0 & 13 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

At first the row one was subtracted from the row 2, then the row one multiplied by 5 was subtracted from the rows three and four. Then the second row of the new matrix was divided by 5, the third row was divided by 15 and the last one by 10. Next action was to subtract the secind row from the following rows to get zeros in them. The last action is to multiply the second row by 2 and add the result to the first row. The last matrix is written in the reduced row echelon form - just look at the first and at the last columnn. This implies that $x_1 = -42x_2 - 6x_3 - 6x_4$ and $x_5 = -13x_2 - x_3 - 7x_4$. The free variables are x_2, x_3, x_4. They can be set arbitrarily and after it one can evaluate x_1 and x_5. This implies that the space of solutions of the system is three dimensional – each sol.ution is defined by three numbers. Now we shall find a basis. Define $\mathbf{v}_1 = (-42, 1, 0, 0, -13)$, $\mathbf{v}_2 = (-6, 0, 1, 0, -1)$ and $\mathbf{v}_3 = (-6, 0, 0, 1, -7)$. These vectors are linearly independent. To see it just look at the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (0, 0, 0, 0, 0)$. Look the the second cordinate to realize that $c_1 = 0$. Then do the same thing with the third and the fourth coordinate to see that also $c_2 = 0 = c_3$. $\square$

b. Let $\mathbf{v} = (-18, 1, -2, -2, 3)$. Does $\mathbf{v}$ belong to $\mathbf{V}$? If $\mathbf{v} \in \mathbf{V}$, find a basis B of $\mathbf{V}$ such that all coordinates of $\mathbf{v}$ relative to B are equal to 1.

Solution. Clearly it follows from $(-18, 1, -2, -2, 3) = \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1(-42, 1, 0, 0, -13) + c_2(-6, 0, 1, 0, -1) + c_3(-6, 0, 0, 1, -7)$ that $c_1 = 1, c_2 = -2 = c_3$. This may be written as $(-18, 1, -2, -2, 3) = (-42, 1, 0, 0, -13) + (12, 0, -2, 0, 2) + (12, 0, 0, -2, 14)$. Therefore a basis we are looking for may be consist of the vectors $(-42, 1, 0, 0, -13)$, $(12, 0, -2, 0, 2)$, $(12, 0, 0, -2, 14)$. There many other possibilities but we were asked to give an example of just one basis. So we are done. □

2. Let $W = \mathrm{lin}((2, 5, 7), (6, 3, 9), (1, 1, 2)) \subseteq \mathbb{R}^3$.

a. Find a system of linear homogeneous equations such that the set of the solutions of the system equals W. How many equations do you need?

Solution. Let us at first try to find the dimension of W. To this end we shall look at the matrix $\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 9 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & 3 \\ 0 & -3 & -3 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. The last matrix is written in a reduced row echelon form. This shows that if an equation $c_1x_1 + c_2x_2 + c_3x_3 = 0$ is satisfied by the three vectors then $c_1 = -c_3$ and $c_2 = -c_3$. Therefore if we do not think of the triple $c_1 = c_2 = c_3 = 0$ then we may write that $x_1 + x_2 - x_3 = 0$. All this shows that W is two dimensional subspace of $\mathbb{R}^3$. One equation is necessary and the one written above suffices for a desription of W which turns out to be a plane through the origin.

b. For what $t \in \mathbb{R}$ does the vector $(t^2, t, 1)$ belong to the space W.

Solution. We have found an equation that discribes the subspace. So we only need to check for what the equationn is satisfied, so $0 = t^2 + t - 1 = (t + \frac{1}{2})^2 - \frac{5}{4}$. Therefore $= -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. There are two numbers that satisfy the condition. We are done.