Consider the Fibonacci sequence again : $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ldots$ A next term is a sum of its two predecessors: $F_{n+2} = F_{n+1} + F_n$. We may look at it in the following way. Out of a pair of given numbers x, y we create a new pair y and $x+y$ starting from $1, 1$. We may say that there is a map from $\mathbb{R}^2$ into itself given by the formula $f(x,y) = (y, x+y)$. The map is linear. We may write $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix} \to \begin{pmatrix} 3 \\ 5 \end{pmatrix} \to \begin{pmatrix} 5 \\ 8 \end{pmatrix}$ and so on. We can see that $\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

This shows that finding a formula for F_n is equivalent to finding a formula for $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$. The subsequent powers of the matrix are: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$,

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^4 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^5 = \begin{pmatrix} 3 & 5 \\ 5 & 8 \end{pmatrix}$. We see the Fibonacci numbers again which is not surprising at all because we only restated the problem. Now we shall try to change the basis in $\mathbb{R}^2$ in an appropriate way. Let us try to find vectors mapped the vectors parallel to them that is such a non-zero vector $\begin{pmatrix} x \\ y \end{pmatrix}$ that $\begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = q \begin{pmatrix} x \\ y \end{pmatrix}$ for some number q (recall that two vectors are parallel if the product of one of them and some number equals to the second one). Two equations must be fulfilled $y = qx$ and $x + y = qy$. This implies that $x + qx = q^2x$. If $x \neq 0$ then $1 + q = q^2$ so $q = \frac{1}{2}(1 \pm \sqrt{5})$. Let $\mathbf{v}_- = \begin{pmatrix} 1 \\ \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$ and $\mathbf{v}_+ = \begin{pmatrix} 1 \\ \frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$, also $q_- = \frac{1}{2}(1-\sqrt{5})$ and $q_+ = \frac{1}{2}(1+\sqrt{5})$. We have $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ q_\pm \end{pmatrix} = q_\pm \begin{pmatrix} 1 \\ q_\pm \end{pmatrix}$. The vectors $\begin{pmatrix} 1 \\ q_- \end{pmatrix}, \begin{pmatrix} 1 \\ q_+ \end{pmatrix}$ are linearly independent so they constitute a basis of $\mathbb{R}^2$. The matrix of the map is $M_B^B = \begin{pmatrix} q_- & 0 \\ 0 & q_+ \end{pmatrix}$. Obviously $(M_B^B)^n = \begin{pmatrix} q_-^n & 0 \\ 0 & q_+^n \end{pmatrix}$.

$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = u_1\begin{pmatrix} 1 \\ q_- \end{pmatrix} + u_2\begin{pmatrix} 1 \\ q_+ \end{pmatrix}$. This means that

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ q_- & q_+ \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ or $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

– we solved the system of linear equations for u_1 and u_2. Using the notation adopted in the prof. Kędzierski's class we may write $M_{st}^B = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$ and $M_B^{st} = \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}$. So we can write $\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}\begin{pmatrix} q_-^n & 0 \\ 0 & q_+^n \end{pmatrix}\begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} =$

$= \begin{pmatrix} q_-^n & q_+^n \\ q_-^{n+1} & q_+^{n+1} \end{pmatrix}\begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{\sqrt{5}-1}{2\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} q_+^{n-1} - q_-^{n-1} & q_+^n - q_-^n \\ q_+^n - q_-^n & q_+^{n+1} - q_-^{n+1} \end{pmatrix}$.

It follows from the above equalities that

$$F_n = \frac{1}{\sqrt{5}}\left(q_+^n - q_-^n\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The change of the basis made the problem trivial (very easy). The whole story was told before without mentioning any basis, no matrices were multiplied. Now it was put into a more general setting. One may imagine that similar actions may be undertaken in other situations.

New problems

35. What are the coordinates of the vector $\mathbf{v}$ relative to a basis B if

a. $\mathbf{v} = \binom{2}{3}$ and $B = \left\{\binom{1}{1}, \binom{1}{-1}\right\}$ **b.** $\mathbf{v} = \binom{3}{3}$ and $B = \left\{\binom{1}{2}, \binom{1}{1}\right\}$

c. $\mathbf{v} = \binom{12}{3}$ and $B = \left\{\binom{0}{2}, \binom{-1}{1}\right\}$ **d.** $\mathbf{v} = \binom{0}{9}$ and $B = \left\{\binom{10}{1}, \binom{-12}{1}\right\}$

36. Find a basis B of $\mathbb{R}^2$ such that the map f defined as $f(x,y) = (2x+5y, 3x+8y)$ relative to standard coordinates is represented by a matrix with zeroes outside of the main diagonal when the coordinates relative to B are used (the basis B is used for both the domain and the range) .

37. Is the map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ linear if $f(x,y) =$

a. $((x+3)^2 - (x+1)^2 - 8, y)$; **b.** $((x+3)^2 - (x+1)^2, y)$;

c. $(2\sqrt[3]{x^3+3x^2+3x+1} - 2(y+1), 5x - 3y)$; **d.** $(|x+1| - |y+1|, 2x)$.

38. Find a formula for the linear map $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ if $\varphi(2,3) = (1,0,1)$ and $\varphi(1,2) = (2,3,3)$.

39. Find a formula for the symmetry of $\mathbb{R}^3$ relative to the plane spanned by the vectors $(3,2,1)$ and $(1,-3,3)$. *A vector (a,b,c) is perpendicular to a vector (u,v,w) if and only if $au+bv+cw=0$.*

40. Let $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map such that $\varphi(3,1)=(1,2,3)$ and $\varphi(5,2)=(2,1,3)$ and $\psi\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a symmetry relative to the plane spanned by the vectors $(3,2,1$ and $(1,-3,3)$. Find the matrix of the map $\psi\circ\varphi$ relative to the standard bases.

41. Let $\varphi\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map such that $\varphi(2,1)=(2,1)$ and $\varphi(1,1)=(-1,-1)$. Find the matrix of φ relative to the basis $\{(2,1),(1,1)\}$ and the matrix of φ relative to the standard basis. Recall that the standard basis consists of $(1,0)$ and $(0,1)$.

How the matrix of $\varphi\circ\varphi$ looks like in both cases?

Solution of the problem 36. We have $M_{st}^{st}(f)=\begin{pmatrix} 2 & 5\\ 3 & 8\end{pmatrix}$. We want to find a basis $B=\{\mathbf{v}_1,\mathbf{v}_2\}$ such that $M_B^B(f)=\begin{pmatrix} c & 0\\ 0 & d\end{pmatrix}$ for some numbers $c,d\in\mathbb{R}$. If the numbers c,d and the vectors $\mathbf{v}_1,\mathbf{v}_2$ exist then $f(\mathbf{v}_1)=c\mathbf{v}_1$ and $f(\mathbf{v}_2)=d\mathbf{v}_2$. The vectors $\mathbf{v}_1,\mathbf{v}_2$ should be linearly independent so none of them equals $(0,0)$. Let $\mathbf{v}_1=\begin{pmatrix} x\\ y\end{pmatrix}$. Then we have $c\mathbf{v}_1=c\begin{pmatrix} x\\ y\end{pmatrix}=f\begin{pmatrix} x\\ y\end{pmatrix}=\begin{pmatrix} 2x+5y\\ 3x+8y\end{pmatrix}$. This means that the equations $(2-c)x+5y=0$ and $3x+(8-c)y=0$ should hold. Subtract the first one multiplied by $8-c$ from the second one multiplied by 5. Then $(15-(2-c)(8-c))x=0$. Since $x\neq 0$ we have $15-(2-c)(8-c)=0$ so $0=c^2-10c+1=(c-5)^2-24$ therefore $c=5\pm 2\sqrt{6}$. Let $c=5+2\sqrt{6}$ and $x=5$. Then $0=(2-5-2\sqrt{6})5+5y=(-3-2\sqrt{6})5+5y$ so $y=3+2\sqrt{6}$. Let $d=5-2\sqrt{6}$ and $x=5$. Then $0=(2-5+2\sqrt{6})5+5y=(-3+2\sqrt{6})5+5y$ so $y=3-2\sqrt{6}$. Then $\begin{pmatrix} 2 & 5\\ 3 & 8\end{pmatrix}\cdot\begin{pmatrix} 5\\ 3+2\sqrt{6}\end{pmatrix}=\begin{pmatrix} 25+10\sqrt{6}\\ 39+16\sqrt{6}\end{pmatrix}=(5+2\sqrt{6})\begin{pmatrix} 5\\ 3+2\sqrt{6}\end{pmatrix}$ and $\begin{pmatrix} 2 & 5\\ 3 & 8\end{pmatrix}\cdot\begin{pmatrix} 5\\ 3-2\sqrt{6}\end{pmatrix}=\begin{pmatrix} 25-10\sqrt{6}\\ 39-16\sqrt{6}\end{pmatrix}=(5-2\sqrt{6})\begin{pmatrix} 5\\ 3-2\sqrt{6}\end{pmatrix}$. Now define $B=\{\mathbf{v}_1,\mathbf{v}_2\}$ with $\mathbf{v}_1=\begin{pmatrix} 5\\ 3+2\sqrt{6}\end{pmatrix}$ and $\mathbf{v}_2=\begin{pmatrix} 5\\ 3-2\sqrt{6}\end{pmatrix}$. The above equalities prove that $M_B^B(f)=\begin{pmatrix} 5+2\sqrt{6} & 0\\ 0 & 5-2\sqrt{6}\end{pmatrix}$ as required. □

Remark. In fact the two long equations at the end which contain matrices are not necessary because they follow immediately from what was said before.

Solution of the problem 39. Let us start with finding a non–zero vector (a, b, c) perpendicular to the vectors $(3, 2, 1)$ and $(1, -3, 3)$. The two equations should be satisfied: $3a + 2b + c =$ and $a - 3b + 3c = 0$. The matrix of this system of the linear homogeneous equations is $\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -8 \\ 1 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -8 \\ 11 & 0 & 9 \end{pmatrix}$. Therefore $11b - 8c = 0$ and $11a + 9c = 0$ so we can set $c = 11$, $b = 8$ and $a = -9$. There are infinitely many other possibilities but they do not differ too much from our choice: one can multiply the chosen vector by any number different from 0.

The symmetry leaves all vectors in the plane spanned by the two vectors unchanged so $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is mapped to itself and also the vector $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ is mapped to itself. The vector $\mathbf{v}_3 = \begin{pmatrix} -9 \\ 8 \\ 11 \end{pmatrix}$ is mapped to $-\mathbf{v}_3 = \begin{pmatrix} 9 \\ -8 \\ -11 \end{pmatrix}$. This implies that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is mapped to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - c_3\mathbf{v}_3$.

This means that the matrix of the symmetry relative to the basis $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We want to find the matrix of the symmetry relative to the standard basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ etc.

If $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ i.e $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. It remains to find $M^{-1} = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}^{-1}$. The columns of $M = \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix}$ are mutually perpendicular so

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \\ -9 & 8 & 11 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & -9 \\ 2 & -3 & 8 \\ 1 & 3 & 11 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 266 \end{pmatrix}.$$ Therefore we may write

$$M^{-1} = \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \text{ so } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$ We are ready to end the solution. We change coordinates from $x's$ to $c's$ then apply the symmetry and then we go back to $x's$ (read it from right to left):

$$\begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 3 \\ -9 & 8 & 11 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & 2 & -1 \\ 1 & -3 & -3 \\ -9 & 8 & -11 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{14} & \frac{2}{14} & \frac{1}{14} \\ \frac{1}{19} & \frac{-3}{19} & \frac{3}{19} \\ \frac{-9}{266} & \frac{8}{266} & \frac{11}{266} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{52}{133} & \frac{72}{133} & \frac{99}{133} \\ \frac{72}{133} & \frac{69}{133} & \frac{-88}{133} \\ \frac{99}{133} & \frac{-88}{133} & \frac{12}{133} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Remark. In this solution it was assumed that the symmetry is a linear map. This is true because in the case discussed above the plane of symmetry contains the origin. This was not proved in the class but the proof is not hard because the sum of two vectors is a diagonal of the parallelogram spanned by them so the definition is not only algebraic but also has a geometrical meaning and symmetry behaves the geometrical properties. A solution of the problem given in the 9:45 class did not use this theorem and therefore an orthogonal projection onto the plane was considered.
In both groups a very stupid computational errors appeared on the board. I hope that they all are removed from this text.

Solution of the problem 40. Let us find the matrix of the of φ. The matrix should have two columns and three rows because φ maps $\mathbb{R}^2$ into $\mathbb{R}^3$. Let it be $\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. We want to find such numbers a, b, c, d, e, f that $\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix}$. We have $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ this follows from the definition of the inverse matrix. Do not be afraid of guessing! Now multiply the equation by $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$. We get

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix}.$$

We may write $\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 3x - 7y \\ 3x - 6y \end{pmatrix}$ and we are done. The last thing is to find the matrix of $\psi \circ \varphi$. The matrix can be obtained as a product of the two matrices (for the matrix of the symmetry ψ look into the solution of the previous problem)

$$\begin{pmatrix} \frac{52}{133} & \frac{72}{133} & \frac{99}{133} \\ \frac{72}{133} & \frac{69}{133} & \frac{-88}{133} \\ \frac{99}{133} & \frac{-88}{133} & \frac{12}{133} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} = \frac{1}{133} \begin{pmatrix} 52 & 72 & 99 \\ 72 & 69 & -88 \\ 99 & -88 & 12 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 3 & -7 \\ 3 & -6 \end{pmatrix} = \begin{pmatrix} \frac{513}{133} & -\frac{1046}{133} \\ -\frac{57}{133} & \frac{117}{133} \\ -\frac{228}{133} & \frac{643}{133} \end{pmatrix}$$