Recall that a basis $\mathcal{B}$ of a vector space V is a set which
consists of linearly inedendent vectors
and spans the whole space V so each element of V is a linear combination of some elements of $\mathcal{B}$.

The simplest example of a basis of $\mathbb{R}^2$ is the set $\{(1,0),(0,1)\}$. But there are infinitely many others e.g. $\{(5,8),(8,13)\}$. The two vectors are linearly independent because if $(0,0) = c_1(5,8) + c_2(8,13)$ then $0 = 5c_1 + 8c_2$ and $0 = 8c_1 + 13c_2$. Therefore $0 = 8 \cdot 0 - 5 \cdot 0 = 8(5c_1 + 8c_2) - 5(8c_1 + 13c_2) = -c_2$ so $c_2 = 0$. This implies that $0 = 5c_1$ so $c_1 = 0$. In the same way we prove that $\mathrm{lin}((5,8),(8,13)) = \mathbb{R}^2$. We shall do it. We want to show that each vector (x,y) may be written as $c_1(5,8) + c_2(8,13)$, so $x = {=}5c_1 + 8c_2$ and $y = 8c_1 + 13c_2$. We need to prove that for any pair of numbers (x,y) there exist numbers c_1, c_2 such that the two above equations hold. Then $-8x + 5y = -8(5c_1 + 8c_2) + 5(8c_1 + 13c_2) = c_2$ and $13x - 8y = 13((5c_1 + 8c_2) - 8(8c_1 + 13c_2) = c_1$ so there is just one possible choice of numbers c_1, c_2. One should mention that if the set $\mathcal{B}$ spans the space V and each vector of V can be expressed as a linear combination of the elements of $\mathcal{B}$ in exactly one way then $\mathcal{B}$ is a basis of V. This can be regarded as another definition of basis of course equivalent to the previous one.

One more thing that should be reminded at the moment is all basis of a vector space V have the same number of elements. This number is called the dimension of the vector space. The dimension of $\mathbb{R}^k$ is k so each basis of $\mathbb{R}^k$ consists of k linearly independent vectors. If one considers a plane contained in $\mathbb{R}^3$ which contains the origin then one considers a two dimensional subspace of $\mathbb{R}^3$. Any basis of such subspace consists of two vectors which are not parallel (all vectors start at the origin). For example, the equation $2x + 3y - 5z = 0$ defines a plane through the origin. The plane is perpendicular to the vector $(2,3,-5)$. An example of a basis of this plane is $\{(3,-2,0),(5,0,2)\}$.

A map $\varphi\colon V \to W$ from a vector space V to a vector space W is called linear if and only if

$\varphi(\mathbf{v}_1 + \mathbf{v}_2) = \varphi(\mathbf{v}_1) + \varphi(\mathbf{v}_2)$ for vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ (additivity) and

$\varphi(c\mathbf{v}) = c\varphi(\mathbf{v})$ for any $c \in \mathbb{R}$ and any $\mathbf{v} \in V$ homogeneity.

It follows right from this definition that a linear combination of vectors is mapped the linear combination of their images with the same coefficients. In particular the zero vector is mapped to the zero vector.

Therefore to define a linear map it is enough to tell what are the images of one basis. If the $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$ is linear then $\varphi(x,y,z) = x\varphi(1,0,0) + y\varphi(0,1,0) + z\varphi(0,0,1) = ax + by + cz$ where $a = \varphi(1,0,0)$, $b = \varphi(0,1,0)$ and $c = \varphi(0,0,1)$. A very important examples are projections onto the axis: $p_1(x,y,z) = x$ $(a = 1, b = c = 0)$, $p_2(x,y,z) = y$ $(a = c = 0, b = 1)$, $p_3(x,y,z) = z$ $(a = b = 0, c = 1)$.

A map φ defined by the formula $\varphi(x,y,z) = (x-y, y-z, z-x)$ is linear this can be checked easily. It maps the space $\mathbb{R}^3$ into itself. So the image (or the range) of the map, i.e. the set of all values of the map, is a subspace of $\mathbb{R}^3$. This image is smaller than $\mathbb{R}^3$. For example $(1,0,0)$ does not belong to this image. In fact the image of the map is a plane P defined by the equation $x_1 + x_2 + x_3 = 0$. It is obvious that P contains the image of the map: $x - y + y - z + z - x = 0$. From the formula $\varphi(x_1 + x_2, x_2, 0) = (x_1, x_2, -x_1 - x_2)$ it follows that the image contains all points of the plane

$x_1 + x_2 + x_3 = 0$. All points of the form (x, x, x) are mapped to one point, namely to $(0, 0, 0)$. This is a straight line which is called a kernel of φ (the kernel consists of all points that are mapped to zero vector). So in this case the image is a plane, the kernel is a straight line. The sum of their dimensions is 3, it is not a random information.

Let $\varphi(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$. The map φ maps the plane $\mathbb{R}^2$ into itself. We might ask a question: which points are mapped to the point $(0, 0)$, or equivalently what is the kernel of φ? This leads to the system of equations $0.6x - 0.8y = 0,\ 0.8x + 0.6y = 0$. They imply that $0 = 0.6(0.6x - 0.8y) + 0.8(0.8x + 0.6y) = x$. Then from $0 = 0.6 \cdot 0 - 0.8y$ it follows that also $y = 0$. So in this case the kernel consists of the origin only. Another property of the map follows from the equality $(0.6x - 0.8y)^2 + (0.8x + 0.6y)^2 = 0.36x^2 - 0.96xy + 0.64y^2 + 0.64x^2 + 0.96xy + 0.36x^2 = = x^2 + y^2$. This means the the distance from the origin to the point (x, y) is equal to the distance from the origin to the point $\varphi(x, y)$. This may be easily generalized to the following: the distance from a point (x, y) to a point (u, v) is the same as the distance from the point $\varphi(x, y)$ to the point $\varphi(u, v)$. The only point (x, y) which is mapped to itself is $(0, 0)$. This can be deduced from the equations $x = 0.6x - 0.8y,\ y = 0.8x + 0.6y$, which are equivalent to $-0.4x = -0.8y,\ 0.4y = 0.8x$ or $x = 2y,\ y = 2x$, so $x = 4x$ so $x = 0$. These informations in fact allow us to say that φ is a rotation about the origin (the point $(1, 0)$ goes to the point $(0.6, 0.8)$. The angle is approximately $53°$, counterclockwise. In fact a rotation about the origin by $\alpha°$ is a linear map that maps a point (x, y) to the point $(x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha)$. This can be checked by examinig the triangle with the vertices $(0, 0), (x, y)$ and $(x \cos\alpha - y \sin\alpha, x \sin\alpha + y \cos\alpha)$.

Is a translation (shift) a linear map? Justify your answer, please!

Prove that if $\varphi(x, y) = (x \cos\alpha + y \sin\alpha, x \sin\alpha - y \cos\alpha)$ then there is a straight line L through the origin (one dimensional vector subspace of $\mathbb{R}^2$) which consists entirely of the fixed points of φ i.e. such points (x, y) that $\varphi(x, y) = (x, y)$. Give an equation of L, please. Prove that for each point (x, y) the points (x, y) and $\varphi(x, y)$ are symmetric relative to L.

One can prove that an isometry of $\mathbb{R}^n$ i.e. a map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ such that the distances from $\mathbf{p}$ to $\mathbf{q}$ and from $\varphi(\mathbf{p})$ to $\varphi(\mathbf{q})$ are equal is linear if and only if $\varphi(\mathbf{0}) = \mathbf{0}$. A proof of this theorem is not a difficult.

Let $\varphi(x, y) = (y, x + y)$. This map is linear – obvious! Clearly if $(x, y) = \varphi(x, y) = (y, x + y)$ then $(x, y) = (0, 0)$ (so there is only one fixed ponit in this case). This map is NOT an isometry because the distance from the origin to $(0, 1)$ is 1 while the distance from the origin to $\varphi(0, 1) = (1, 1)$ is $\sqrt{2}$. One can ask a question: does there exist a line through the origin which mapped onto itself? The question can be restated as follows: is there a vector $\mathbf{v}$ which is mapped to a vector parallel to $\mathbf{v}$ (i.e. to itself). A vector parallel to $\mathbf{v}$ may be written as $\lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$. So our problem is: does there exist a vector $\mathbf{v} \in \mathbb{R}^2$ and a number $\lambda \in \mathbb{R}$ such that $\varphi(\mathbf{v}) = \lambda\mathbf{v}$. Write $\mathbf{v} = (x, y)$. We have now 2 equations: $y = \lambda x,\ x + y = \lambda y$. If $x = 0$ then $y = 0$. If $x \neq 0$ then $x + \lambda x = \lambda^2 x$ hence $1 + \lambda = \lambda^2$. Therefore either $\lambda = \frac{1}{2}(1 - \sqrt{5}) \in (-1, 0)$ or $\lambda = \frac{1}{2}(1 + \sqrt{5}) > 1$. There are two lines through the origin that are mapped by φ onto themselves. The line $y = \frac{1}{2}(1 - \sqrt{5})x$ is squized and flipped due to $-1 < \frac{1}{2}(1-\sqrt{5}) < 0$ while the line $y = \frac{1}{2}(1+\sqrt{5})x$ is expanded due to $\frac{1}{2}(1+\sqrt{5}) > 1$. How the image of the square with the vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ looks like? What is the area of this image?