Find (if there exists) a vector $\gamma \in \mathbb{R}^3$, such that the system $(1,0,1),(2,1,0),\gamma$ constitutes a basis of the space R^3 and the coordinates of the vector $(7,3,5)$ are $3,1,2$ in the basis.

Solution. Let $\gamma = (x,y,z)$. We want to find numbers x,y,z such that $(7,3,5) = 3(1,0,1)+(2,1,0)+2(x,y,z)$. We can write this vector equation as three equations: $3+2+2x = 7$, $1+2y = 3$, $3+2z = 5$. This implies that $x = 1, y = 1, z = 1$. One needs to check whether or not the vectors $(1,0,1),(2,1,0),(1,1,1)$ constitute a basis of $\mathbb{R}^3$

Homework Problem 3. Prove that if $x_1 < x_2 < x_3$ then for any real numbers y_1, y_2, y_3 there exist real numbers a,b,c such that $y_j = ax_j^2 + bx_j + c$. How many triples a,b,c exist? Do there exist numbers x_1, x_2, x_3,y_1, y_2, y_3 with $x_1 < x_2 < x_3$ such that $a = 0$?

Solution. We have to analyze the system of equations

$$(1)\qquad \begin{cases} ax_1^2 + bx_1 + c &= y_1 \\ ax_2^2 + bx_2 + c &= y_2 \\ ax_3^2 + bx_3 + c &= y_3 \end{cases}$$

We see three equations with three unknowns a,b,c. The matrix of the system is

$$\begin{pmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{pmatrix}$$

We want to have number 1 at the left upper corner. The order of the unknowns is unimportant. Therefore we may write the three columns on the left in any way (we have never done it up to now. but we are allowed to swap the columns corresponding to the unknowns).

$$(2)\qquad \begin{pmatrix} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{pmatrix}$$

We are ready to perform row reduction. Subtract the first row from the others

$$\begin{pmatrix} 1 & x_1 & x_1^2 & y_1 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & y_2 - y_1 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 & y_3 - y_1 \end{pmatrix}$$

Since $x_2 - x_1 \neq 0 \neq x_3 - x_1$ we may divide the rows by these numbers:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & y_1 \\ 0 & 1 & x_2 + x_1 & \frac{y_2 - y_1}{x_2 - x_1} \\ 0 & 1 & x_3 + x_1 & \frac{y_3 - y_1}{x_3 - x_1} \end{pmatrix}$$

Subtract the second row from the third one

$$\begin{pmatrix} 1 & x_1 & x_1^2 & y_1 \\ 0 & 1 & x_2+x_1 & \frac{y_2-y_1}{x_2-x_1} \\ 0 & 0 & x_3-x_2 & \frac{y_3-y_1}{x_3-x_1}-\frac{y_2-y_1}{x_2-x_1} \end{pmatrix}$$

Now we can divide the third row by $x_3-x_2 \neq 0$

$$\begin{pmatrix} 1 & x_1 & x_1^2 & y_1 \\ 0 & 1 & x_2+x_1 & \frac{y_2-y_1}{x_2-x_1} \\ 0 & 0 & 1 & \frac{y_3-y_1}{(x_3-x_1)(x_3-x_2)}-\frac{y_2-y_1}{(x_2-x_1)(x_3-x_2)} \end{pmatrix}$$

The matrix is written in the echelon form (not reduced yet). The last row corresponds to the equation $a = 0\cdot c+0\cdot b+1\cdot a = \frac{y_3-y_1}{(x_3-x_1)(x_3-x_2)}-\frac{y_2-y_1}{(x_2-x_1)(x_3-x_2)}$. There is no other choice for a. This means that the system determines a uniquely. Once a is found we can find b from the second equation which looks like this $b + \textit{something known} = \textit{the known number}$. We still need c but it can be found quickly from the first equation. This proves that there is the unique triple c, b, a that solves the system. We might easily find the unknowns although some work was necessary (few transformations with quite big fractions).

There is another possibility. The polynomial $d(x-x_2)(x-x_3)$ assumes the value 0 for $x = x_2$ and for $x = x_3$ and for no other value of x unless $d = 0$. Let $d = \frac{1}{(x_1-x_2)(x_1-x_3)}$ so the polynomial takes the form $\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$ and now its values for $x = x_1$, $x = x_2$, $x - x_3$ are 1, 0 and 0, respectively. Multiply it by y_1 to obtain $y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$. The obtained function assumes values y_1, 0, 0 at x_1, x_2, x_3, respectively. We may consider the polynomial $y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}+y_2\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}+y_3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$. It is easy to see that this polynomial assumes the value y_j for $x = x_j$ as required. As proved above there it is the only one polynomial of degree < 3 that does it. If the points (x_j, y_j) for $j = 0; 1; 2$ lie on one straight line then $a = 0$ (if $y_1 = y_2 = y_3$ then the straight line exists and it is horizontal hence $a = b = 0$). The problem is completely solved. One more remark. From this long story and uniqueness of the polynomial it follows that $a = \frac{y_3-y_1}{(x_3-x_1)(x_3-x_2)}-\frac{y_2-y_1}{(x_2-x_1)(x_3-x_2)} = \frac{y_1}{(x_1-x_2)(x_1-x_3)}+\frac{y_2}{(x_2-x_1)(x_2-x_3)}+\frac{y_3}{(x_3-x_1)(x_3-x_2)}$. One might check it by solving the system (1) up to the very end instead of showing the solution..

The result may be generalized. We may talk about an arbitrary number (finite) of points on the plane. This is stated in the theorem below.

Theorem. Given any $n+1$ points (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n) with $x_0 < x_1 < x_2 < \ldots < x_n$, $n \in \mathbf{N}$ in the plane there are real numbers $a_0, a_1, \ldots, a_n$ such that each all $j \in \{0, 1, 2, \ldots, n\}$ the following equality $y_j = a_0 + a_1x_j + a_2x_j^2 + \ldots + a_nx_j^n$ holds. The numbers $a_0, a_1, \ldots, a_n$ are uniquely defined by the $n+1$ equations written above.

For $n = 1$ this means that given any two points (x_0, y_0), (x_1, y_1) with $x_0 < x_1$ there is a linear function the graph of which contains these points, recall that the graph of the linear function is a straight line. In our case the straight line is not vertical (no graph contains two different points with the same x–coordinate). The homework problem contained this theorem for $n = 3$. For $n = 2$ it means that for any three points on the plane no two of them lying on one vertical line either there is a straight line which contains all three points or there is a parabola (with a vertical axis of symmetry) which contains the three given points. One case excludes another one. One can prove this statement applying row operations to a $n + 1 \times n + 1$ matrix

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}$$

Subtract the first frow from the other rows

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 0 & x_1 - x_0 & x_1^2 - x_0^2 & \dots & x_1^n - x_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_0 & x_n^2 - x_0^2 & \dots & x_n^n - x_0^n \end{pmatrix}$$

the numbers $x_1 - x_0$, $x_2 - x_0$, $\dots$, $x_n - x_0$ are different from 0 so can divide the rows by them. We obtain

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 0 & 1 & x_1 + x_0 & \dots & x_1^{n-1} + x_1^{n-2}x_0 + \dots + x_0^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & x_n + x_0 & \dots & x_n^{n-1} + x_n^{n-2}x_n + \dots + x_0^{n-1} \end{pmatrix}$$

The left-hand side of the equation corresponding to the second row can be written as follows
$a_1 + a_2(x_1 + x_0) + a_3(x_1^2 + x_1x_0 + x_0^2) + \dots + a_n(x_1^{n-1} + x_1^{n-2}x_0 + \dots + x_0^{n-1}) =$
$= (a_1 + a_2x_0 + a_3x_0^2 + \dots + a_nx_0^{n-1}) + (a_2 + a_3x_0 + \dots + a_nx_0^{n-2})x_1 + \dots + (a_{n-1} + a_nx_0)x_1^{n-1} + a_nx_1^{n-1}$.
In the same way the left-hand sides of the next equations can be written.

Let $\tilde{a}_1 = a_1 + a_2x_0 + a_3x_0^2 + \dots + a_nx_0^{n-1}$, $\tilde{a}_2 = a_2 + a_3x_0 + a_4x_0^2 + \dots + a_nx_0^{n-2}$,..., $\tilde{a}_n = a_n$. Then the above left-hand side looks like that $\tilde{a}_1 + \tilde{a}_2x_1 + \dots + \tilde{a}_nx_1^{n-1}$. The left-hand side of the next equation is $\tilde{a}_1 + \tilde{a}_2x_2 + \dots + \tilde{a}_nx_2^{n-1}$ etc. There are some right-hand sides but they are unimportant for us at the moment. One can see that we have reduced the problem to the same type problem but with n points instead of $n + 1$. Therefore we are able to find the numbers $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$. Once they are found we find subsequently $a_n, a_{n-1}, \dots, a_1$. Then a_0 from the very first equation.

So if we assume that we proved the theorem for n points it will also be true for $n + 1$. This we can do provided the theorem we are proving holds for n points. So we are done.

PROBLEMS FOR HOME 2.

21. Solve the systems of linear equations.

$$\begin{cases} 2x-y-z &= 4 \\ 3x+4y-2z &= 11 \\ 3x-2y+4z &= 11 \end{cases} \qquad \begin{cases} 3x+2y+z &= 5 \\ 2x+3y+z &= 1 \\ 2x+y+3z &= 11 \end{cases}$$

$$\begin{cases} 2x-y-z &= 4 \\ 3x+4y-2z &= 11 \\ x+16y-2z &= 17 \end{cases} \qquad \begin{cases} 3x+2y+z &= 5 \\ 2x+3y+z &= 1 \\ 4x+y+z &= 13 \end{cases}$$

Solution. We shall operate on the rows of the appropriate matrices one after another.

$$\begin{pmatrix} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 22 & -2 & 20 \\ 0 & 16 & 4 & 20 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 22 & -2 & 20 \\ 0 & 16 & 4 & 20 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 11 & -1 & 10 \\ 0 & 4 & 1 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 1 & 4 & 5 \\ 0 & 4 & 1 & 5 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & -15 & -15 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$ so $x = 3$, $y = z = 1$. The first system has been solved.

$$\begin{pmatrix} 3 & 2 & 1 & 5 \\ 2 & 3 & 1 & 1 \\ 2 & 1 & 3 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 3 & 1 & 1 \\ 2 & 1 & 3 & 11 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 5 & 1 & -7 \\ 0 & 3 & 3 & 3 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 0 & -4 & -12 \\ 0 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$ so $x = 2$, $y = -2$, $z = 3$. The second system has been solved.

$$\begin{pmatrix} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 1 & 16 & -2 & 17 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 16 & -2 & 17 \\ 0 & -33 & 3 & -30 \\ 0 & -44 & 4 & -40 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 16 & -2 & 17 \\ 0 & 11 & -1 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & -6 & 0 & -3 \\ 0 & -11 & 1 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ so $x - 6y = -3$, $-11y + z = -10$ or $x = 6y - 3$, $z = 11y - 10$. Given any number y we can find x and z. They are uniquely defined through the above formulas. The system has infinitely many solutions. In fact there is a straight line consisting entirely of the solutions to

the system and these are all solutions to this system. The line passes through the point $(-3, 0, -10)$ and is parallel to the vector $[6, 1, 11]$ i.e. $(x, y, z) = (-3, 0, -10) + y(6, 1, 11)$. We might also say that the line passe through the points $(-3, 0, -10)$ and $(-3, 0, -10) + (6, 1, 11) = (3, 1, 1)$. This ends the discussion of the third system.

$$\begin{pmatrix} 3 & 2 & 1 & 5 \\ 2 & 3 & 1 & 1 \\ 4 & 1 & 1 & 13 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 2 & 3 & 1 & 1 \\ 4 & 1 & 1 & 13 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 5 & 1 & -7 \\ 0 & 5 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 5 & 1 & -7 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

At this moment we can stop our action because it is easily seen that the system is inconsistent: the last equation may be read $0 = 0 \cdot x + 0 \cdot y + 0 \cdot z = 4$. A contradiction.

One may also come to this conclusion in a different way. Add the second and the third equation to get $6x + 4y + 2z = 14$. Then divide it by 2: $3x + 2y + z = 7 \neq 5 = 3x + 2y + z$ and we are done.

22. Solve the systems of the linear equations where λ is a given number, the result may depend on λ.

$$\begin{cases} -x+y &= \lambda x \\ -x-3y &= \lambda y \end{cases} \qquad \begin{cases} 2x+\ 6y+\ 4z &= \lambda x \\ -3x-20y-14z &= \lambda y \\ 6x+35y+24z &= \lambda z \end{cases}$$

Let us now investigate the first system $\begin{cases} -x+y &= \lambda x \\ -x-3y &= \lambda y \end{cases}$.

Take all terms to the left hand side of the equations. $\begin{cases} -(1+\lambda)x+y &= 0 \\ -x-(3+\lambda)y &= 0 \end{cases}$.

$x=y=0$ is an obvious solution for each λ. We need to discover other solutions if there are any. Let us play our usual game with the rows of the matrix.

$\begin{pmatrix} -(1+\lambda) & 1 & 0 \\ -1 & -(3+\lambda) & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1+(1+\lambda)(3+\lambda) & 0 \\ 1 & (3+\lambda) & 0 \end{pmatrix}$. If $1+(1+\lambda)(3+\lambda)\neq 0$ then $y=0$ and then the second equation implies that $x=0$ so $(0,0)$ is the unique solution of the system. If $0=1+(1+\lambda)(3+\lambda)=4+4\lambda+\lambda^2=(2+\lambda)^2$ then $\lambda=-2$. For $\lambda\neq -2$ the system has unique solution $x=0=y$. For $\lambda=-2$ there are infinitely many solutions. Corresponding to each number y is $x=-y$ so the solutions treated as points of the plane form a straight line through the origin. Each vector $[x,-x]$ with $x\neq 0$ is called *an eigenvector* of the matrix $\begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix}$.

PROBLEMS FOR HOME 3

31. Do the points $(8,13)$, $(13,21)$ i $(21,34)$ lie on one straight line?

32. Do the points $A=(0,0,0)$, $B=(-7,30,-45)$, $C=(-6,24,-33)$ and $D=(-2,8,10)$ lie on one plane (two dimensional)?

33. Do the points $(0,0,0)$, $(-27,-36,-21)$, $(-2,-6,-2)$ and $(30,60,45)$ lie on one plane?

34. Solve the systems of the linear equations where λ is a given number, the result may depend on λ.

$$\begin{cases} y &= \lambda x \\ x+2y &= \lambda y \end{cases} \qquad \begin{cases} -7x-6y-\ 2z &= \lambda x \\ 30x+24y+\ 8z &= \lambda y \\ -45x-33y-10z &= \lambda z \end{cases}$$

$$\begin{cases} y &= \lambda x \\ x+y &= \lambda y \end{cases} \qquad \begin{cases} -27x-2y+45z &= \lambda x \\ -36x-6y+60z &= \lambda y \\ -21x-2y+35z &= \lambda z \end{cases}$$