Let us consider a system of linear equations

$$\begin{cases} 3x_1 & + & 6x_2 & - & 5x_3 & = & 0 \\ x_1 & - & 5x_2 & + & 3x_3 & = & 0 \\ 13x_1 & - & 2x_2 & - & 3x_3 & = & 0 \end{cases}$$

We can consider the matrix

$$\begin{pmatrix} 3 & 6 & -5 & 0 \\ 1 & -5 & 3 & 0 \\ 13 & -2 & -3 & 0 \end{pmatrix}$$

made of the coefficients standing in front of the unknowns of the system above. We shall solve the system playing the standard game.

$$\begin{pmatrix} 3 & 6 & -5 & 0 \\ 1 & -5 & 3 & 0 \\ 13 & -2 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 3 & 6 & -5 & 0 \\ 13 & -2 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 21 & -14 & 0 \\ 0 & 63 & -42 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We swapped the first two rows to obtain 1 in the left upper corner of the matrix, then we subtracted the first row multiplied by 3 from the second row to obtain 0 below 1, at the same time we subtracted the first row multiplied by 13 from the third one to obtain one more zero below 1. In the next step the second row was divided by 21 while the last row was divided by 63, then the second row was subtracted from the last one. The last action was to add the second row multiplied by 5 to the first one to obtain 0 above 1 in the second row.

Let us write now the system of the linear equations defined by the last matrix:

$$\begin{cases} 1 \cdot x_1 & + & 0 \cdot x_2 & - & \frac{1}{3} \cdot x_3 & = & 0 \\ 0 \cdot x_1 & + & 1 \cdot x_2 & - & \frac{2}{3} \cdot x_3 & = & 0 \end{cases}$$

This can be rewritten as $x_1 = \frac{1}{3}x_3, x_2 = \frac{2}{3}x_3$. We see now that the system has infinitely many solutions. So it is consistent but underdetermined. At the begining there were three equations but after transformations we ended up with only 2 equations with the three unknowns. We can now find many triples of reals that satisfy the initial system: we pick up any real number x_3 and evaluate x_1 and x_2. The unknown x_3 is called a free variable because we may set its value without any restrictions. After that the values of x_1 and x_2 are uniquely defined. x_1 and x_2 are basic variables. All solutions of the system are of the form $(\frac{x_3}{3}, \frac{2x_3}{3}, x_3)$. This can be written also in the form $(x_1, 2x_1, 3x_1) = x_1(1, 2, 3)$. Now x_1 is a free variable, x_2, x_3 are basic variables.

We want to see the set of all solutions of the system of the equations. It contains the point $(0, 0, 0)$, also the point $(1, 2, 3)$ and many others such as $(-2 - 4 - 6)$, $(\sqrt{7}, 2\sqrt{7}, 3\sqrt{7})$ etc. In fact this set is the straight line which contains these two points. The line is parallel to the vector $[1, 2, 3]$.

Denote by r_1 the first row of the initial matrix. r_2, r_3 are the next rows. What did before is

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \to \begin{pmatrix} r_2 \\ r_1 \\ r_3 \end{pmatrix} \to \begin{pmatrix} r_2 \\ r_1 - 3r_2 \\ r_3 - 13r_2 \end{pmatrix} \to \begin{pmatrix} r_2 \\ \frac{1}{21}(r_1 - 3r_2) \\ \frac{1}{63}(r_3 - 13r_2 \end{pmatrix} \text{ and we noticed}$$

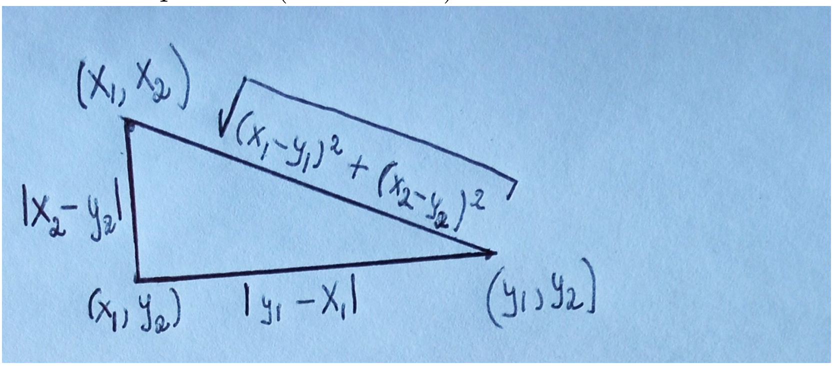
that $\frac{1}{21}(r_1 - 3r_2) = \frac{1}{63}(r_3 - 13r_2)$. This implies that $3(r_1 - 3r_2) = r_3 - 13r_2$ or $3r_1 + 4r_2 = r_3$. This shows that the last equation follows from the first two so it adds no new infromation to them. This means that in fact we deal with the system of **two** equations with **three** unknowns. Therefore it is not surprising that the system has infinitely many solutions.

DEFINITON

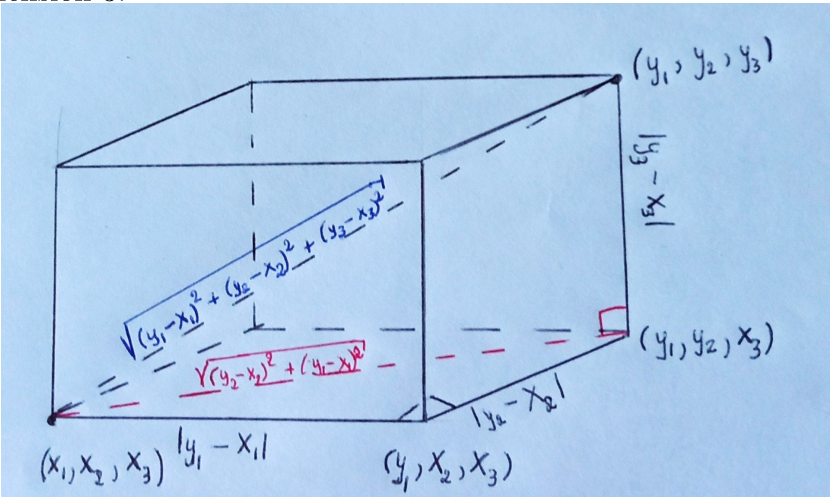
The vector $[x_1, x_2, \dots, x_n]$ is parallel to a vector $[a_1, a_2, \dots, a_n] \neq [0, 0, \dots, 0]$ if and only if there exists a real number t such that

$$[x_1, x_2, \dots, x_n] = t[a_1, a_2, \dots, a_n]. \quad \square$$

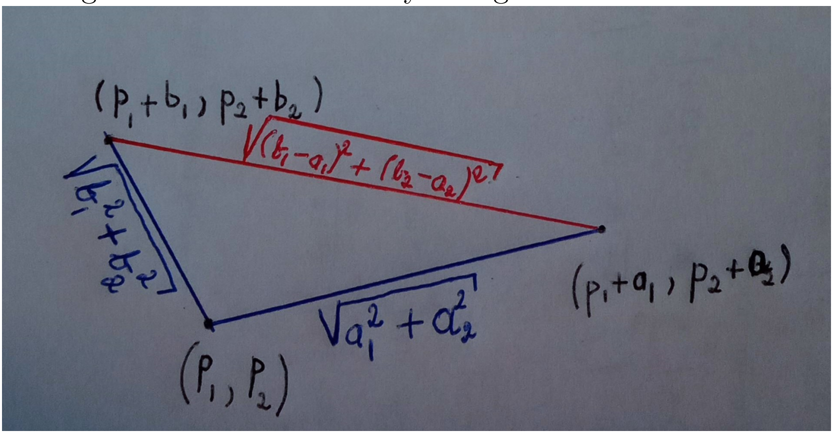
An immediate corollary is that the zero vector is parallel to all vectors. The vectors $[2 - 3]$ and $4, -6]$ are parallel because $[4, -6] = 2[2, -3]$. The vectors $[1, -2, 3]$ and $[-5, 10, -15]$ are parallel because $[-5, 10, -15] = -5[1, -2, 3]$. Obviously we may talk about parallel vectors only if they have the same number of components (coordinates). Otherwise the definition has no sense.



The distance between points $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ is equal to $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}$. In case of the plane (each point has 2 coordinates) it follows from the Pythagorean theorem, it is true also in dimension 3.



For higher dimensions this may be regarded as a definition of the distance.



The vectors $[a_1, a_2, \ldots, a_n]$ and $[b_1, b_2, \ldots, b_n]$ are perpendicular iff (it is an abreviation for *if and only if*) when the angle $\sphericalangle \mathbf{bpa}$ in the triangle with the vertices $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, $\mathbf{a} = (\mathrm{p}_1 + \mathrm{a}_1, \mathrm{p}_2 + \mathrm{a}_2, \ldots, \mathrm{p}_\mathrm{n} + \mathrm{a}_\mathrm{n})$ and $\mathbf{b} = (\mathrm{p}_1 + \mathrm{b}_1, \mathrm{p}_2 + \mathrm{b}_2, \ldots, \mathrm{p}_\mathrm{n} + \mathrm{b}_\mathrm{n})$ is right (equal to 90°). By the Pythagorean theorem this happens iff

$$(a_1^2 + a_2^2 + \cdots + a_n^2) + (b_1^2 + b_2^2 + \cdots + b_n^2) = (b_1 - a_1)^2 + (b_2 - a_2)^2 + \ldots + (b_n - a_n)^2.$$

Applying the formula $(b - a)^2 = b^2 - 2ba + a^2$ n times and simplifying the obtained expression we obtain $0 = -2b_1a_1 - 2b_2a_2 - \ldots - 2b_na_n$ so

$$b_1a_1 + b_2a_2 + \ldots b_na_n = 0.$$

Let us assume that the points (x_1, x_2) and (y_1, y_2) lie on the straight line $2x + 5y = 7$. This means that the equations $2x_1 + 5x_2 = 7$ and $2y_1 + 5y_2 = 7$ are satisfied. Subtract the first one from the second one to get

$$2(y_1 - x_1) + 5(y_2 - x_2) = 7 - 7 = 0.$$

Therefore the vector $[2, 5]$ is perpendicular to the vector $[y_1 - x_1, y_2 - x_2]$. Since the later one is an arbitrary vector whose initial and end points lie on the line this means that the vector $[2, 5]$ is perpendicular to the line. In the same way one proves that e.g. the vector $[3, 6, -5]$ is perpendicular to the plane $3x_1 + 6x_2 - 5x_3 = 10$. To be precise: if $3x_1 + 6x_2 - 5x_3 = 10$ and $3y_1 + 6y_2 - 5y_3 = 10$, so if the points (x_1, x_2, x_3) and (y_1, y_2, y_3) lie on the plane then $3(y_1 - x_1) + 6(y_2 - x_2) - 5(y_3 - x_3) = 0$, this means that the vector $[y_1 - x_1, y_2 - x_2, y_3 - x_3]$ is perpendicular to the vector $[3, 6, -5]$. Since the points (x_1, x_2, x_3) and (y_1, y_2, y_3) were chosen arbitrarily this means that the vector $[3, 6, -5]$ is perpendicular to the plane.

Let us consider a system of linear equations

$$\begin{cases} 3x_1 + 6x_2 - 5x_3 = 10 \\ x_1 - 5x_2 + 3x_3 = 0 \\ 13x_1 - 2x_2 - 3x_3 = 0 \end{cases}$$

We can consider the matrix

$$\begin{pmatrix} 3 & 6 & -5 & 10 \\ 1 & -5 & 3 & 0 \\ 13 & -2 & -3 & 0 \end{pmatrix}$$

made of the coefficients standing in front of the unknowns of the system above. We shall simplify the matrix using row reduction.

$$\begin{pmatrix} 3 & 6 & -5 & 10 \\ 1 & -5 & 3 & 0 \\ 13 & -2 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 3 & 6 & -5 & 10 \\ 13 & -2 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 21 & -14 & 10 \\ 0 & 63 & -42 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{10}{21} \\ 0 & 1 & -\frac{2}{3} & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & 3 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{10}{21} \\ 0 & 0 & 0 & -\frac{10}{21} \end{pmatrix}.$$

This is a contradiction because the last row corresponds to the equation $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -\frac{10}{21}$ which is not possible of course. This system is inconsistent.

Now we shall try to understand the geometry of the system. Each of the three equations describes a plane in the three dimensional space. The planes are not parallel since the vectors $[3, 6, -5]$, $[1, -5, 3]$, $[13, -2, -3]$ which are perpendicular to the corresponding planes are not parallel. Therefore every two planes intersect along some straight line. We shall find these lines.

Let us start with the first two planes. $\begin{cases} 3x_1 + 6x_2 - 5x_3 = 10 \\ x_1 - 5x_2 + 3x_3 = 0 \end{cases}$. Subtract the second equation multiplied by 3 from the first one. The result is $\begin{cases} 21x_2 - 14x_3 = 10 \\ x_1 - 5x_2 + 3x_3 = 0 \end{cases}$ Divide the first equation by 21: $\begin{cases} x_2 - \frac{2}{3}x_3 = \frac{10}{21} \\ x_1 - 5x_2 + 3x_3 = 0 \end{cases}$ Now add the first equation multiplied by 5 to the second one $\begin{cases} x_2 - \frac{2}{3}x_3 = \frac{10}{21} \\ x_1 - \frac{1}{3}x_3 = \frac{50}{21} \end{cases}$ This means that the intersection of the two planes consists of the points of the sort $(\frac{1}{3}x_3+\frac{50}{21}, \frac{2}{3}x_3+\frac{10}{21}, x_3) = (\frac{50}{21}, \frac{10}{21}, 0) + \frac{1}{3}x_3(1,2,3)$ so it is a line parallel to the vector $(1,2,3)$.

Now the last two planes. $\begin{cases} x_1 - 5x_2 + 3x_3 = 0 \\ 13x_1 - 2x_2 - 3x_3 = 0 \end{cases}$ Subtract the first of the two equations multiplied by 13 from the second one
$\begin{cases} x_1 - 5x_2 + 3x_3 = 0 \\ 63x_2 - 42x_3 = 0 \end{cases}$ Divide the second equation by 63:
$\begin{cases} x_1 - 5x_2 + 3x_3 = 0 \\ x_2 - \frac{2}{3}x_3 = 0 \end{cases}$ Now multiply the second equation by 5 and add the result to the first one: $\begin{cases} x_1 - \frac{1}{3}x_3 = 0 \\ x_2 - \frac{2}{3}x_3 = 0 \end{cases}$
This shows that the common points of the two planes are of the form $(\frac{1}{3}x_3, \frac{2}{3}x_3, x_3) = \frac{1}{3}x_3(1,2,3)$ so they form a straight line parallel to the vector $[1,2,3]$. It is easy to see that the line obtained now is parallel to the previous one and that the two lines are disjoint (theoretically it could be the same line).

The last part of the story is the intersection of the first and the third plane. Consider the system of two equations $\begin{cases} 3x_1 + 6x_2 - 5x_3 = 10 \\ 13x_1 - 2x_2 - 3x_3 = 0 \end{cases}$
Using matrix notation we see that

$$\begin{pmatrix} 3 & 6 & -5 & 10 \\ 13 & -2 & -3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -\frac{5}{3} & \frac{10}{3} \\ 13 & -2 & -3 & 0 \end{pmatrix} \to$$

$$\begin{pmatrix} 1 & 2 & -\frac{5}{3} & \frac{10}{3} \\ 0 & -28 & \frac{56}{3} & -\frac{130}{3} \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -\frac{5}{3} & \frac{10}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{65}{42} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{5}{21} \\ 0 & 1 & -\frac{2}{3} & \frac{65}{42} \end{pmatrix}.$$

So $x_1 = \frac{1}{3}x_3 + \frac{5}{21}$, $x_2 = \frac{2}{3}x_3 + \frac{65}{42}$ or $(x_1, x_2, x_3) = (\frac{5}{21}, \frac{65}{42}, 0) + \frac{1}{3}x_3(1, 2, 3)$. Once again it turned out that the intersection is a straight line parallel to the two lines found before.

The students should prove that the three lines are disjoint knowing that either the two lines coincide or they are disjont (we know that they are parallel). **This is your homework!**

We shall consider one more system. Namely

$$\begin{cases} 3x_1 + 2x_2 + 3x_3 + 4x_4 = 8 \\ x_1 + x_2 + x_3 + 2x_4 = 8 \\ 5x_1 + 3x_2 + 6x_3 + 3x_4 = 9 \end{cases}$$

The matrix representing the system is $\begin{pmatrix} 3 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 8 \\ 5 & 3 & 6 & 3 & 9 \end{pmatrix}$. We are going to simplify the matrix by managing the rows of it, as we usually do.

$$\begin{pmatrix} 3 & 2 & 3 & 4 & 8 \\ 1 & 1 & 1 & 2 & 8 \\ 5 & 3 & 6 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 & 8 \\ 3 & 2 & 3 & 4 & 8 \\ 5 & 3 & 6 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 & 8 \\ 0 & -1 & 0 & -2 & -16 \\ 0 & -2 & 1 & -7 & -31 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & -2 & 1 & -7 & -31 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -3 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 1 & 0 & 5 & 7 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 3 & -9 \\ 0 & 1 & 0 & 2 & 16 \\ 0 & 0 & 1 & -3 & 1 \end{pmatrix}.$$

The first two were swapped, the second row was multiplied by -1, the second row was multiplied by 2 and added to the third one, the third row was subtracted from the first one, the second row was subtracted from the first one. A the end we the matrix in the reduced echelon form wich ends our actions. The result may be written in the following way: $x_1 = -3x_4 - 9$, $x_2 = -2x_4 + 16$, $x_3 = 3x_4 + 1$. x_4 may be an arbitrary number (it is a free variable), once we set the value of x_4 the values of x_1, x_2, x_3 are uniquely defined. One may say that the solutions of the system are points of $\mathbb{R}^4$ (four dimensional space) of the form $(-3x_4 - 9, -2x_4 + 16, 3x_4 + 1, x_4) = (-9, 16, 1, 0) + x_4(-3, -2, 3, 1)$ so they form a straight line in the four dimensioanal space. The line contains the point $(-9, 16, 1, 0)$ and it is parallel to the vector $(-3, -2, 3, 1)$.