Let us consider the sequence 1, 1, 2, 3, 5, 8, 13, 21, ... Let us denote $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$,... We evaluate the next term of the sequence adding its two predecessors i.e. we can write that $F_{n+2} = F_{n+1} + F_n$. The sequence is named after Fibonacci (Leonardo of Pisa), an Italian mathematician born in twelve century. Fibonacci considered population of rabbits. The statement of the problem was given in the following way.

There is a pair of young rabbits a male and a female at the beginning. After a month they decide to do what many mammals like to do (even some catholic priests who declare that they do not do it). After the pregnancy the female rabbit gives birth to a new pair of rabbits (it might happen theoretically). New pair is too young for doing anything but the old ones repeat their action. So after a month we have two pairs of rabbits new born and the old one. After two months these two pairs still live but there is a third pair. New born do not do anything but the adults ... So after the third month the old rabbit females give birth to the next pairs of rabbits. So there are now 5 pairs of rabbits. The situation repeats. New born are new born but the adults behave in a standard way. As a result after one more month new little rabbits try to see the world. The hypothesis is that each female gives birth to a new pair of rabbits one female and one male.

This is how Fibonacci described this sequence. The question that appeared in his book *Liber Abaci* was: how many pairs of rabbits be after one year. So it was quite easy even at that timeat least for educated people. But mathematicians want a formula for F_n so that they could evaluate e.g. $F_{10\,000}$ without evaluating all previous terms of the sequence.

One may at first look at the set V of all possible sequences $a_1, a_2, a_3, \ldots$ for which

$$a_{n+2} = a_{n+1} + a_n. \tag{1}$$

We do not assume anything about the initial terms.

We can start e.g. from 5 and -2, the sequence now is $5, -2, 3, 1, 4, 5, 9, 14, \ldots$ If we multiply this sequence by a number the new one will also satisfy the same equation, e.g. we may use -7 to obtain $-35, 14, -21, -7, -28, -35, -63, -98, \ldots$ If we have 2 sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ that satisfy the equation (1) then their sum $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ also does.

The sequences are entirely determined by their first two terms and the equation that does not change. All this means that the set of all these sequences is a vector (linear) space. It is not very strange that one may identify V with a plane which consists of pairs of real numbers. There arises a question: are there among these sequences some of types known e.g. from schools? Arithmetical? Geometrical? Let us check it.

Let $a_n = a_{n-1} + d$ (an arithmetical sequence is considered). By the equation (1) we have

$$a_{n+1} + d = a_n + d + a_{n-1} + d \text{ for } n = 1, 2, 3, \ldots$$

i.e. $a_1 + (n+1)d = a_1 + nd + a_1 + (n-1)d$. Simplify the equation to get $0 = a_1 + (n-2)d$. The LHS does not depend on n so the RHS does not to. This means that $d = 0$ and this implies that $a_1 = 0$. We have proved that the only arithmetic sequence that satisfies the equation consists of zeros only. Not a very interesting sequence.

Let us think now of a geometric sequence: $a_n = a_1 q^{n-1}$. The equation is $a_1 q^{n+1} = a_1 q^n + a_1 q^{n-1}$ and it should hold for $n = 1, 2, 3, \ldots$. Let us assume that $a_1 \neq 0 \neq q$ and simplify the equation. We get $q^2 = q + 1$ or $q^2 - q - 1 = 0$ so either $q = \frac{1}{2}(1 - \sqrt{5}\,)$ or $q = \frac{1}{2}(1 + \sqrt{5}\,)$. We proved that only two ratios are allowed and now we know that all sequences of the form

$$c_1 \left(\frac{1}{2}(1 - \sqrt{5}\,)\right)^{n-1} + c_2 \left(\frac{1}{2}(1 + \sqrt{5}\,)\right)^{n-1} \tag{2}$$

satisfy the equation $a_{n+2} = a_{n+1} + a_n$. Is that all? The answer is YES and the proof is quite easy. Assume that the numbers a_1, a_2 are given . Are there numbers c_1, c_2 such that $a_1 = c_1 + c_2$ and $a_2 = c_1 \cdot \frac{1}{2}(1 - \sqrt{5}\,) + c_2 \cdot \frac{1}{2}(1 + \sqrt{5}\,)$? We obtained the system of two linear equations with the two unknowns c_1, c_2. The matrix of the system is $\begin{pmatrix} 1 & 1 & a_1 \\ \frac{1}{2}(1-\sqrt{5}) & \frac{1}{2}(1+\sqrt{5}) & a_2 \end{pmatrix}$. Subtract the first row multiplied by $\frac{1}{2}(1 - \sqrt{5})$ from the second one to obtain $\begin{pmatrix} 1 & 1 & a_1 \\ 0 & \sqrt{5} & a_2 - a_1 \cdot \frac{1}{2}(1-\sqrt{5}) \end{pmatrix}$. Therefore $c_2 = \frac{1}{\sqrt{5}} \left(a_2 - a_1 \cdot \frac{1}{2}(1 - \sqrt{5})\right)$ and $c_1 = a_1 - \frac{1}{\sqrt{5}} \left(a_2 - a_1 \cdot \frac{1}{2}(1 - \sqrt{5})\right)$. This means that we can write any sequence satisfying the equation $a_{n+2} = a_{n+1} + a_n$ in the form (2). If $a_1 = a_2 = 1$ then

$$c_1 = 1 - \frac{1}{\sqrt{5}} \left(1 - \frac{1}{2}(1 - \sqrt{5})\right) = 1 - \frac{1}{\sqrt{5}} \cdot \frac{1}{2}(1 + \sqrt{5}) = \frac{1}{2} - \frac{1}{2\sqrt{5}} = -\frac{1}{2\sqrt{5}}(1 - \sqrt{5})$$

and

$$c_2 = \frac{1}{\sqrt{5}} \left(1 - 1 \cdot \frac{1}{2}(1 - \sqrt{5})\right) = \frac{1}{2\sqrt{5}}(1 + \sqrt{5}).$$

This implies that in this case we obtain $a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1}{2}(1 + \sqrt{5}\,)\right)^n - \left(\frac{1}{2}(1 - \sqrt{5}\,)\right)^n \right)$. We proved that $F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1}{2}(1 + \sqrt{5}\,)\right)^n - \left(\frac{1}{2}(1 - \sqrt{5}\,)\right)^n \right)$. This is called Binet's formula. It is a standard situation: the theorem bears the name of the person who proved it but this person was not the first to prove it, because it was known before his birth. It was discovered in the XVIII century more than 500 years after the question had been asked. The problem was to apply a correct method and took a lot of time to invent it. With a correct approach it is not hard. The main difficulty is to try to solve it in an appropriate way. Looking at a vector space of all sequences was not obvious at all. It became standard later on, much later. Once one looks at the problem this way the question of choosing a system of coordinates in the plane becomes natural. It is equivalent to looking for a basis of the linear space. In the discussed case it consisted of geometrical sequences.

One may notice that the vectors $[1, \frac{1}{2}(1 - \sqrt{5})]$ and $[1, \frac{1}{2}(1 + \sqrt{5})]$ are perpendicular but this was not used in the solution of the problem.

Another comment is. How could one guess the formula (2) without having some idea not necessarily an obvious one?