LINEAR ALGEBRA

Eigenvalues and eigenvectors additional informations

If you notice a mistake let me know so I will be able to remove it.

Last changes made January 20-th, 9:30

There are few more facts about eigenvalues and eigenvectors you hear of. Let me recall the definition once again.If

$$(eig) \qquad M\mathbf{v} = \lambda\mathbf{v} \quad \text{and} \quad \mathbf{v} \neq (0, 0, \dots, 0)$$

then λ is called an eigenvalue of M and $\mathbf{v}$ is called an eigenvector associated to λ.

Easy corollaries are. If λ is an eigenvalue of M then 5λ is an eigenvalue of $5M$ with the same set of the eigenvectors associated to the eigenvalue: $M\mathbf{v} = \lambda\mathbf{v} \Rightarrow (5M)\mathbf{v} = 5(M\mathbf{v}) = 5(\lambda\mathbf{v}) = (5\lambda)\mathbf{v}$. The number 5 may be replaced with with an arbitrary real number.

If λ is an eigenvalue of M then λ^2 is an eigenvalue of the matrix $M^2 = M \cdot M$ and the eigenvectors of M associated to λ are the eigenvectors of M^2 associated to λ^2: $M \cdot M \cdot \mathbf{v} = M(\lambda\mathbf{v}) = \lambda M\mathbf{v} = \lambda \cdot \lambda \cdot \mathbf{v} = \lambda^2\mathbf{v}$. Let $M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The characteristic equation is $0 = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} =$ $= (1-\lambda)(\lambda^2+1)$ so the only eigenvalue is 1 and it is a single eigenvalue. The eigenspace associated to it is $\operatorname{lin}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$. $M^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The eigenvalues of M^2 are 1 and a double eigenvalue -1. $M^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$. Therefore M^4 has a unique eigenvalue namely 1 and this eigenvalue is a triple eigenvalue. The eigenspace associated to it is the whole $\mathbb{R}^3$. In fact the map $\mathbf{x} \longrightarrow M\mathbf{x}$ is the rotation abou the z-axis by $\frac{\pi}{2}$ radians or equivalently by 90°. Therefore the points fromn z–axis do not move under the map which means that the vectors of the form $\left(\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}\right)$ are eigenvectors associated to 1. It is also clear that the map $\mathbf{x} \longrightarrow M^2\mathbf{x}$ is the rotation about the z–axis by π radians or equivalently by 180° or it is the symmetry with respect to the z–axis. This implies right away that the z–axis is the eigenspace associated to 1 while the plane defined bhy the equation $z = 0$ is the eigenspace associated to -1. Also it follows immediately that $M^4 = I$ is the identity map so the eigenspace associated to 1 is the whole $\mathbb{R}^3$. As everybody sees the eigenspace for the same eigenvalue of the map may be smaller then that of its square (in the example above the eigenspace associated to 1 was one–dimenskional for M^2 while it was three-dimensional for M^4.

If C is an invertible matrix then

$$|M - \lambda I| = |CC^{-1}| \cdot |M - \lambda I| = |C| \cdot |M - \lambda IJ| \cdot |C^{-1}| = |C(M - \lambda I)C^{-1}| = |CMC^{-1} - \lambda I|.$$

This proves that the characterstic polynomials of M and of CMC^{-1} are equal. If $n\times n$ matrix M has n real eigenvalues (counted with thei multiplicities) then there exists a matrix C such that the matrix CMC^{-1} is upper–triangular (below the main diagonal there are zeros only) – this is an important theorem with quite long proof. The entries on the main diagonal of CMC^{-1} are the eigenvalues of both M and CMC^{-1}. Let they be $\lambda_1,\lambda_2,\ldots,]\lambda_n$. Then the characteristic polynomial may be written as $(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)\ldots(\lambda_n-\lambda)$. This implies that

$$\lambda_1\lambda_2\lambda_3\ldots\lambda_n=|CMC^{-1}|=|M|$$

so the determinant of the magtrix equals to the product of its all eigenvalues provided there are n eigenvalues. Another corollary is that the sum $\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n$ is the coefficient in front of $(-\lambda)^{n-1}$ in the characteristic polynomial. It is not hard to see that if $M=(a_{ij})_{1\le i,l,\le n}$ the this coefficient equals to the $(-1)^{n-1}(a_{11}+a_{22}+a_{33}+\ldots+a_{nn})$ so

$$\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n=a_{11}+a_{22}+a_{33}+\ldots+a_{nn}.$$

The quantity $\mathrm{tr(M)=a_{11}+a_{22}+a_{33}+\ldots+a_{nn}}$ is called the trace of the matrix and as shown above is independent of the basis chosen: $\mathrm{tr(M)=tr(CMC^{-1})}$. This is important theorem for many reasons. One, not the most important, is that when a student computes the eigenvalues and the sum of the obtained number differs from the trace of the matrix the person knows that an error was made.

Let us look into the January 14 test problems.

1. Let $A(t)=\begin{pmatrix}2-t & t & 0\\ -t & t+2 & 0\\ -t & t & 2\end{pmatrix}$.

Find the eigenvalues of the matrix $A(t)$ and the eigenspaces associated to them.

For what $t\in\mathbb{R}$ there exists a basis of $\mathbb{R}^3$ consisting of the eigenvectors of $A(t)$.

Find a basis B of $\mathbb{R}^3$ such that $M(A(1))_B^B=\begin{pmatrix}2&1&0\\0&2&0\\0&0&2\end{pmatrix}$ or prove that such a basis does not exist.

Solution. The characteristic equation is $0=\begin{vmatrix}2-t-\lambda & t & 0\\ -t & t+2-\lambda & 0\\ -t & t & 2-\lambda\end{vmatrix}=$

$=(2-\lambda)\begin{vmatrix}2-t-\lambda & t\\ -t & t+2-\lambda\end{vmatrix}=(2-\lambda)\Big((2-t-\lambda)(t+2-\lambda)+t^2\Big)=$

$=(2-\lambda)\Big((2-\lambda-t)(2-\lambda+t)+t^2\Big)=(2-\lambda)\Big((2-\lambda)^2-t^2+t^2\Big)=(2-\lambda)^3.$

This implies that 2 is a triple eigenvalue of the matrix. By the theorems known from prof. Kędzierski's class we know that the dimension of the eigenspace associated to the eigenvalue may be either 1 or 2 or 3. Let us write the equation for the eigenvectors.

$$\begin{pmatrix} 2-t & t & 0 \\ -t & t+2 & 0 \\ -t & t & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \begin{pmatrix} -t & t & 0 \\ -t & t & 0 \\ -t & t & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are two different cases. If $t = 0$ the the equation is satisfied by all vectors in $\mathbb{R}^3$ so the eigenspace is the whole space $\mathbb{R}^3$. Obviously in this case there a basis consisting of the eigenvectors e.g. the standard basis of $\mathbb{R}^3$ or any other basis of $\mathbb{R}^3$. A different situation is for $t \neq 0$. Now there is one equation for the eigenvectors $-tx + ty = 0$ equivalent to $x - y = 0$. It is an equation of a (two–dimensional) plane in $\mathbb{R}^3$. Let $w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. This shows that each eigenvector is of the form $c_1 w_1 + c_2 w_2$ with $c_1, c_2 \in \mathbb{R}$.

Let us look at the last question. $A(1) = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$. Lets us take a vector from outside of the eigenspace e.g. $w_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We see that

$$A(1)w_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = 2w_3 + w_4 \text{ with}$$

$w_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$. Clearly w_4 is an eigenvector of $A(1)$: $A(1)w_4 = 2w_4$. Set $B = (w_4, w_3, w_1)$. We have $A(1)w_4 = 2w_4$, $A(1)w_3 = 2w_3 + w_4$, and $A(1)w_1 = 2w_1$ as required in the third part of the problem.

2. Let $V = \text{lin}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ with $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -7 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \\ -3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Find numbers a, b, c, d such that $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \quad ax_1 + bx_2 + cx_3 + dx_4 = 0 \right\}$.

Find a point $\mathbf{q}$ symmetric to the point $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the subspace V and the orthogonal projection $\mathbf{r}$ of $\mathbf{p}$ onto V. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Solution. We want to find numbers a, b, c, d such that $a+2b+4c-7d = 0$, $2a+2b-c-3d = 0$ and $-3a+b+c+d = 0$. Apply the row reduction process to the matrix $\begin{pmatrix} 1 & 2 & 4 & -7 \\ 2 & 2 & -1 & -3 \\ -3 & 1 & 1 & 1 \end{pmatrix} \to$

$$\to \begin{pmatrix} 1 & 2 & 4 & -7 \\ 0 & -2 & -9 & 11 \\ 0 & 7 & 13 & -20 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & -7 \\ 0 & -2 & -9 & 11 \\ 0 & 0 & -37 & 37 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & -7 \\ 0 & -2 & -9 & 11 \\ 0 & 0 & -1 & 1 \end{pmatrix} \to$$

$\to \begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ so $a = b = c = d$, e.g. $a = 1$ and the equation is $x_1 + x_2 + x_3 + x_4 = 0$. The vector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is perpendicular to the three–dimensional subspace V of $\mathbb{R}^3$, because $\mathbf{v} \cdot \mathbf{v}_1 = 0$, $\mathbf{v} \cdot \mathbf{v}_2 = 0$ and $\mathbf{v} \cdot \mathbf{v}_3 = 0$. Let us project $\mathbf{p}$ onto $V^\perp = \mathrm{lin}(\mathbf{v})$. We obtain $\frac{\mathbf{p}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$. Therefore $\mathbf{r} = \mathbf{p} - \frac{\mathbf{p}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} - \frac{10}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$ is the projection of $\mathbf{p}$ onto V. The point $\mathbf{q}$ symmetric to the point $\mathbf{p}$ with respect to V is $\mathbf{q} = \mathbf{p} + 2(\mathbf{r} - \mathbf{p}) = 2\mathbf{r} - \mathbf{p} = \begin{pmatrix} -4 \\ -3 \\ -2 \\ -1 \end{pmatrix}$. The end. □

1. Let $A(t) = \begin{pmatrix} t+3 & t+1 & 0 \\ -t-1 & 1-t & t+1 \\ 0 & 0 & 2 \end{pmatrix}$.

Find the eigenvalues of the matrix $A(t)$ and the eigenspaces associated to them.
For what $t \in \mathbb{R}$ there exists a basis of $\mathbb{R}^3$ consisting of the eigenvectors.
Find a basis B of $\mathbb{R}^3$ such that $M(A(0))_B^B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ or prove that such a basis does not exist.

Solution. The characteristic equation is $0 = \begin{vmatrix} t+3-\lambda & t+1 & 0 \\ -t-1 & 1-t-\lambda & t+1 \\ 0 & 0 & 2-\lambda \end{vmatrix} =$

$= (2-\lambda)\begin{vmatrix} t+3-\lambda & t+1 \\ -t-1 & 1-t-\lambda \end{vmatrix} = (2-\lambda)\Big((t+3-\lambda)(1-t-\lambda)+(t+1)^2\Big) =$

$= (2-\lambda)\Big(\lambda^2-\lambda(t+3+1-t)+(t+3)(1-t)+(t+1)^2\Big) = (2-\lambda)\Big(\lambda^2-4\lambda+4\Big) = (2-\lambda)^3$. The number 2 is a triple eigevalue of $A(t)$. The dimension of the dorresponding eigenspace is either 1 or 2 or 3. The equation for the eigenvectors is $\begin{pmatrix} t+1 & t+1 & 0 \\ -t-1 & -1-t & t+1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

If $t+1=0$ it is satisfied for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$. If $t \neq -1$ then $x+y=0$ and $-x-y+z=0$ so $y=-x$ and $z=0$. In this case the eigenspace is one–dimensional. It is spanned by the vector $w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$. The basis of $\mathbb{R}^3$ consisting of the eigenvectors exists only for $t=-1$.

$A(0) = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Suppose that a basis B exists. Let $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. We see that $J\mathbf{e}_1 = 2\mathbf{e}_1$, $J\mathbf{e}_2 = 2\mathbf{e}_2 + \mathbf{e}_1$ and $J\mathbf{e}_3 = 2\mathbf{e}_3 + \mathbf{e}_2$. Let us try to find a vector w_2 such that $A(0)w_2 = 2w_2 + w_1$. A vector $w_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = w_1$ will be useful. This implies that $x+y=1$ and $-x-y+z=-1$ so $z=0$ and $x=1-y$. Let $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Now we need a vector w_3 such that $A(0)w_3 = 2w_3 + w_2$ i.e. we want to solve for $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the equation $\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = w_2$. This implies that $x+y=1$ and $-x-y+z=0$ so $z=1$. Let $w_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The (ordered) basis B exists for example $B = (w_1, w_2, w_3)$. □

2. Let $V = \mathrm{lin}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Find numbers a, b, c, d such that $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \quad ax_1 + bx_2 + cx_3 + dx_4 = 0 \right\}$.

Find a point $\mathbf{q}$ symmetric to the point $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the subspace V and the orthogonal projection $\mathbf{r}$ of $\mathbf{p}$ onto V.

Solution. It is easy to see that $\mathbf{v}_1 + \mathbf{v}_3 = 2\mathbf{v}_2$. The vectors are linearly **dependent.** The vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, just apply the definition to prove it. This implies that $V = \mathrm{lin}(\mathbf{v}_1, \mathbf{v}_2)$. Therefore V is two–dimensional. Therefore it cannot be described as required, one non–trivial linear equation in $\mathbb{R}^4$ describes a three–dimensional subspace of $\mathbb{R}^4$. There are ininitely many quadruplets a, b, c, d such that $ax_1 + bx_2 + cx_3 + dx_4 = 0$ and $a + b + c + d = 1$ for all points of V but in each such case the equation is satisfied by many points from outside of V. Let us find all $(a, b, c, d) \neq (0, 0, 0, 0)$ for which $ax_1 + bx_2 + cx_3 + dx_4 = 0$ for all $\mathbf{x} \in V$. We shall apply row reduction to the matrix $\begin{pmatrix} 1 & -1 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to$ $\to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{pmatrix}$. This implies that $b = d$ and $a = -c - 2d$. As we said there many possibilities. Let $d = -1$ and $c = 1$. Then $b = -1$ and $a = 1$. The equation is $x_1 - x_2 + x_3 - x_4 = 0$. There are many more non–equivalent to the above. For example let $d = 1$ and $c = 0$. Then $b = 1$ and $a = -2$. We obtain the equation $-2x_1 + x_2 + x_4 = 0$. If $x_1 = x_2 = 0$ and $x_3 = 1 = x_4$ then the first equation is satisfied while the second is not so they are not equivalent. Now we shall find the projection of p onto V. This means that we shall find numbers c_1, c_2 such that the vector $\mathbf{p} - (c_1\mathbf{v}_1 + c_2 vg_2)$ will be orthogonal to V or it will be orthogonal to both vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. We want the equations $0 = \mathbf{v}_1 \cdot (\mathbf{p} - (c_1\mathbf{v}_1 + c_2\mathbf{v}_2))$ and $0 = \mathbf{v}_2 \cdot (\mathbf{p} - (c_1\mathbf{v}_1 + c_2\mathbf{v}_2))$ to be satisfied. We have $\mathbf{v}_1 \cdot \mathbf{v}_1 = 12$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 8$ and $\mathbf{v}_2 \cdot \mathbf{v}_2 = 6$. Also $\mathbf{p} \cdot \mathbf{v}_1 = 14$ and $\mathbf{p} \cdot \mathbf{v}_2 = 12$. The equations take form $12c_1 + 8c_2 = 14$ and $8c_1 + 6c_2 = 12$ or $12c_1 + 8c_2 = 14$ and $12c_1 + 9c_2 = 18$. Subtract them to get $c_2 = 4$ and then $c_1 = -\frac{3}{2}$. We may write $\mathbf{r} = -\frac{3}{2}\mathbf{v}_1 + 4\mathbf{v}_2 = \frac{1}{2}\begin{pmatrix} 5 \\ 3 \\ 5 \\ 7 \end{pmatrix}$.

Then $\mathbf{q} = \mathbf{p} + 2(\mathbf{r} - \mathbf{p}) = 2\mathbf{r} - \mathbf{p} = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. □

1. Let $A(t) = \begin{pmatrix} t+4 & t+1 & 0 \\ -t-1 & 2-t & 0 \\ t+1 & t+1 & 3 \end{pmatrix}$.

Find the eigenvalues of the matrix $A(t)$ and the eigenspaces associated to them.
For what $t \in \mathbb{R}$ there exists a basis of $\mathbb{R}^3$ consisting of the eigenvectors.
Find a basis B of $\mathbb{R}^3$ such that $M(A(0))_B^B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ or prove that such a basis does not exist.

Solution. Solve the characteristic equation $0 = \begin{vmatrix} t+4-\lambda & t+1 & 0 \\ -t-1 & 2-t-\lambda & 0 \\ t+1 & t+1 & 3-\lambda \end{vmatrix} =$
$= (3-\lambda)\begin{vmatrix} t+4-\lambda & t+1 \\ -t-1 & 2-t-\lambda \end{vmatrix} = (3-\lambda)\Big(\lambda^2 - \lambda(t+4+2-t) + (t+4)(2-t) + (t+1)^2\Big) =$
$= (3-\lambda)(\lambda^2 - 6\lambda + 9) = (3-\lambda)^3$. The number 3 is a triple eigevalue of $A(t)$. The dimension of the corresponding eigenspace is either 1 or 2 or 3. The equation for the eigenvectors is $\begin{pmatrix} t+1 & t+1 & 0 \\ -t-1 & -1-t & 0 \\ t+1 & t+1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. For $t+1 = 0$ the eigenspace consists of all points of $\mathbb{R}^3$, so in this case there is a basis consisting of the eigenvectors (any basis of $\mathbb{R}^3$). If $t \neq -1$ then the eigenspace is described by $x + y = 0$ so it is a two-dimensional subspace of $\mathbb{R}^3$ so the basis consisting of the eigenvectors does not exist (three linearly independent eigenvectors should be chosen but it is impossible). $A(0) = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ Let $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. It is NOT an eigenvector.

We have $\begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Let $w_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$,
$w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. One has $A(0)w_1 = 3w_1$, $A(0)w_2 = 3w_2 + w_1$ and $A(0)w_3 = 3w_3$ so the basis B exists, e.g. $B = (w_1, w_2, w_3)$. The solution is now complete. □

2. Let $V = \mathrm{lin}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$.

Find numbers a, b, c, d such that $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \quad ax_1 + bx_2 + cx_3 + dx_4 = 0 \right\}$.

Find a point $\mathbf{q}$ symmetric to the point $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the subspace V and the orthogonal projection $\mathbf{r}$ of $\mathbf{p}$ onto V.

Solution. It is easy to see that $\mathbf{v}_2 + \mathbf{v}_3 = 2\mathbf{v}_1$. The vectors are linearly **dependent.** The vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent, just apply the definition to prove it. This implies that $V = \mathrm{lin}(\mathbf{v}_1, \mathbf{v}_2)$. Therefore V is two–dimensional. Therefore it cannot be described as required, one non–trivial linear equation in $\mathbb{R}^4$ describes a three–dimensional subspace of $\mathbb{R}^4$. There are ininitely many quadruplets a, b, c, d such that $ax_1 + bx_2 + cx_3 + dx_4 = 0$ and $a + b + c + d = 1$ for all points of V but in each such case the equation is satisfied by many points from outside of V. Let us find all $(a, b, c, d) \neq (0, 0, 0, 0)$ for which $ax_1 + bx_2 + cx_3 + dx_4 = 0$ for all $\mathbf{x} \in V$. We shall apply row reduction to the matrix $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$. This implies that $b = -2d$ and $a = -c - d$. As we said there many possibilities. Let $d = -1 = c$. Then $b = 2 = a$. The equation is $2x_1 + 2x_2 - x_3 - x_4 = 0$. There are many more non–equivalent to the above. For example let $d = 1$ and $c = 0$. We have now $b = -2$ and $a = -1$. The equation is $-x_1 - 2x_2 + x_4 = 0$. It is not equivalent to the previous one. Check it by yourself, please.

1. Let $A(t) = \begin{pmatrix} t+4 & t+1 & 0 \\ -t-1 & 2-t & t+1 \\ 0 & 0 & 3 \end{pmatrix}$.

Find the eigenvalues of the matrix $A(t)$ and the eigenspaces associated to them.
For what $t \in \mathbb{R}$ there exists a basis of $\mathbb{R}^3$ consisting of the eigenvectors of $A(t)$.

Find a basis B of $\mathbb{R}^3$ such that $M(A(0))_B^B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ or prove that such a basis does not exist.

Solution. We start with the characteristic equation $0 = \begin{vmatrix} t+4-\lambda & t+1 & 0 \\ -t-1 & 2-t-\lambda & t+1 \\ 0 & 0 & 3-\lambda \end{vmatrix} =$

$$=(3-\lambda)\begin{vmatrix} t+4-\lambda & t+1 \\ -t-1 & 2-t-\lambda \end{vmatrix} = (3-\lambda)\Big((t+4-\lambda)(2-t-\lambda)+(t+1)^2\Big) =$$

$= (3-\lambda)\Big(\lambda^2-\lambda(t+4+2-t)+(t+4)(2-t)+(t+1)^2\Big)(3-\lambda)(\lambda^2-6t+9) = (3-\lambda)^2$. The number 3 is a triple eigenvalue of the matrix. The dimension of the associated eigenspace is either 1 or 2 or 3. The equation for the eigenvectors is $\begin{pmatrix} t+4 & t+1 & 0 \\ -t-1 & 2-t & t+1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Transform it to $\begin{pmatrix} t+1 & t+1 & 0 \\ -t-1 & -1-t & t+1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. It $t=-1$ then all elements of $\mathbb{R}^3$ are eigenvectors so the basis consisting of them exists. If $t \neq -1$ then the equations are $(t+1)x+(t+1)y = 0$ and $-(t+1)x-(t+1)y(t+1)z = 0$. The are equivalent to $x+y = 0$ and $-x-y+z = 0$. The eigenspace is described by the equations $x+y = 0$ and $z = 0$. Therefore its dimension is 1. So the basis made of the eigenvectors does not exist. We have $A(0) = \begin{pmatrix} 4 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Let $J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Both matrices have the same triple eigenvalue 3. The eigenspace associated to 3 in both cases has dimension 1. We may notice that $J\mathbf{e}_1 = 3\mathbf{e}_1$, $J\mathbf{e}_2 = 3\mathbf{e}_2+\mathbf{e}_1$ and $J\mathbf{e}_3 = 3\mathbf{e}_3+\mathbf{e}_2$. The question is whether or not there are such vectors for $A(0)$. There is not much choice for the eigenvector. Let $w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Does there exist a vector w_2 such that $A(0)w_2 = 3w_2+w_1$? To answer the question we solve the (matrix) equation $(A(0)-3I)\mathbf{x} = w_1$. This can be written as follows $\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. It is equivalent to $x+y = 1$ and $x+y = 1$ and $-x-y+z = -1$. Therefore $z = 0$. We may set $x = 1$ so $y = 0$. Let $w_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. We see that $A(0)w_2 = 3w_2+w_1$. Next question is: does there exist a vector w_3 such that $A(0)w_3 = 3w_3+w_2$? We know that the question is equivalent to the system of equations $\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Without matrices it can be written as follows $x+y = 1$ and $-x-y+z = 0$. This implies that $z = -1$. Set $x = 1$ so

$y = 0$ and define $w_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. The required equation are satisfied . If $B = (w_1, w_2, w_3)$ then $M(A(0))_B^B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. □

2. Let $V = \mathrm{lin}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Find numbers a, b, c, d such that $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \quad ax_1 + bx_2 + cx_3 + dx_4 = 0 \right\}$.

Find a point $\mathbf{q}$ symmetric to the point $\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the subspace V and the orthogonal projection $\mathbf{r}$ of $\mathbf{p}$ onto V.

Solution. We want the following equations to be satisfied: $a+b+c+d = 0$, $2a+b+c-2d = 0$ and $a - b - c + d = 0$. We shall simplify the matrix of the system. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -2 \\ 1 & -1 & -1 & 1 \end{pmatrix} \to$

$\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -4 \\ 0 & -2 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -2 & 0 & -4 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. Therefore $a = -d$, $b = -2d$ and $c = -b = 2d$. Let $d = -1$. Then the equation is $x_1 + 2x_2 - 2x_3 - x_4 = 0$. We know now that V is a three–dimensional space orthogonal to the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix}$.

The orthogonal projection $\mathbf{r}$ of $\mathbf{p}$ onto V is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: $\mathbf{r} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Therefore $(\mathbf{p} - \mathbf{r}) \cdot \mathbf{v}_j = 0$ for $j = 1, 2, 3$. We have $\mathbf{p} \cdot \mathbf{v}_1 = 10$, $\mathbf{p} \cdot \mathbf{v}_2 = -5$, $\mathbf{p} \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_1 = 4$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 10$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$ and $\mathbf{v}_3 \cdot \mathbf{v}_3 = 4$. Therefore $10 = \mathbf{p} \cdot \mathbf{v}_1 = 4c_1$, $-5 = \mathbf{p} \cdot \mathbf{v}_2 = 10c_2$ and $0 = \mathbf{p} \cdot \mathbf{v}_3 = 4c_3$ so $c_1 = \frac{5}{2}$,

$c_2 = -\frac{1}{2}$ and $c_3 = 0$. Thus $\mathbf{r} = \frac{5}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 = \frac{1}{2}\begin{pmatrix} 3 \\ 6 \\ 4 \\ 7 \end{pmatrix}$. Thus $\mathbf{q} = \mathbf{p} + 2(\mathbf{r} - \mathbf{p}) = 2\mathbf{r} - \mathbf{p} = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 4 \end{pmatrix}$.

Over. □

Dear students, when you are working with a linear space you should know its dimension. The rank od the matrix is the maximal number of the linearly independent rows which is equal to the maximal number of the linearly independent columns which is equal to the maximal size of the determinant of a matrix obtained by crossing out several rows and several columns. It is much easier to do the job having the above in your mind.