Linear Algebra

Eigenvalues and eigenvectors

All students were asked to write down a definition of an eigenvalue and of an eigenvector of a matrix. The definition can be writtenas follows:

if M is an $n \times n$ matrix, $\lambda \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n$ are such that

$$(eig) \qquad M\mathbf{v} = \lambda\mathbf{v} \quad \textbf{and} \quad \mathbf{v} \neq (0,0,\ldots,0)$$

then λ is called an eigenvalue of M and $\mathbf{v}$ is called an eigenvector associated to λ. Many people forgot of writing $\mathbf{v} \neq (0,0,\ldots,0)$. It is a serious error. This happened in many papers in which the authors tried to avoid the formula (eig) writing it using many words. There is no reason for avoiding math formulas when studying math.

Some people used names eigenfunction and refered to differential equation. Formally speaking you were not asked of so called differential operators so this was not an answer to my question (although I accepted it).

The definition of an eigenvalue and an eigenvector of a linear map $\varphi \colon V \to V$ is the same: a number λ is an eigenvalue if φ if and only if there exists a non-zero vector $\mathbf{v} \in V$ such that $\varphi(\mathbf{v}) = \lambda\mathbf{v}$. The vector $\mathbf{v}$ is called an eigenvector associated to λ.

If the dimension V is a finite , A is a basis of V then the eigenvalues of the map φ and of the matrix $M(\varphi)_A^A$ are the same (this follows from the theorems proved by prof. Kędzierski at his classes). The basis in V regarded as a domain of φ and the basis of V regarded as a range of φ must coincide because otherwise if φ is one to one map, $A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of V and $B = \{\varphi(\mathbf{v}_1), \varphi(\mathbf{v}_2), \ldots \varphi(\mathbf{v}_n)\}$ then $M(\varphi)_A^B$ is an identity matrix.

The eigenvalues of a matrix M are roots of the characteristic polynomial in λ of M: $|M - \lambda I|$ or equivalently of the equation $|M - \lambda I| = 0$. Let us consider three matrices

$$M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

It is easy to see that in the three cases the characteristic equation is $(3-\lambda)^3 = 0$. This proves that each of the matrices M_1, M_2, M_3 has one (triple) eigenvalue 3.

Let us find eigenvectors associated to M_1. The equation is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Clearly all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfy it. This means that each non–zero vector of $\mathbb{R}^3$ is an eigenvector associated with 3.

Now consider the matrix M_2. The equation is $3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 3x_2 \\ 3x_3 \end{pmatrix}$.

This implies that $3x_1 = 3x_1 + x_2$ so $x_2 = 0$. Obviously $3\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$. This means that all vectors of the form $\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$ with $x_1 \neq 0$ or $x_2 \neq 0$ are eigenvectors associated to 3. In this case the eigenspace associated to 3 is given by the equation $x_2 = 0$ so its dimension is 2.

In the third case the equation is $3\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 3x_2 + x_3 \\ 3x_3 \end{pmatrix}$. This implies that $x_2 = 0$ and $x_3 = 0$. It is easy to see that $3\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$ so in this case the eigenspace associated to 3 is given by the equations $x_2 = 0$ and $x_3 = 0$ slo its dimension is 1. This is the simplest example showing that the eigenspace associated to the eigenvalue may be of any dimension greater than 0 and not bigger than the multiplicity of the eigenvalue.

100. Let $0 < \alpha < 2\pi$. $M = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$. Prove that the cosine of the angle between a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\cos\alpha$. For what $\alpha \in (0,2)$ the matrix M has a real eigenvalue?

Solution. The characteristic equation is $0 = \begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{vmatrix} = (\cos\alpha - \lambda)^2 + \sin^2\alpha$. This implies that $\sin\alpha = 0$ and $\cos\alpha - \lambda = 0$ – the sum of squares of real numbers is 0 if and only if all squared numbers are zeroes. Therefore there exists an integer n such that $\alpha = n\pi$. Then $\cos\alpha = (-1)^n$ so the only candidates for the eigenvalues are ± 1. If n is even then necessarilly $\lambda = 1$ and the matrics becomes $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so the number 1 becomes a double eigenvalue and the corresponding eigenspace is the whole plane ($\mathbb{R}^2$). If n is odd then the matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ so the number -1 becomes a double eigenvalue and the corresponding eigenspace is the whole plane ($\mathbb{R}^2$).

The length of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ equals $\sqrt{x^2 + y^2}$. The length of the vector

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix}$$

equals $\sqrt{(x\cos\alpha - y\sin\alpha)^2 + (x\sin\alpha + y\cos\alpha)^2} = \sqrt{x^2 + y^2}$. The scalar product of these vectors is equal to the product of the lengths of the vectors and of the cosine of the angle they make. This scalar product is $x(x\cos\alpha - y\sin\alpha) + y(x\sin\alpha + y\cos\alpha) = (x^2 + y^2)\cos\alpha$ so the cosine of the angle is $\frac{(x^2+y^2)\cos\alpha}{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}} = \cos\alpha$. This shows that unless $\cos\alpha = \pm 1$ or $x = 0 = y$ the two vectors are not parallel so the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is not an eigenvector corresponding to **a real** eigenvalue of M. □

101. Let a sequence $a_1, a_2, a_3, \ldots$ be defined by the formulae $a_{n+2} = -2a_{n+1} + 2a_n$, $a_1 = 1$ and $a_2 = 4$. Check that the linear map defined by the matrix $M = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}$ assigns the vector $\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix}$ to the vector $\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$. Find the eigenvalues and the eigenvectors of M and the explicit formulae for the matrix M^n and a_n.

Solution. We have $\begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ 2a_n - 2a_{n+1} \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix}$ by the definition of this sequence. The characteristic equation is $0 = \begin{vmatrix} 0-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = \lambda(\lambda+2) - 2 = $ $= (\lambda+1)^2 - 3$ so $\lambda_1 = -1+\sqrt{3}$ and $\lambda_2 = -1-\sqrt{3}$. If $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector associated to λ_1 then $\lambda_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2x_1 - 2x_2 \end{pmatrix}$ so $(-1+\sqrt{3})x_1 = x_2$ (the second equation is equivalent to ther first one), e.g. $x_1 = 1$ and $x_2 = \sqrt{3}-1$, so $\mathbf{v}_1 = \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix}$ is an eigenvector associated to $-1+\sqrt{3}$. In the same way we show that $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{3}-1 \end{pmatrix}$ is an eigenvector associated to $-1-\sqrt{3}$. Let $A = \{\mathbf{v}_1, \mathbf{v}_2\}$, $\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then

$M(\varphi)_A^A = \begin{pmatrix} \sqrt{3}-1 & 0 \\ 0 & -\sqrt{3}-1 \end{pmatrix}$, $M_A^{st} = \begin{pmatrix} 1 & 1 \\ \sqrt{3}-1 & -\sqrt{3}-1 \end{pmatrix}$ so $M_{st}^A = (M_A^{st})^{-1} =$

$= -\frac{1}{2\sqrt{3}} \begin{pmatrix} -1-\sqrt{3} & -1 \\ 1-\sqrt{3} & 1 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & 1 \\ \sqrt{3}-1 & -1 \end{pmatrix}$. Therefore

$M^n = (M(\varphi)_{st}^{st})^n = (M_A^{st}(M(\varphi)_A^A)M_{st}^A)^n = M_A^{st}(M(\varphi)_A^A)^n M_{st}^A =$

$= \begin{pmatrix} 1 & 1 \\ \sqrt{3}-1 & -\sqrt{3}-1 \end{pmatrix} \cdot \begin{pmatrix} (\sqrt{3}-1)^n & 0 \\ 0 & (-\sqrt{3}-1)^n \end{pmatrix} \cdot \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & 1 \\ \sqrt{3}-1 & -1 \end{pmatrix} =$

$= \begin{pmatrix} (\sqrt{3}-1)^n & (-\sqrt{3}-1)^n \\ (\sqrt{3}-1)^{n+1} & (-\sqrt{3}-1)^{n+1} \end{pmatrix} \cdot \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}+1 & 1 \\ \sqrt{3}-1 & -1 \end{pmatrix} =$

$= \frac{1}{2\sqrt{3}} \begin{pmatrix} 2(\sqrt{3}-1)^{n-1} - 2(-\sqrt{3}-1)^{n-1} & (\sqrt{3}-1)^n - (-\sqrt{3}-1)^n \\ 2(\sqrt{3}-1)^n - 2(-\sqrt{3}-1)^n & (\sqrt{3}-1)^{n+1} - (-\sqrt{3}-1)^{n+1} \end{pmatrix}$.

$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = M^{n-1} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} (2\sqrt{3}-1)(\sqrt{3}-1)^{n-2} + (2\sqrt{3}+1)(-\sqrt{3}-1)^{n-2} \\ (2\sqrt{3}-1)(\sqrt{3}-1)^{n-1} + (2\sqrt{3}+1)(-\sqrt{3}-1)^{n-1} \end{pmatrix}$. □

102. Let V be a set consisting of all functions in x of the form $w(x)e^x$ with $w(x) = a + bx + cx^2$, $a, b, c \in \mathbb{R}$. Prove that V is a linear space over $\mathbb{R}$. Prove that the dimension of V is 3. Find a basis of V as simple as you can. For $f \in V$ define $\varphi(f)(x) = f'(x) - 2f(x)$ and $\psi(f)(x) = f'(x) - f(x)$. Prove that $\varphi(f), \psi(f) \in V$ and that φ, ψ are linear maps . Find their eigenvalues and eigenvectors (in this case they are frequently called eigenfunctions because the elements of V are functions).

Solution. If the functions $w(x)e^x$ and $w_1(x)e^x$ are in V then there are $a, a_1, b, b_1, c, c_1 \in \mathbb{R}$ such that $w(x) = a + bx + cx^2$ and $w_1(x) = a_1 + b_1x + c_1x^2$ the for $\alpha, \beta \in \mathbb{R}$ we have $\alpha w(x)e^x + \beta w_1(x)e^x = ((a\alpha + a_1\beta) + (b\alpha + b_1\beta)x + (c\alpha + c_1\beta)x^2)\, e^x$. This means that a linear combination of functions from V is an element of V. This proves that V is a linear space. $dim(V) = 3$ because the functions $1 \cdot e^x$, xe^x and x^2e^x are linearly independent (a polynomial of degree at most 2 is equal to 0 if and only if all its coefficients are zeros) and each element of V is a linear combination of them. This means that $A = \{e^x, xe^x, x^2x^x\}$ is a basis of V. There are infinitely many of other bases of V. We were supposed to find one of them.

If $f(x) = w(x)e^x = (a + bx + cx^2)e^x$ then

$(\varphi(f))(x) = (w(x)e^x)' - 2w(x)e^x = (w'(x) - w(x))e^x = ((b - a) + (2c - b)x - cx^2)e^x$.

This proves $M(\varphi)_A^A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. There is one triple eigenvalue -1. An eigenspace associated to it is spanned by e^x so it consists of all functions of the form ae^x with $a \in \mathbb{R}$.

$(\psi(f))(x) = w'(x)f(x) = (b + 2cx)e^x$. Therefore $M(\psi)_A^A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. In this case there is one (triple) eigenvalue 0. As above the eigenspace consists of all functions of the form $ae^x, a \in \mathbb{R}$. $\square$

103. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. Let $V = \text{lin}(\mathbf{v}_1, \mathbf{v}_2)$ and let L be a straight line through the origin perpendicular to V. Let φ be the symmetry relative to V and ψ the symmetry relative to L. Let P_V be the orthogonal projection of the space $\mathbb{R}^3$ onto its subspace V and P_L the orthogonal projection of $\mathbb{R}^3$ onto L.

Find the eigenvalues and eigenvectors of all four maps defined above.

Find the explicit formulae for the maps φ, P_V, ψ and P_L using the standard basis of $\mathbb{R}^3$.

Solution. We shall use the formulae $\mathbf{v}_1 \cdot \mathbf{v}_1 = 1^2 + 1^2 + 1^2 = 3$, $\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \cdot 3 + 1 \cdot 0 + 1 \cdot 1 = 4$ and $\mathbf{v}_2 \cdot \mathbf{v}_2 = 3^2 + 0^2 + 1^2 = 10$. Orthogonal projection P_V of $\mathbb{R}^3$ onto V is defined as follows: let $\mathbf{p} \in \mathbb{R}^3$ be an arbitrary point, let $L(\mathbf{p})$ be a straight line through $\mathbf{p}$ parallel to L i.e. perpendicular (orthogonal) to V, the line $L(\mathbf{p})$ meets the plane V at one point $P_V(\mathbf{p})$ (so $\{P_V(\mathbf{p})\} = L(\mathbf{p}) \cap V$) which is called an orthogonal projection of $\mathbf{p}$ onto V. Notice that if $\mathbf{p}_1 \neq \mathbf{p}_2$ then the line through $\mathbf{p}_1$ and $\mathbf{p}_2$ is perpendicular to V if and only if $P_V(\mathbf{p}_1) = P_V(\mathbf{p}_2)$. We shall find a formula for $P_V(\mathbf{p})$. Since $P_V(\mathbf{p}) \in V$ there are numbers $a_1, a_2 \in \mathbb{R}$ such that $P_V(\mathbf{p}) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$. The vector $\mathbf{p} - P_V(\mathbf{p})$ is perpendicular to V which means that it is perpendicular to $\mathbf{v}_1$ and to $\mathbf{v}_2$. This is equivalent to $(\mathbf{p} - P_V(\mathbf{p})) \cdot \mathbf{v}_1 = 0 = (\mathbf{p} - P_V(\mathbf{p})) \cdot \mathbf{v}_2$ (both equations must hold). The equations may be rewritten: $\mathbf{p} \cdot \mathbf{v}_1 = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \cdot \mathbf{v}_1 = {=}a_1\mathbf{v}_1 \cdot \mathbf{v}_1 + a_2\mathbf{v}_2 \cdot \mathbf{v}_1 = 3a_1 + 4a_2$ and $p \cdot \mathbf{v}_2 = (a_1\mathbf{v}_1 + a_2\mathbf{v}_2) \cdot \mathbf{v}_2 = a_1\mathbf{v}_1 \cdot \mathbf{v}_2 + a_2\mathbf{v}_2 \cdot \mathbf{v}_2 = 4a_1 + 10a_2$. This implies that $5\mathbf{p} \cdot \mathbf{v}_1 - 2\mathbf{p} \cdot \mathbf{v}_2 = 15a_1 - 8a_1 = 7a_1$ so $a_1 = \frac{5}{7}\mathbf{p} \cdot \mathbf{v}_1 - \frac{2}{7}\mathbf{p} \cdot \mathbf{v}_2$ and $3\mathbf{p} \cdot \mathbf{v}_2 - 4\mathbf{p} \cdot \mathbf{v}_1 = 30a_2 - 16a_2 = 14a_2$ so $a_2 = \frac{3}{14}\mathbf{p} \cdot \mathbf{v}_2 - \frac{2}{7}\mathbf{p} \cdot \mathbf{v}_1$. Therefore the projection of the point $\mathbf{p}$ onto the plane V is the point $\left(\frac{5}{7}\mathbf{p} \cdot \mathbf{v}_1 - \frac{2}{7}\mathbf{p} \cdot \mathbf{v}_2\right)\mathbf{v}_1 + \left(\frac{3}{14}\mathbf{p} \cdot \mathbf{v}_2 - \frac{2}{7}\mathbf{p} \cdot \mathbf{v}_1\right)\mathbf{v}_2$. By the definition of a linear map the projection P_V is linear.

Now let us find a formula for the projection P_L onto L. Let $V(\mathbf{p})$ be the plane through $\mathbf{p}$ parallel to V i.e. perpendicular (orthogonal to L). It intersects the line L at the point $P_L(\mathbf{p})$ so $\{P_L(\mathbf{p})\} = L \cap V(\mathbf{p})$. We need a vector $\mathbf{v}_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ perpendicular to V so the equations $a + b + c = \mathbf{v}_1 \cdot \mathbf{v}_3 = 0 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 3a + c$ should be satisfied. This implies that $2a - b = 0$. Let $a = 1$. Then $b = 2$ and $c = -a - b = -3a = -3$ so $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. Of course this vector may be multiplied by an arbitrary real number different from 0. There exists a number α such that $P_L(\mathbf{p}) = \alpha\mathbf{v}_3$. Therefore $\mathbf{p} - \alpha\mathbf{v}_3$ is a vector perpendicular to $\mathbf{v}_3$ so $0 = \mathbf{v}_3 \cdot (\mathbf{p} - \alpha\mathbf{v}_3) = \mathbf{v}_3 \cdot \mathbf{p} - \alpha\mathbf{v}_3 \cdot \mathbf{v}_3$. Thus $\alpha = \frac{\mathbf{v}_3 \cdot \mathbf{p}}{\mathbf{v}_3 \cdot \mathbf{v}_3}$ and $P_L(p) = \frac{\mathbf{v}_3 \cdot \mathbf{p}}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$. Again one can easily see that the map P_L is linear in $\mathbf{p}$.

We have found the formulae for both projections.

We note that $P_V(\mathbf{p}) = \mathbf{p}$ for all $\mathbf{p} \in V$ so the number 1 is an eigenvalue of P_V and V consists the eigenvectors associated to 1 and of zero–vector. This implies that the multiplicity of the eigenvalue 1 is at least 2 (the dimension if the eigenspace does not exceed the multiplicity of the eigenvalue). Also $P_V(\mathbf{p}) = 0$ for all $\mathbf{p} \in L$. Therefore 0 is an eigenvalue of P_V and the eigenspace associated to 0 is L. Therefore the multiplicity of 1 is 2 and the multiplicity of 0 is 1 (the sum of multiplicities of all eigenvalues of the linear map from $\mathbb{R}^3$ into $\mathbb{R}^3$ is at most 3).

$P_L(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in L$ and $P_L(\mathbf{v}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \mathbf{v}$ for all $\mathbf{v} \in V$. So in) is a double eigenvalue of P_L and 1 is a single eigenvalue of P_L.

We shall find the matrices of both projections relative to the standard basis of $\mathbb{R}^3$. Let us start with the basis $A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. From the story of the eigenvaluers and of the eigenvectors it follows that $M(P_V)_A^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $M(P_L)_A^A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The following equalities hold $M_A^{st} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}$ and $M_{st}^A = (M_A^{st})^{-1} = \frac{1}{14}\begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix}$. We get

$$M(P_V)_{st}^{st} = M_A^{st} M(P_V)_A^A M_{st}^A = \frac{1}{14}\begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} =$$

$$= \frac{1}{14}\begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 13 & -2 & 3 \\ -2 & 10 & 6 \\ 3 & 6 & 5 \end{pmatrix} \quad \text{oraz}$$

$$M(P_V)_{st}^{st} = M_A^{st} M(P_V)_A^A M_{st}^A = \frac{1}{14}\begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} =$$

$$= \frac{1}{14}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{pmatrix}.$$

The conclusion is $P_V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 13 & -2 & 3 \\ -2 & 10 & 6 \\ 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 13x - 2y + 3z \\ -2x + 10y + 6z \\ 3x + 6y + 5z \end{pmatrix}$ and

$$P_L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} x + 2y - 3z \\ 2x + 4y - 6z \\ -3x - 6y + 9z \end{pmatrix}.$$

Now it is time for the symmetries. Since $\mathbf{v}_1, \mathbf{v}_2 \in V$ the two formulae $\varphi(\mathbf{v}_1) = \mathbf{v}_1$ and $\varphi(\mathbf{v}_2) = \mathbf{v}_2$. The vector $\mathbf{v}_3$ is perpendicular to the plane V so $\varphi(\mathbf{v}_3) = -\mathbf{v}_3$. This implies that

$M(\varphi)_A^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ so $M(\varphi)_{st}^{st} = M_A^{st} M(\varphi)_A^A M_{st}^A =$

$$= \frac{1}{14} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} =$$

$$= \frac{1}{14} \begin{pmatrix} 1 & 3 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 12 & -4 & 6 \\ -4 & 6 & 12 \\ 6 & 12 & -4 \end{pmatrix}.$$

$\psi(\mathbf{v}_3) = \mathbf{v}_3$ because the symmetry relative to the line L leaves the points of L where they have been (does not move them at all). $\psi(\mathbf{v}_1) = -\mathbf{v}_1$ and $\psi(\mathbf{v}_2) = -\mathbf{v}_2$ by the definition of the symmetry relative to the line. Therefore $M(\varphi)_A^A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This implies that

$M(\psi)_{st}^{st} = M_A^{st} M(\psi)_A^A M_{st}^A =$

$$= \frac{1}{14} \begin{pmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} =$$

$$= \frac{1}{14} \begin{pmatrix} -1 & -3 & 1 \\ -1 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} -2 & 10 & 6 \\ 5 & -4 & -1 \\ 1 & 2 & -3 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} -12 & 4 & -6 \\ 4 & -6 & -12 \\ -6 & -12 & 4 \end{pmatrix}.$$

The eigenvalues of φ are 1 and -1. The number 1 is a double eigenvalue because the dimension of the associated eigenspace is 2 and -1 is a single eigenvalues as before there is no room for higher multiplicities because the dimension of the whole space is 3 and the sum of multiplicities of all eigenvalues of the map is less or equal to the dimension of the space.

The eigenvalues of ψ are also 1 and -1 but in this case 1b is a single eigenvalue while -1 is a double eigenvalue, the justification of the statement as above. □

Remark. $\varphi = P_V - P_L$ and $\psi = P_L - P_V$. This statement is essentially obvious. We could have started with this equation and justify it geometrically. This would shorten the solution. One more observation: if you take a look at the matrices of φ and ψ then you may notice that the columns are mutually perpendicular and have the same length $= 1$ (do not forget of $\frac{1}{14}$ in front of the matrix!). It is so because any symmetry preserves lengths of all vectors. Then same is true for the rows of the matrices.

104. Let $L = \text{lin}(\mathbf{v})$ with $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find a formula for the rotations about L by $\frac{\pi}{2}$ radians and by π radians. Find the eigenvalues and the eigenvectors of both rotations.

Solution. Let V be a linear subpace orthogonal to L. V contains of all points $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x + 2y + 3z = 0$. Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in V perpendicular one to another. Let $\mathbf{v}_1' = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. The using standard row reductions we can find a vector $\mathbf{v}_2$ peprpendicular to $\mathbf{v}$ and to $\mathbf{v}_1'$. This leads to the formula $\mathbf{v}_2 = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$. Let $\mathbf{v}_1 = \sqrt{14}\mathbf{v}_1'$. Therefore the vectors $\mathbf{v}_1, \mathbf{v}_2$ have the same length $\sqrt{42}$. The rotation about L maps $\mathbf{v}$ to $\mathbf{v}$ (rotation axis consists of the points which are not moved by the rotation, so called fixed points). Let us assume that $\mathbf{v}_1$ after the rotation falls onto $\mathbf{v}_2$ (there are two possibilities: either $\mathbf{v}_1 \to \mathbf{v}_2$ or $\mathbf{v}_2 \to \mathbf{v}_1$, we considder the first one, the other one can be considered in the manner). There $\mathbf{v}_1 \to \mathbf{v}_2$ and $\mathbf{v}_2 \to -\mathbf{v}_1$. Let R be this rotaion and let $A = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2\}$. Clearly $M(R)_A^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.
Instead of solving the characteristic equation it is enough to notice that the only vectors turned onto vectors parallel to themselvers are those parallel to the line L and they are not moced at all. Therefore 1 is a single eigenvalue and the eigenspace associated to it is L. We need also a formulas for this rotation which means that we need to find the matrix $M(R)_{st}^{st}$.
$M_A^{st} = \begin{pmatrix} 1 & \sqrt{14} & 5 \\ 2 & \sqrt{14} & -4 \\ 3 & -\sqrt{14} & 1 \end{pmatrix}$. Therefore

$$M_{st}^A = (M_A^{st})^{-1} = \frac{-1}{42\sqrt{14}} \begin{pmatrix} -3\sqrt{14} & -6\sqrt{14} & -9\sqrt{14} \\ -14 & -14 & 14 \\ -5\sqrt{14} & 4\sqrt{14} & -\sqrt{14} \end{pmatrix} = \frac{1}{42} \begin{pmatrix} 3 & 6 & 9 \\ \sqrt{14} & \sqrt{14} & -\sqrt{14} \\ 5 & -4 & 1 \end{pmatrix}.$$

This implies that the rotation matrix equals $M(R)_{st}^{st} = M_A^{st} \cdot M(R)_A^A \cdot M_{st}^A =$

$$= \frac{1}{42} \begin{pmatrix} 1 & \sqrt{14} & 5 \\ 2 & \sqrt{14} & -4 \\ 3 & -\sqrt{14} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 \\ \sqrt{14} & \sqrt{14} & -\sqrt{14} \\ 5 & -4 & 1 \end{pmatrix} =$$

$$= \frac{1}{42} \begin{pmatrix} 1 & 5 & -\sqrt{14} \\ 2 & -4 & -\sqrt{14} \\ 3 & 1 & \sqrt{14} \end{pmatrix} \begin{pmatrix} 3 & 6 & 9 \\ \sqrt{14} & \sqrt{14} & -\sqrt{14} \\ 5 & -4 & 1 \end{pmatrix} =$$

$$=\frac{1}{42}\begin{pmatrix} 3 & 6+9\sqrt{14} & 9-6\sqrt{14} \\ 6-9\sqrt{14} & 12 & 18+3\sqrt{14} \\ 9+6\sqrt{14} & 18-3\sqrt{14} & 27 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 2+3\sqrt{14} & 3-2\sqrt{14} \\ 2-3\sqrt{14} & 4 & 6+\sqrt{14} \\ 3+2\sqrt{14} & 6-\sqrt{14} & 9 \end{pmatrix}.$$

We are done with rotation by $\frac{\pi}{2}$ radians. The last thing to do is to take care of rotaion by π radians about the same axis. It is enough to notice that this rotation is simply $R \circ R$ therefore it suffices to square the obtained matrix , i.e. multiply it by itself. I leave this task for the students. □

105. Let $A = \begin{pmatrix} 5+t & t & 0 \\ 0 & 5 & t \\ -t & -t & 5-t \end{pmatrix}$, $B = \begin{pmatrix} 5+t & t & 0 \\ -t & 5-t & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Find the eigenvalues of A, B, C?

Find a basis consisting of the eigenvectors or prove that such a basis does not exist.

Does there exist an invertible matrix M such that $A = MBM^{-1}$?

Does there exist an invertible matrix M such that $A = MCM^{-1}$?

Solution. Let start from the matrix C. Clearly $C \cdot \mathbf{v} = 5\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$. The only eigenvalue is 5 and the eigenspace associated to it is $\mathbb{R}^3$. Of course 5 is a triple eigenvalue.

Now we shall take care of B. To find the charecteristic polynomkial it suffices to expand the determinant of $B-\lambda I$ with respect to the third row. We get $(5-\lambda)((5+t-\lambda)(5-t-\lambda)+t^2) = (5-\lambda)((5-\lambda)^2-t^2+t^2) = (5-\lambda)^3$. The only eigenvalue in this case is 5, it is a triple eigenvalue of B. We have to find all vectors $\mathbf{v}$ such that $B\mathbf{v} = 5\mathbf{v}$. This leads to the system of equations

$$\begin{cases} (t+5)x + ty = 5x; \\ -tx + (5-t)y = 5y; \\ 5z = 5z. \end{cases}$$

After simplifications we get $tx + ty = 0$. This implies that for $t \neq 0$ the eigenspace is given by the equation $x + y = 0$ so it is a 2–dimensional linear subspace of $\mathbb{R}^3$. For $t = 0$ the matrix B equals to C.

The last is matrix A. The characteristic equation is $0 = \begin{vmatrix} 5+t-\lambda & t & 0 \\ 0 & 5-\lambda & t \\ -t & -t & 5-t-\lambda \end{vmatrix} =$

$=(5+t-\lambda)((5-\lambda)(5-t-\lambda)+t^2) - t \cdot t^2 = (5-\lambda+t)((5-\lambda)^2 - t(5-\lambda)+t^2) - t^3 =$
$= (5-\lambda)^3$. There also in this case the number 5 is a triple eigenvalue of the matrix. For $t = 0$ the matrix A equals to C. Assume now that $t \neq 0$. The equations for the eigenvectors are

$$\begin{cases} (t+5)x + ty = 5x; \\ 5y + tz = 5y; \\ -tx - ty + (5-t)z = 5z. \end{cases}$$

From the second equation and $t \neq 0$ it follows that $z = 0$. Thden the first and the third equations takde form $x + y = 0$. This means that in this case the eigenspace associated to 5 is one–dimensional – it is given by the equations $x + y = 0$ and $z = 0$.

This implies that for $t \neq 0$ the matrix M does not exist (the dimensions of the eigenspaces differ). For $t = 0$ the matrices A, B, C are equal. It is easy to see that if M is invertible then $A = MBM^{-1} = MCM^{-1}$

106. Let $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$.

Find a non-zero vector $\mathbf{v}_4$ orthogonal to the subspace $V = \text{lin}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Find a vector $\mathbf{w}$ symmetric to $\mathbf{v}_4$ relative to V and the orthogonal projection of $\mathbf{v}_4$ onto V.

Solution. Let $\mathbf{v}_4 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. The equations $\mathbf{v}_1 \cdot \mathbf{v}_4 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_4 = 0$ and $\mathbf{v}_3 \cdot \mathbf{v}_4 = 0$ may be written as

$$\begin{cases} -a + b + c - d = 0; \\ -2a + 2b - 2c + d = 0; \\ a + b - c = 0. \end{cases}$$

Add the first equation to the last one to get $2b = d$. Subtract the third equation multiplied by 2 from the second one to get $4a = d$. Thus $b = 2a$ and $c = a + b = 3a$. Let $a = 1$. Then $\mathbf{v}_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Since $\mathbf{v}_4$ is orthogonal (perpendicular) to V its projection onto V is $\mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. The vector $-\mathbf{v}_4 = \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix}$ is symmetric to $\mathbf{v}_4$ with respect to V. This is because the origin is the midpoint of the straight line segment with ends $\mathbf{v}_4$ and $-\mathbf{v}_4$ and this segment is perpendicular to V. □

107. (Problem 49 of 60 Problems) Let $W = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : \quad x_1 - x_2 + 2x_3 = 0 \right\}$. Find a formula expressing the orthogonal projection of $\mathbb{R}^3$ onto W and a formula expressing the orthogonal symmetry with respect to W.

Solution. W is a linear subspace of $\mathbb{R}^3$ of dimension $3 - 1 = 2$. It means that it is a plane through the origin. The plane is perpendicular to the vector $\mathbf{w} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Given $\mathbf{x} \in \mathbb{R}^3$ we shall find a number t such that the vector $\mathbf{x} - t\mathbf{w}$ is orthogonal to $\mathbf{w}$ (then $t\mathbf{w}$ will be the orthogonal projection of $\mathbf{x}$ onto the line $\text{lin}(\mathbf{w})$. The equality $0 = (\mathbf{x} - t\mathbf{w}) \cdot \mathbf{w} = \mathbf{x} \cdot \mathbf{w} - t\mathbf{w} \cdot \mathbf{w}$

must be fulfilled. Therefore $t = \frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$ so the orthogonal projection of $\mathbf{x}$ onto L is $\mathbf{y} = \frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$. Therefore $\mathbf{z} = \mathbf{x} - \mathbf{y} = \mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$ is the orthogonal projection of $\mathbf{x}$ onto W ($\mathbf{z} \in W$ and $\mathbf{x} - \mathbf{z} = \frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$ is orthogonal to W). We have $\mathbf{z} = \mathbf{x} + (\mathbf{z} - \mathbf{x})$ so the point symmetric to $\mathbf{x}$ with respdect to W is $\mathbf{q} = \mathbf{x} + 2(\mathbf{z} - \mathbf{x}) = 2\mathbf{z} - \mathbf{x}$. We may write

$$\mathbf{q} = \mathbf{x} - 2\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - 2\frac{x_1 - x_2 + 2x_3}{1+1+4}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2x_1 + x_2 - 2x_3 \\ x_1 + 2x_2 + 2x_3 \\ -2x_1 + 2x_2 - x_3 \end{pmatrix}.$$

This may be written as follows

$$\mathbf{q} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

This ends the solution. □